The Psi function and its zeros on the complex plane
Version 2.0

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https://digilander.libero.it/tr7mail#Psi

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The Psi function

On June 21 2024 [Ref. 1], I defined the **Psi function** as the sum of reciprocals of factorials to the *s* power:

$$\Psi(s) = \sum_{n=1}^{\infty} \frac{1}{n!^s} \qquad s \in \mathbb{C} \tag{1}$$

The zeros of the Psi function on the complex plane

I found interesting studying **the zeros of the Psi function** from a numerical point of view with MATLAB/Octave, Mathematica and a my multi-thread C++ software using Microsoft Visual Studio Community 2022 and some arbitrary-precision arithmetic/mathematical libraries.

I found numerically that **the zero with maximum Re(s)** in the area $0 < \mathrm{Im}(s) \leq 10^9$ is:

$$s\_max \approx 0.726346524 + i \cdot 928745093.078332215$$

Sigma_sup calculation

On June 23, 2024, I asked a question on the MathOverflow website, asking if it was possible to calculate:

$$\sigma_{\sup} := \sup\{\mathrm{Re}(s) : \Psi(s) = 0\} \qquad (2)$$

After a few days of theoretical work, on July 6, 2024 I myself posted an answer with an algorithm to calculate the solution, see [Ref. 2]:

https://mathoverflow.net/q/473764

I have also presented an Octave script to calculate it. This is the value approximated to 12 significant decimal places:

$$\sigma_{\sup} \approx 0.726347508576$$

So the zeros of the Psi function are restricted in the **critical strip**:

$$0 < \mathrm{Re}(s) < \sigma_{\sup}$$

just as we know that the non-trivial zeros of the **Riemann's Zeta function** are restricted in the **critical strip**:

$$0 < \mathrm{Re}(s) < 1$$

```
(*   sigma_sup  algorithm        version with arbitrary precision result   *)
Nmax = 200;
Cn = {1}; kn = {0};
For[n = 2, n ≤ Nmax, n = n + 1, If[PrimeQ[n], If[Cn⟦n - 1⟧ == 1, AppendTo[kn, 1], AppendTo[kn, 0]]; AppendTo[Cn, -1], PF = FactorInteger[n];
  For[m = 1;
   somma = 0, m ≤ Length[PF], m = m + 1, somma = somma + kn⟦PF⟦m⟧⟦1⟧⟧ * PF⟦m⟧⟦2⟧];  AppendTo[kn, Mod[somma, 2]];
  If[kn⟦n⟧ == 0, AppendTo[Cn, Cn⟦n - 1⟧], AppendTo[Cn, -Cn⟦n - 1⟧]]]]
```

```
In[•]:= Cn
Out[•]=
{1, -1, -1, -1, -1, 1, -1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1, -1, 1, -1, 1, -1, 1,
 1, -1, 1, 1, -1, -1, -1, 1, -1, -1, -1, -1, -1, 1, -1, -1, -1, 1, 1, 1, -1, 1, 1, 1, -1, -1, -1, -1, -1, -1, 1, 1, 1, -1,
 -1, -1, -1, -1, -1, 1, -1, -1, -1, 1, -1, 1, -1, -1, -1, -1, -1, 1, 1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1, 1, -1, 1, 1, 1,
 -1, 1, -1, 1, -1, -1, -1, -1, -1, 1, -1, 1, -1, -1, -1, 1, 1, -1, 1, -1, -1, 1, -1, 1, 1, 1, -1, 1, 1, -1, -1, -1, -1, 1,
 1, -1, -1, 1, -1, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1, 1, -1, -1, -1, -1, 1, 1, -1, -1, 1, -1, 1, -1, -1, 1, 1, -1, -1, -1,
 -1, 1, -1, -1, -1, -1, 1, 1, 1, -1, -1, -1, -1, -1, -1, 1, -1, -1, -1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 1, -1, 1}
```

```
In[•]:= kn
Out[•]=
{0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0,
 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 1, 1,
 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 1,
 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1}
```

```
In[•]:= N[1 / 200 ! ^0.72634]
Out[•]=
```

4.98189×10^{-273}

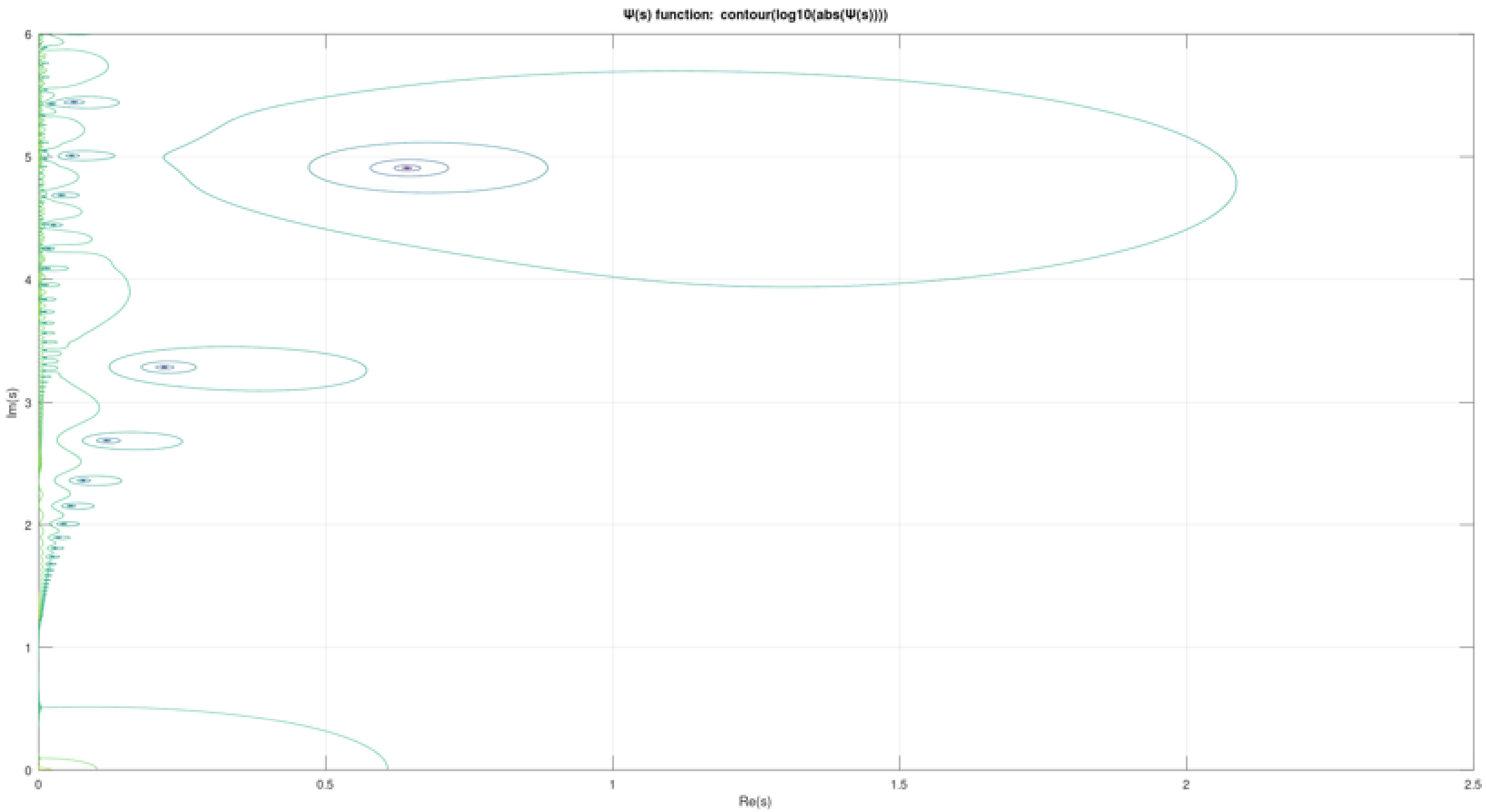
```
In[•]:= (*   sigma_sup   *)
NSolve[ {Sum[Cn⟦n⟧ * n ! ^ -SigmaSup, {n, 1, Nmax}] == 0, SigmaSup > 1 / 10, SigmaSup < 1}, SigmaSup, WorkingPrecision → 200]
Out[•]=
{{SigmaSup →
   0.726347508576201145941640262269523250850134334300641278184683634126562991783232991193408923590644698306930893715608287072339740688014
    89637235900420989499716627030139187568755886609785376580289304168236 5}}
```

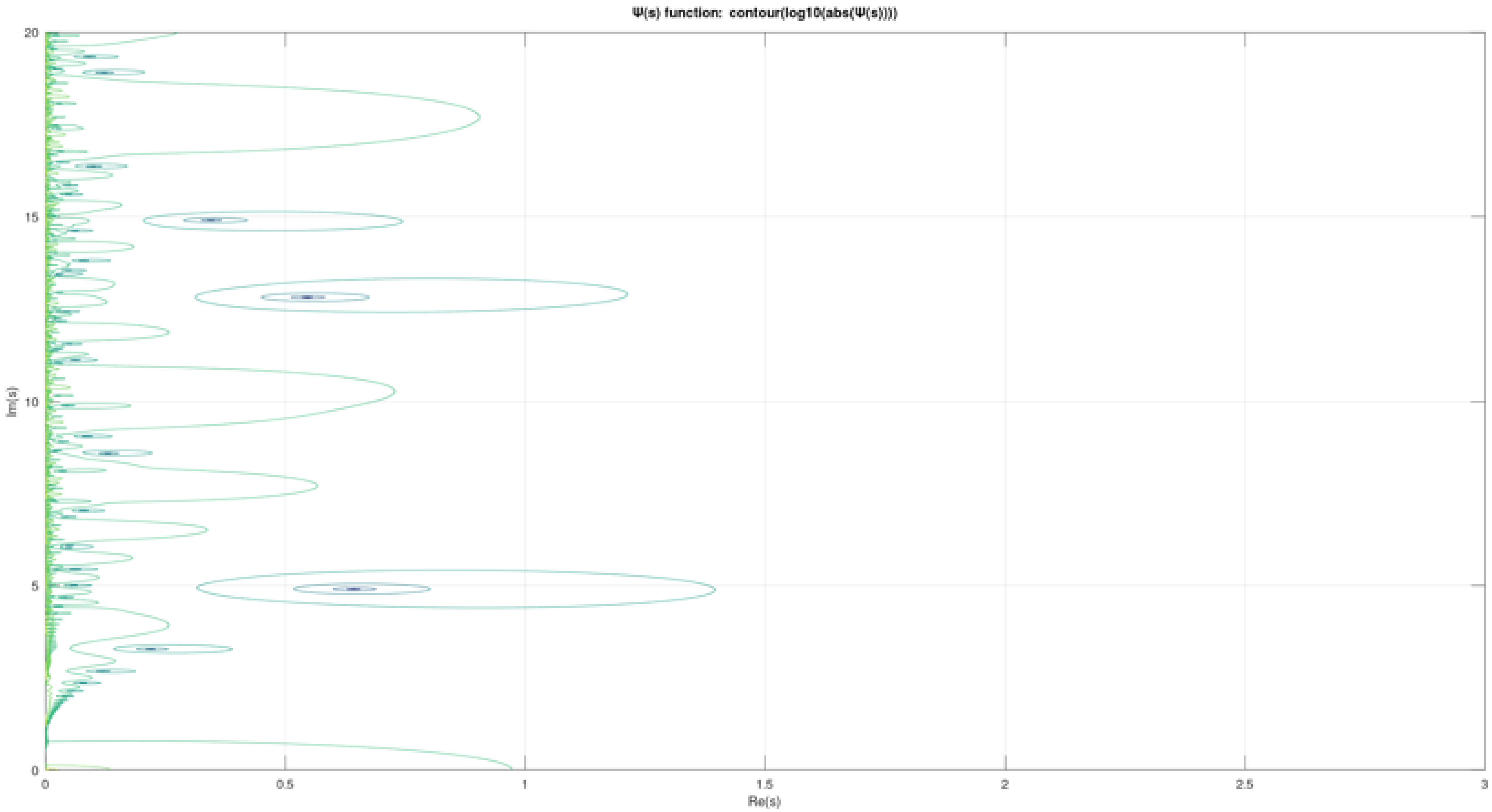
A proposal for a new mathematical constant

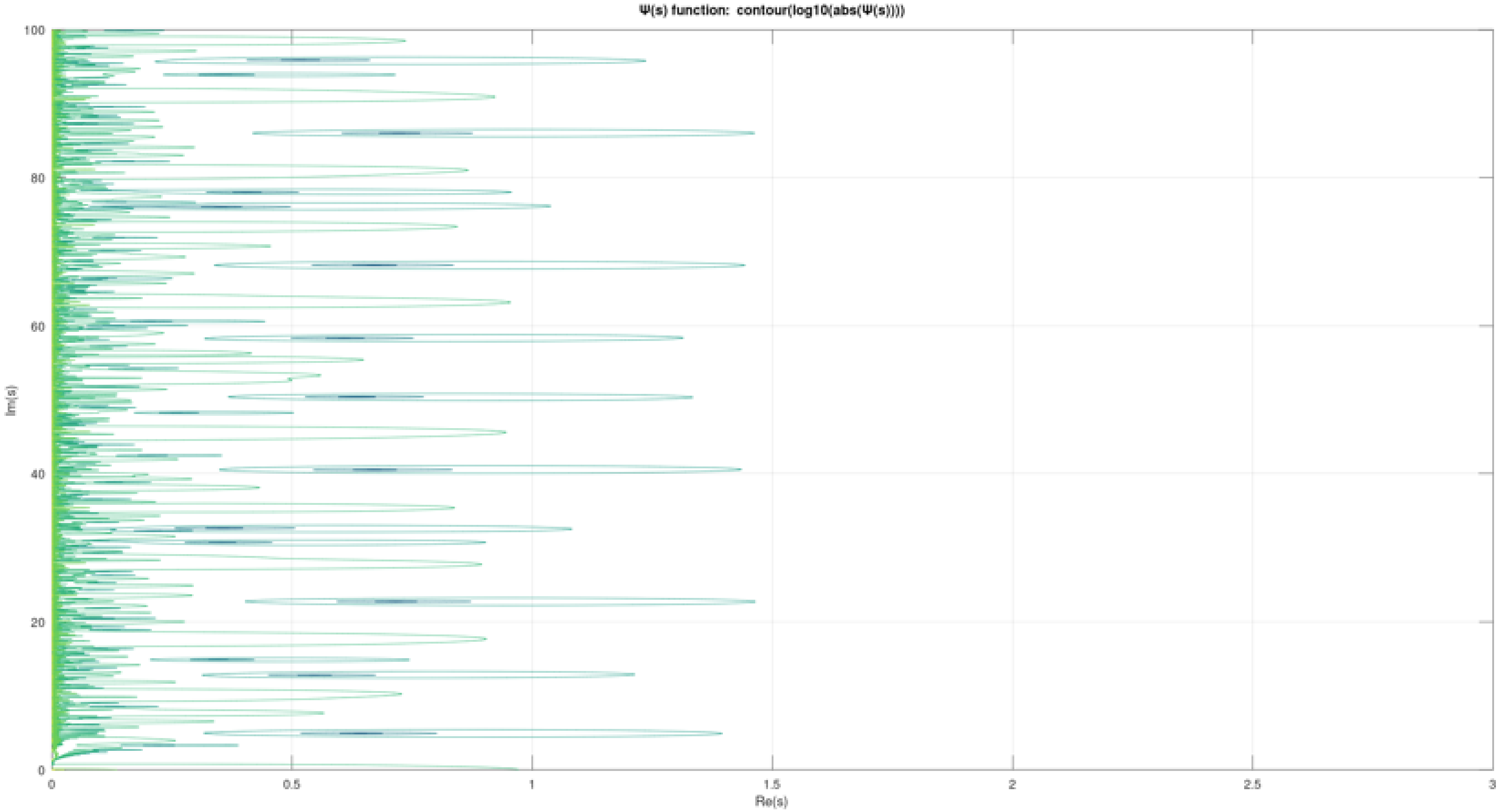
Below is a sequence of graphs of the levels (contour) of the function log10(abs(psi(s))) for s belonging to the complex plane.

In the first graph, the **first zero in the semi-plane Re(s)** $\geq \frac{1}{2}$ is in evidence.
His position is approximately $s_1 \simeq$ **0.6418158643 + 4.9068764351i** .

Im(s_1**) ≃ 4.9068764351** seems to be **a new mathematical constant**.
So I registered it on the **OEIS.org website** [Ref. 3]: https://oeis.org/A373204 .

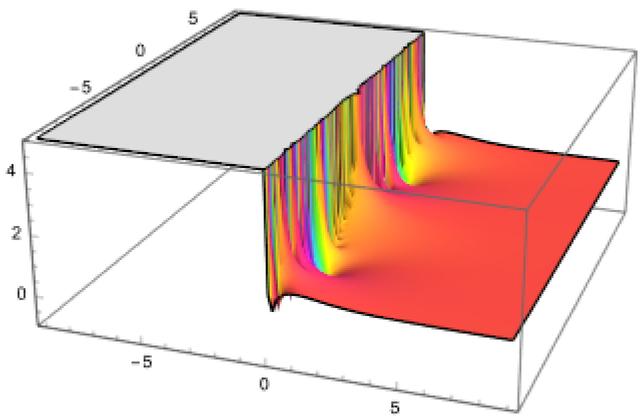

Ψ(s) function: contour(log10(abs(Ψ(s))))
Im(s)
Re(s)
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5
6


Ψ(s) function: contour(log10(abs(Ψ(s))))
Im(s)
Re(s)
0
5
10
15
20
0
0.5
1
1.5
2
2.5
3


Ψ(s) function: contour(log10(abs(Ψ(s))))
100
80
60
40
20
0
Im(s)
0
0.5
1
1.5
2
2.5
3
Re(s)

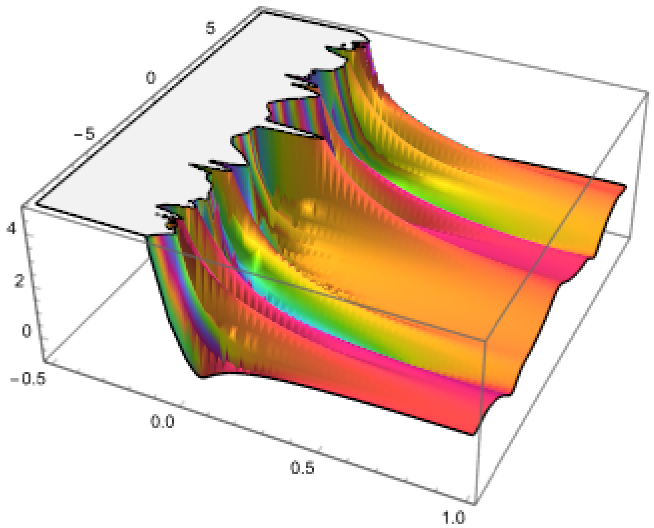
```
(*   Approximate Psi(s) function   *)

Psi[s_, nmax_] := ParallelSum[1 / n ! ^s, {n, 1, nmax}]



ComplexPlot3D[Psi[s, 10], {s, -9 - 9 I, 9 + 9 I}, PlotRange → {-1, 5}]
```

Out[◦]=




```
ComplexPlot3D[Psi[s, 10], {s, -1 / 2 - 9 I, 1 + 9 I}, PlotRange → {-1, 5}]
```

Out[◦]=



```
(*   Examples of real values   *)

N[Psi[1000, 200], 100]    (*   limit   Psi(s) ──Re(s)→ ∞──→ 1   *)
Out[∘]=
1.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000

N[Psi[1, 200], 100]     (*   Psi(1) = e − 1   *)
Out[∘]=
1.718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178525166427

In[∘]:= N[Psi[1 / 2, 200], 100]
Out[∘]=
2.469506314521047562475636744660150257689756183994969199677925701971835488003408834038662508412007135

In[∘]:= N[Psi[1 / 4, 300], 100]
Out[∘]=
3.706322173479391073244427601647255813774406138034280338206804109345001835671964744961441193290389664

In[∘]:= N[Psi[5 / 1000, 20 000], 100]
Out[∘]=
58.35677582171687522306150932691040732414573671813989447675578698932963030813328134227727344209568700

In[∘]:= N[Psi[1 / 1000, 20 000], 100]
Out[∘]=
212.8934344404598864421628936787745427802332320971500245174100206444701514489283865814643740128295603
```

```
(*   The s1 zero   *)

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 50]
Out[•]=
{x → 0.64181586433314113248112139394951847372318338379012, y → 4.9068764351428513475351082583558535315328564648993}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 150]
Out[•]=
{x →
  0.641815864333141132481121393949518473723183383790117174605186222762580517064969791362762127145058179633534712208669467283714546798848\
   631625316442136767,
 y →
  4.906876435142851347535108258355853531532856464899337635202889524870080968491604060111065196944920181067326293799784810249860650082932\
   55799891424877982}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 200]
Out[•]=
{x →
  0.641815864333141132481121393949518473723183383790117174605186222762580517064969791362762127145058179633534712208669467283714546798848\
   63162531644213676681386342214014473677286328546523427080926215482659, y →
  4.906876435142851347535108258355853531532856464899337635202889524870080968491604060111065196944920181067326293799784810249860650082932\
   5579989142487798169771964786789243286585094776786735969645818985011}
```

```
(*   Other examples of zeros   *)

In[•]:= FindRoot[{Re[Psi[x + y * I, 2000]], Im[Psi[x + y * I, 2000]]}, {{x, 1336 / 1 000 000}, {y, 1 + 4705 / 1 000 000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.0013361234465891396568602622971086091779190211534074, y → 1.0047047214630754990772104173473400204889209870526}

In[•]:= FindRoot[{Re[Psi[x + y * I, 20 000]], Im[Psi[x + y * I, 20 000]]}, {{x, 1336 / 1 000 000}, {y, 1 + 4705 / 1 000 000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.0013361234423282565356094057650499049578042780328027, y → 1.0047047214631007857421115748599516575637423334916}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 670 / 1000}, {y, 40 + 516 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.67022547274446564754942561921853627239573382162093, y → 40.516540228521088324200375633146033781651332005560}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 721 / 1000}, {y, 86 + 2 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.72159477459801208832092300307574122028151736384671, y → 86.002031717653971712641931481424395869298288479074}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 200 / 1000}, {y, 512 + 787 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.19977206033939305741111784335367081276729571786498, y → 512.78788254499022664698311303017535684680849482170}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 611 / 1000}, {y, 1019 + 180 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.61144000424789254536789403748651182040962005969469, y → 1019.1796100394148909618609953176907836874492265833}
```

0.6919053669 + 149 998.1372808481 i

```
In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 692 / 1000}, {y, 149 998 + 137 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.69190536689323821234606604538539980100012880561469, y → 149 998.13728084803537276334254075480663772664388012}
```

0.726240868445895 + 108 744.911314453 i

(* The zero with maximum Re(s) (Im(s)<150000) *)

```
In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]}, {{x, 726 / 1000}, {y, 108 744 + 911 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.72624086844586790634634422045970079610134134053575, y → 108 744.91131445345894385570366150252207947684867118}
```

```
s = 0.726346524 + 928 745 093.078332215 * I
(*     Im(s) ≤ 10^9   *)
(*   zero with maximum Re(s)   *)
```

In[∘]:=
```
FindRoot[{Re[Psi[x + y * I, 2000]], Im[Psi[x + y * I, 2000]]}, {{x, 726 / 1000}, {y, 928 745 093 + 078 / 1000}}, WorkingPrecision → 70]
t = %[[2]][[2]]
```

Out[∘]=

{x → 0.7263465240704593823610249313772206410122178177821511169269650494313870,
y → 9.287450930783322146495079966918712988899515528150392732819424179206443 × 10^8}

Out[∘]=

9.287450930783322146495079966918712988899515528150392732819424179206443 × 10^8

In[∘]:=
```
t
TableForm[Table[{n, N[Cos[t * Log[n !]], 8], N[Sin[-t * Log[n !]], 8], If[Abs[t * Log[n] / Pi - Round[t * Log[n] / Pi]] < 0.02,
    N[t * Log[n] / Pi - Round[t * Log[n] / Pi], 6]], If[Abs[t * Log[n] / Pi - Round[t * Log[n] / Pi]] < 0.02, Round[t * Log[n] / Pi]],
   If[Abs[t * Log[n] / Pi - Round[t * Log[n] / Pi]] < 0.02, FactorInteger[Round[t * Log[n] / Pi]]]}, {n, 2, 20}],
  TableHeadings → {None, {"n", "C_n(t)", "S_n(t)", "err", "k_n(t)", "k_n factors"}}]
```

Out[∘]=

9.287450930783322146495079966918712988899515528150392732819424179206443 × 10^8

Out[∘]//TableForm=

n	C_n(t)	S_n(t)	err	k_n(t)	k_n factors
2	−0.99999986	0.00052808127	0.000168093	204 914 231	9749 1 21 019 1
3	−0.99999899	−0.0014225763	−0.000620914	324 781 372	2 2 81 195 343 1
4	−0.99999993	−0.00036641422	0.000336187	409 828 462	2 1 9749 1 21 019 1
5	−0.99999015	0.0044376574	0.00152919	475 796 110	2 1 5 1 47 579 611 1
6	0.99999545	−0.0030150906	−0.000452820	529 695 603	3 2 58 855 067 1
7	−0.99999436	−0.0033598765	−0.00202922	575 266 975	5 2 23 010 679 1
8	0.99999842	0.0017756380	0.000504280	614 742 693	3 1 9749 1 21 019 1
9	0.99998389	0.0056769246	−0.00124183	649 562 744	2 3 81 195 343 1

10	−0.99999994	−0.00034478758	0.00169728	680 710 341	3 1 109 1 769 1 2707 1
11	−0.86172998	−0.50736717	Null	Null	Null
12	−0.86127579	−0.50813778	−0.000284727	734 609 834	2 1 7 1 23 1 541 1 4217 1
13	0.99371908	−0.11190345	Null	Null	Null
14	0.99435638	−0.10609139	−0.00186113	780 181 206	2 1 3 1 7 1 23 1 743 1 1087 1
15	0.99404961	−0.10892828	0.000908274	800 577 482	2 1 6599 1 60 659 1
16	0.99381730	−0.11102779	0.000672374	819 656 924	2 2 9749 1 21 019 1
17	−0.94297978	−0.33285002	Null	Null	Null
18	0.94185164	0.33602901	−0.00107373	854 476 975	5 2 11 1 131 1 23 719 1
19	0.46143296	0.88717508	Null	Null	Null
20	0.46662407	0.88445575	0.00186538	885 624 572	2 2 7 1 47 1 672 967 1

In[*]:=
```
(*   Factorial   *)
Table[{n, n!}, {n, 1, 35}] // Column
```

Out[*]=
```
{1, 1}
{2, 2}
{3, 6}
{4, 24}
{5, 120}
{6, 720}
{7, 5040}
{8, 40 320}
{9, 362 880}
{10, 3 628 800}
{11, 39 916 800}
{12, 479 001 600}
{13, 6 227 020 800}
{14, 87 178 291 200}
{15, 1 307 674 368 000}
{16, 20 922 789 888 000}
{17, 355 687 428 096 000}
{18, 6 402 373 705 728 000}
{19, 121 645 100 408 832 000}
{20, 2 432 902 008 176 640 000}
{21, 51 090 942 171 709 440 000}
{22, 1 124 000 727 777 607 680 000}
{23, 25 852 016 738 884 976 640 000}
{24, 620 448 401 733 239 439 360 000}
{25, 15 511 210 043 330 985 984 000 000}
{26, 403 291 461 126 605 635 584 000 000}
{27, 10 888 869 450 418 352 160 768 000 000}
{28, 304 888 344 611 713 860 501 504 000 000}
{29, 8 841 761 993 739 701 954 543 616 000 000}
{30, 265 252 859 812 191 058 636 308 480 000 000}
{31, 8 222 838 654 177 922 817 725 562 880 000 000}
{32, 263 130 836 933 693 530 167 218 012 160 000 000}
{33, 8 683 317 618 811 886 495 518 194 401 280 000 000}
{34, 295 232 799 039 604 140 847 618 609 643 520 000 000}
{35, 10 333 147 966 386 144 929 666 651 337 523 200 000 000}
```

```
(*   Some examples of contributions to the summation   *)
```

In[ ]:= `N[1 / (200 !) ^ (1 / 2)]`

Out[ ]=

3.56087×10^{-188}

In[ ]:= `N[1 / (200 !) ^ (1 / 4)]`

Out[ ]=

1.88703×10^{-94}

In[ ]:= `N[1 / (200 !) ^ (1 / 1000)]`

Out[ ]=

0.421797

In[ ]:= `N[1 / (2000 !) ^ (1 / 1000)]`

Out[ ]=

1.83857×10^{-6}

In[ ]:= `N[1 / (20 000 !) ^ (1 / 1000)]`

Out[ ]=

4.59981×10^{-78}

In[ ]:= `N[1 / (1 000 000 000 !) ^ (1 / 1 000 000 000)]`

The “main sequence of zeros” leading to the origin of the complex plane

I also numerically calculated some zeros of the Psi function present in the “main sequence” which seems to start from the zero s_1, creating an arc which tends towards the origin of the complex plane (see next figure) and which become more and more dense as they approach the origin.

Since the *n*-th term contributes, for large *n*:

$$\frac{1}{n!^{\sigma}} = e^{-\sigma \cdot \log(n!)} \approx e^{-\sigma \cdot (n \cdot \log n - n)}$$

and since:

```
In[◦]:= n = 1000000000;
σ = 1 / 1000000000;
N[E^(-σ * (n * Log[n] - n))]
```

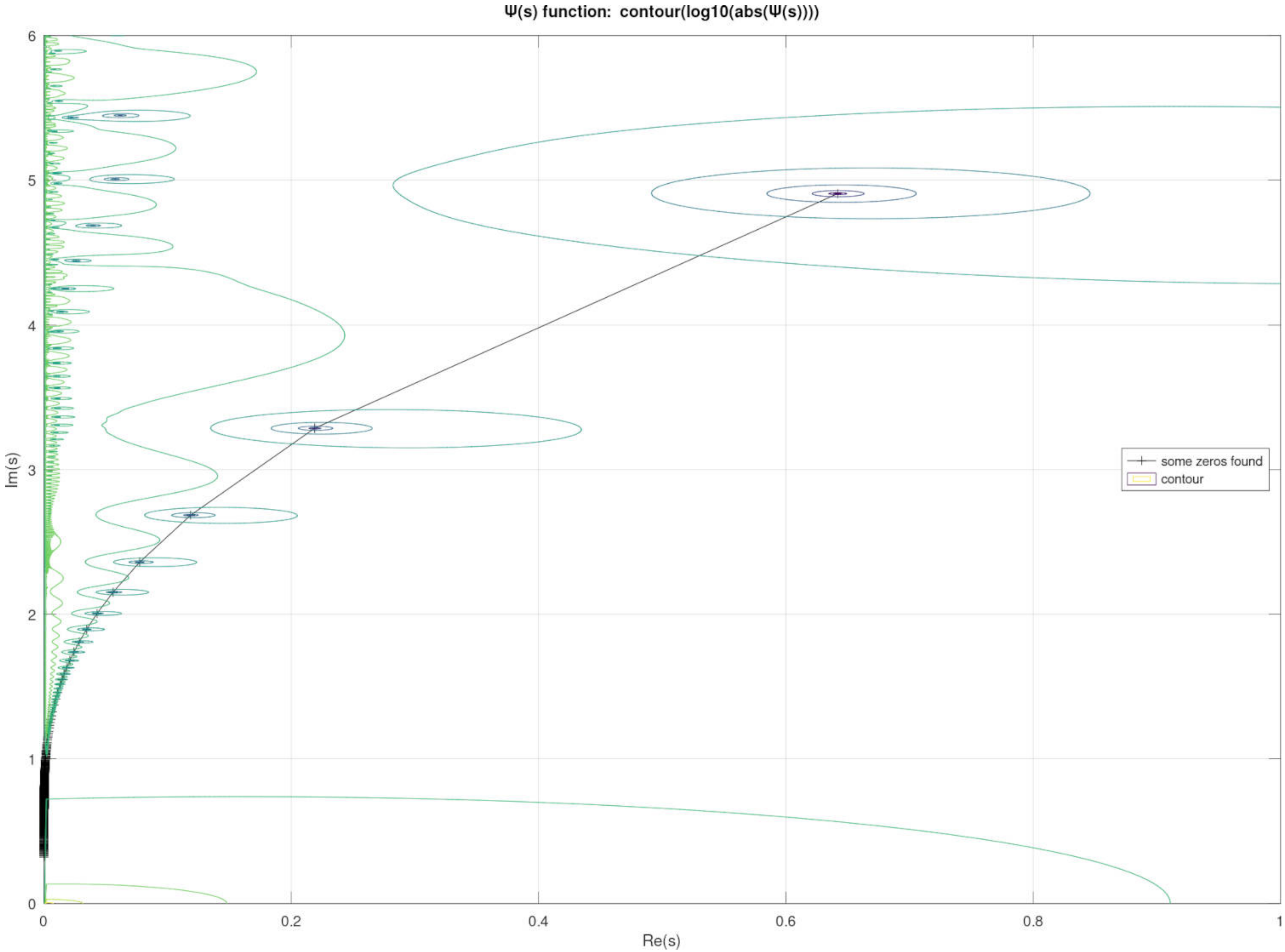
```
Out[◦]=
2.71828 × 10^-9
```

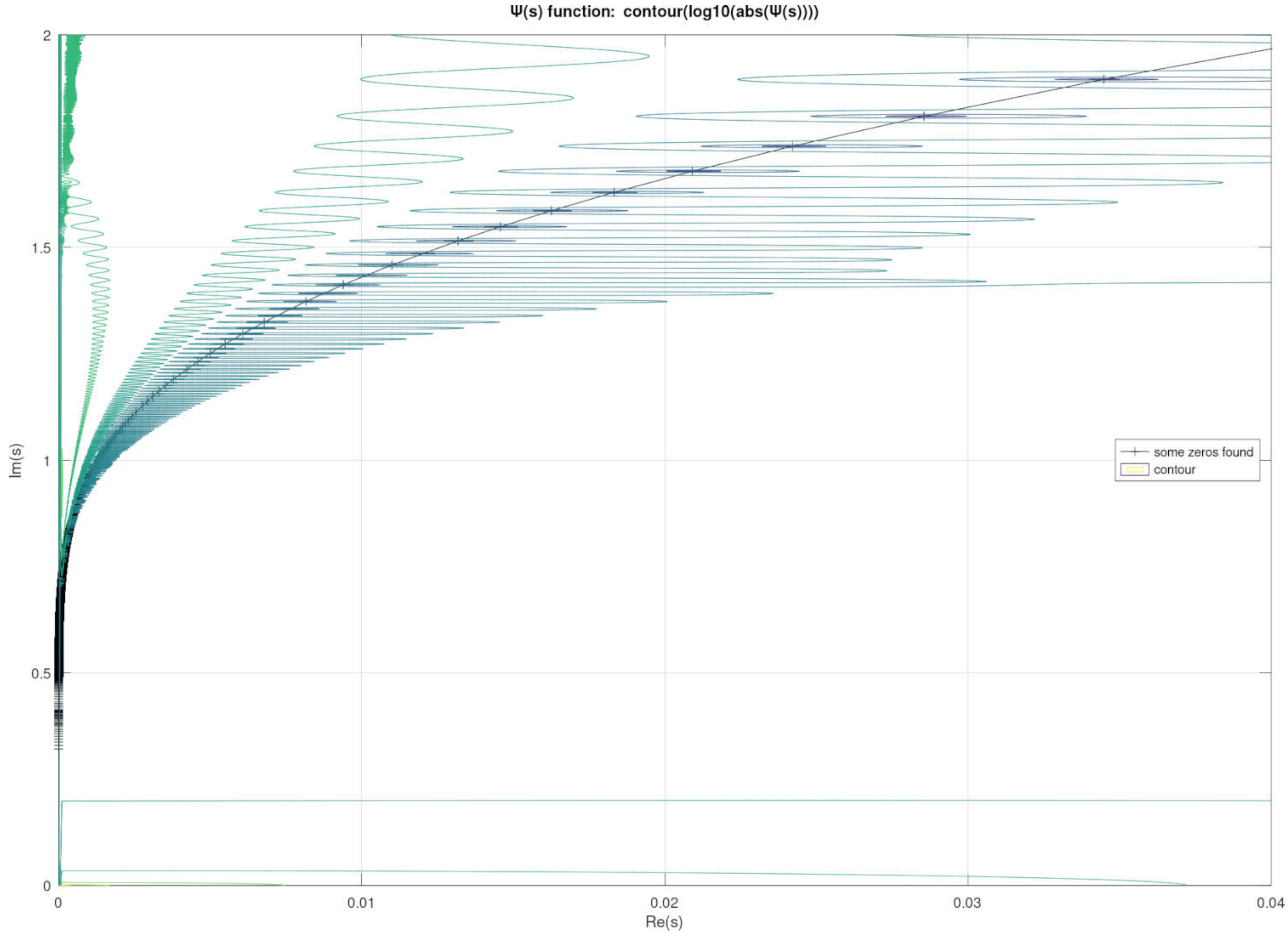
so to calculate zeros up to $\sigma = \mathrm{Re}(s)$ of the order of 10^{-9} I had to take into account the terms up to $n = 10^9$ (a billion terms!) in the summation of the Psi function:

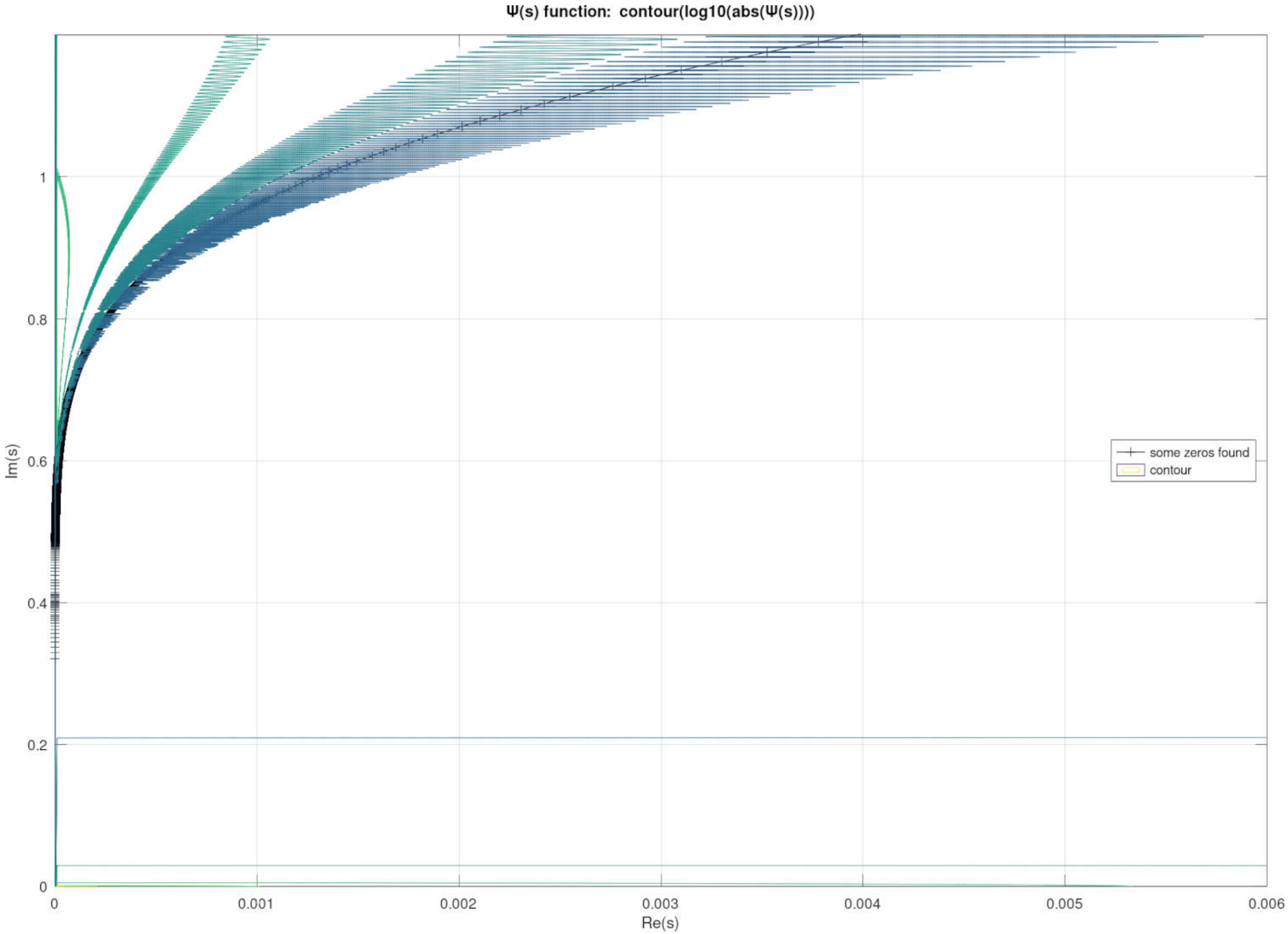
$$\Psi(s) = \sum_{n=1}^{\infty} \frac{1}{n!^{s}} = \sum_{n=1}^{\infty} e^{-s \cdot \log n!} = \sum_{n=1}^{\infty} e^{-\sigma \cdot \log n!} [\cos(t \cdot \log n!) - i * \sin(t \cdot \log n!)] \qquad s = \sigma + i \cdot t \tag{3}$$

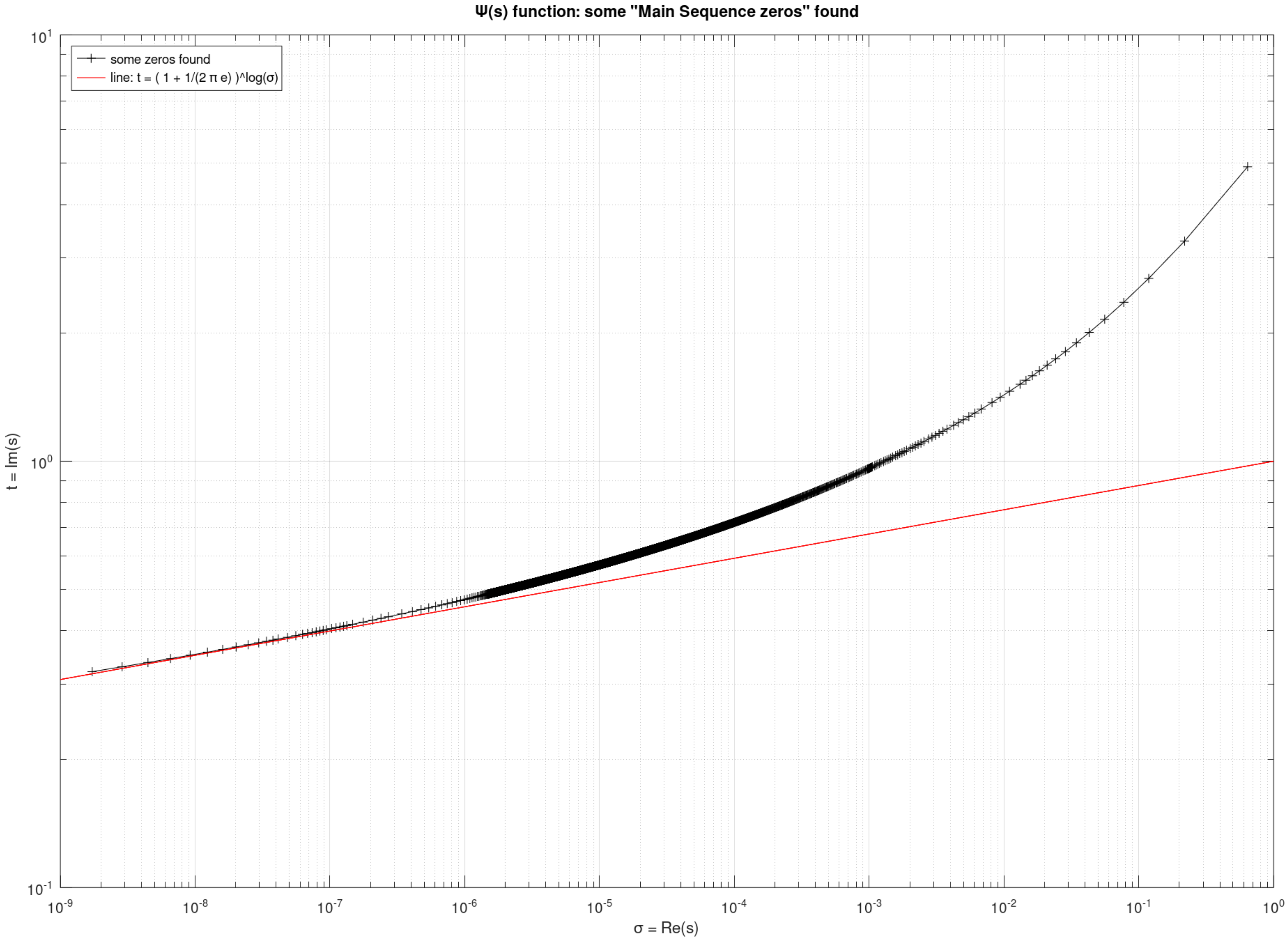
To speed up the calculation, I pre-calculated a log(*n*!) table in multiple precision (quad-double (equivalent to about 64 decimal digits of mantissa)) and I wrote my own multi-threaded C++ software, with a zero-finding algorithm, to exploit all the cores of the PC microprocessor.

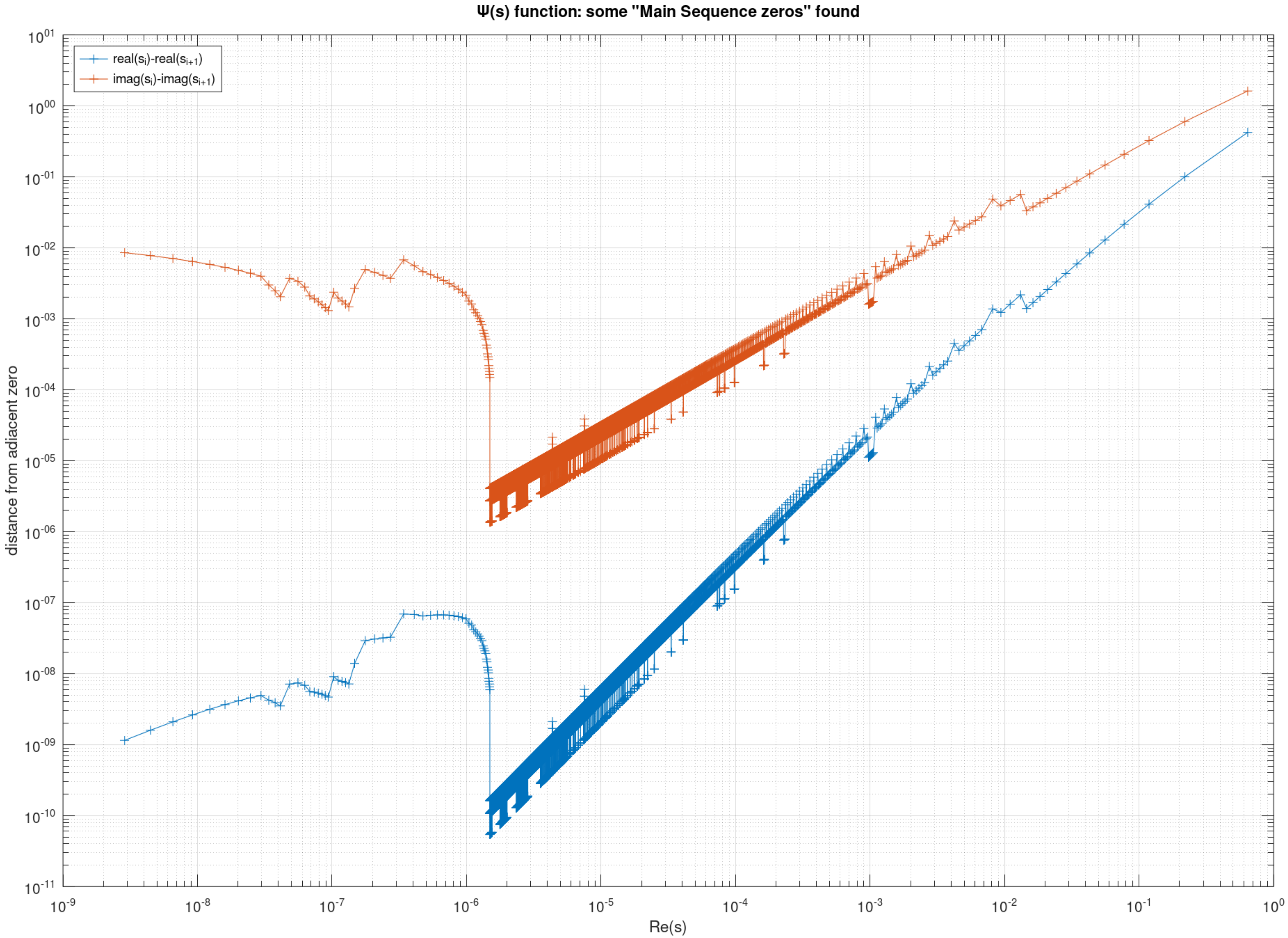
Below are a series of graphs of some “main sequence” zeros that I found.


Ψ(s) function: contour(log10(abs(Ψ(s))))
Im(s)
Re(s)
0
1
2
3
4
5
6
0
0.2
0.4
0.6
0.8
1
some zeros found
contour


Ψ(s) function: contour(log10(abs(Ψ(s))))
2
1.5
1
0.5
0
Im(s)
0
0.01
0.02
0.03
0.04
Re(s)
some zeros found
contour


Ψ(s) function: contour(log10(abs(Ψ(s))))
Im(s)
Re(s)
0
0.2
0.4
0.6
0.8
1
0.001
0.002
0.003
0.004
0.005
0.006
some zeros found
contour


Ψ(s) function: some "Main Sequence zeros" found
some zeros found
line: t = (1 + 1/(2 π e))^log(σ)
t = Im(s)
σ = Re(s)
10^1
10^0
10^-1
10^-9
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1
10^0



Ψ(s) function: some "Main Sequence zeros" found
real(s_i)-real(s_i+1)
imag(s_i)-imag(s_i+1)
distance from adiacent zero
Re(s)

The Psi_0 function

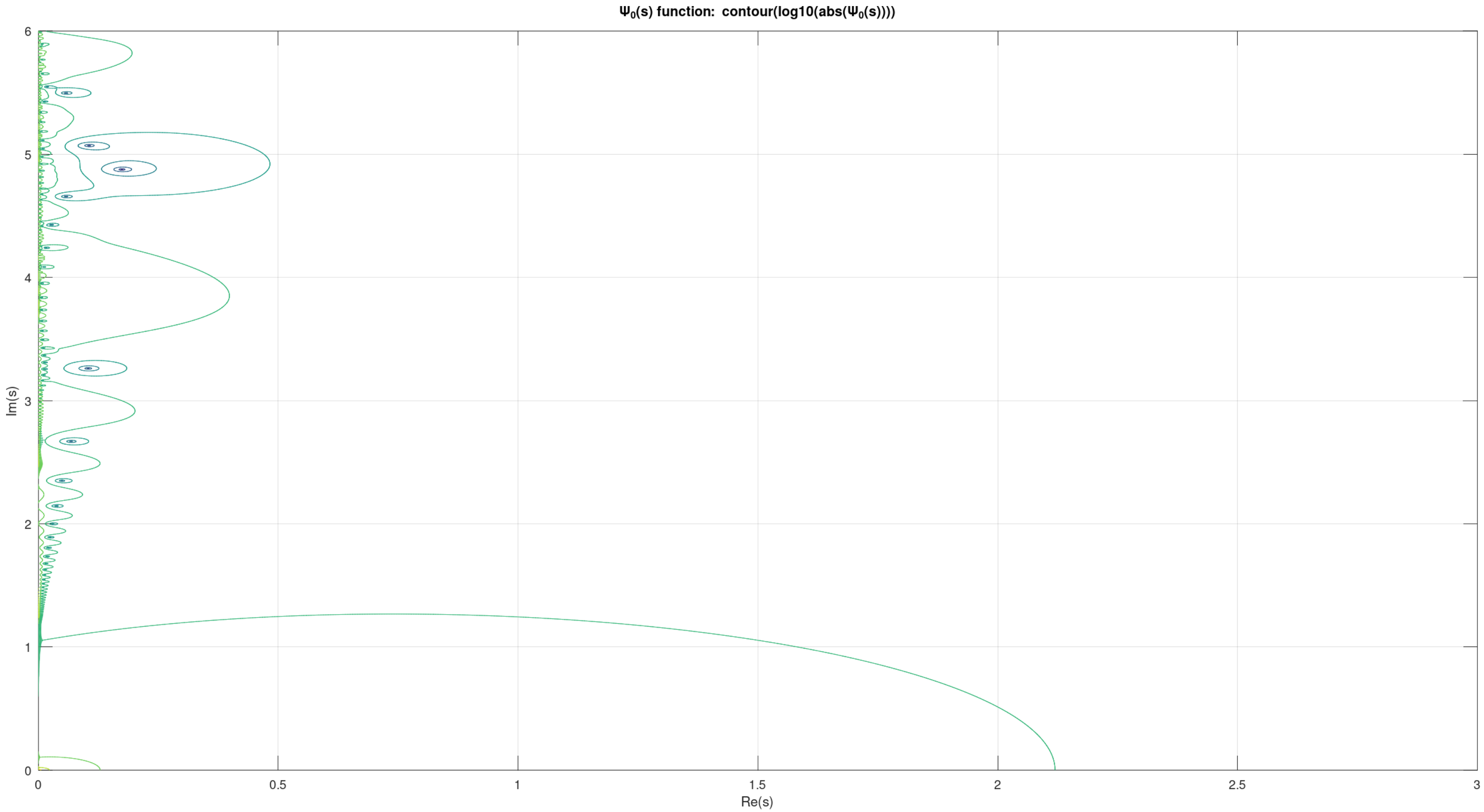
Defining the **Psi_0 function** as the sum of reciprocals of factorials to the s power, **but starting from *n*=0** :

$$\Psi_0(s) = \sum_{n=0}^{\infty} \frac{1}{n!^s} \qquad s \in \mathbb{C} \tag{4}$$

we have

$$\sigma_{\text{sup0}} := \sup \{ \operatorname{Re}(s) : \Psi_0(s)=0 \}$$

$$\sigma_{\text{sup0}} \approx 0.275793268746$$


Ψ0(s) function: contour(log10(abs(Ψ0(s))))
Im(s)
Re(s)
0
0.5
1
1.5
2
2.5
3
1
2
3
4
5
6

Conclusions

By definition, in some mode, the **Psi function** is a “part” of the **Riemann’s Zeta function**.
The zeros of the Psi function are in the **critical strip** $\mathbf{0 < Re(s) < \sigma_{sup} \approx 0.726347508576}$.
I proposed as **new mathematical constant** the imaginary part of the first zero of the Psi function for Re(s)$\geq\frac{1}{2}$: $\mathbf{Im(s_1) \approx 4.9068764351}$.
I’m sorry, but I have not found reference materials really regarding a function like the Psi function and the study of its zeros.
The zeros of the Zeta function are related to the distribution of prime numbers, as demonstrated by the legendary genius of Riemann in 1859, who knows if it will be the same for the Psi function.

References

1) The Psi function and its zeros on the complex plane - June 21 2024 https://digilander.libero.it/tr7mail/Psi_function_zeros.pdf
2) https://mathoverflow.net/q/473764
3) https://oeis.org/A373204

4) https://oeis.org/A058303
5) https://mathworld.wolfram.com/FactorialSums.html
6) https://mathworld.wolfram.com/RiemannZetaFunction.html