Synaptic algebras as models for quantum mechanics

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Abstract

This lecture is based on a joint work with David Foulis [3, 4, 5]. A synaptic algebra, introduced by Foulis in [1], is both a special Jordan algebra and an order-unit normed space satisfying certain natural conditions suggested by the partially ordered Jordan algebra of bounded Hermitian operators on a Hilbert space. The adjective synaptic, borrowed from biology, is meant to suggest that such an algebra coherently ties together the notions of a Jordan algebra, a spectral order-unit space, a convex effect algebra and an orthomodular lattice. All these structures appeared in the mathematical foundations of quantum mechanics.

In this lecture, we first review some basic properties of a synaptic algebra. In particular, we focus on the interaction between a synaptic algebra and its orthomodular lattice of projections. Our aim is to show that a synaptic algebra can host the probability measures and it can serve as a value algebra for quantum observables. We show that each element determines and is determined by a one-parameter family of projections – its spectral resolution. It follows that elements of a synaptic algebras correspond to quantum mechanical observables, and as an (Archimedean) order-unit space, a synaptic algebra has a rich supply of states.

It is known that soon after introducing Hilbert-space based foundations for quantum mechanics [7], von Neumann had begun to focus on what is now called a type II_1 factor as the appropriate mathematical basis for quantum mechanics. Later on, it was discovered that type III factors occur naturally in relativistic quantum field theory [6]. In this lecture we study the type I/II/III decomposition theory for a synaptic algebra. Whereas the orthomodular lattice (OML) of projections in a von Neumann algebra or a JW-algebra [8] is complete, the OML of projections in a synaptic algebra need not be complete. Our aim in this part is twofold. First we study equivalence of projections based on the symmetries in the synaptic algebra, and show that a synaptic algebra with a complete lattice of projections has sufficiently many properties in common with a JW-algebra to enable Topping's proof of his version of a type I/II/III decomposition theorem. Our second aim is to show how the type-theory developed in [2] for effect algebras applies to a synaptic algebra satisfying the much weaker central orthocompleteness condition. Our main tool in the study of type decompositions is the notion of a type determining (TD) subset of the projection lattice.

References

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