The Psi function and its zeros on the complex plane

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The Psi function

I defined the **Psi function** as the sum of reciprocals of factorials to the s power:

$$\psi(s) := \sum_{n=1}^{\infty} \frac{1}{(n!)^s} \qquad s \in \mathbb{C} \tag{1}$$

The zeros of the Psi function on the complex plane

I found interesting studying **the zeros of the Psi function** from a computational point of view with MATLAB/Octave and Mathematica.

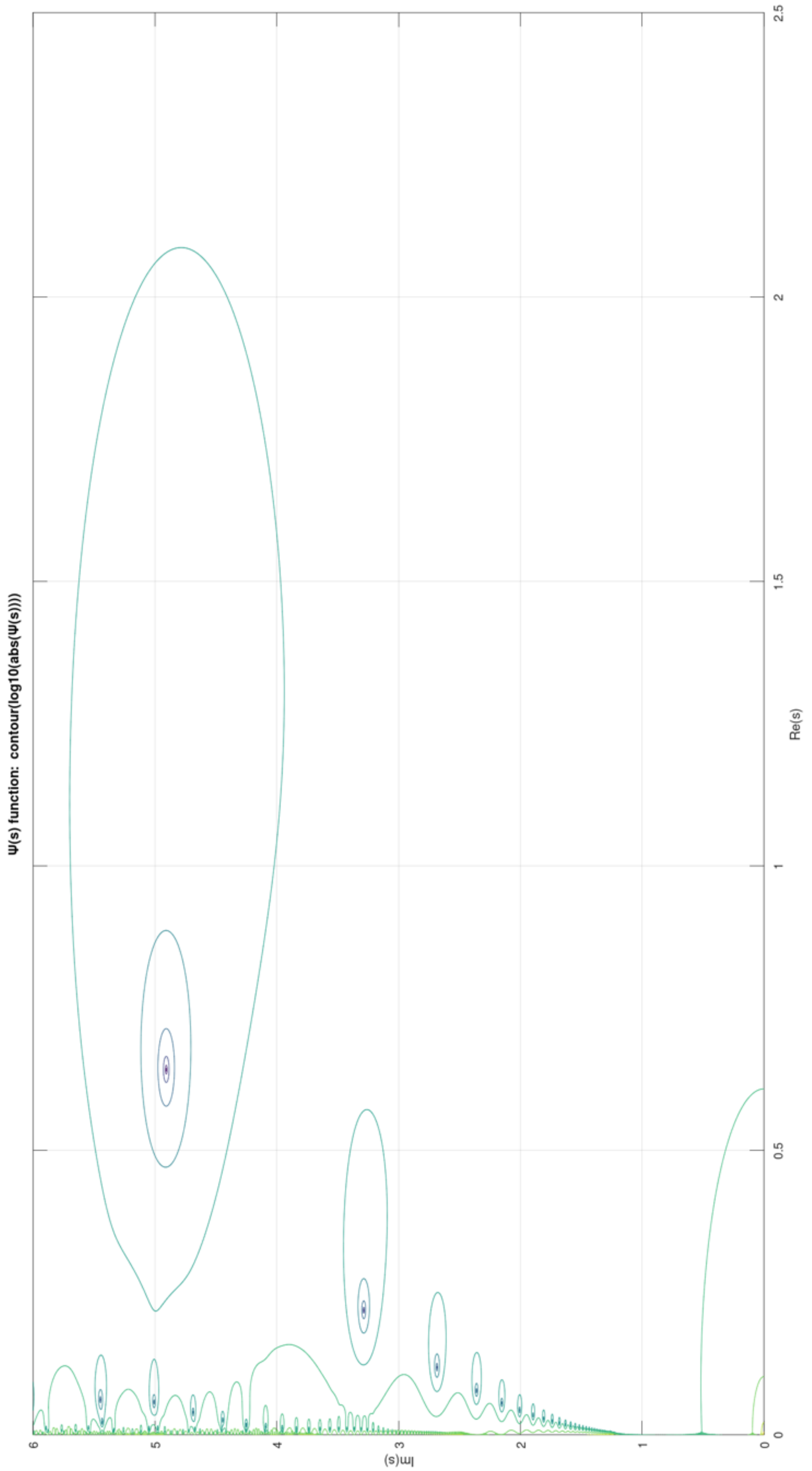
I formulate two conjectures:
Conjecture 1: the non-trivial zeros of the Psi function are all in the **critical strip** $0 < \mathrm{Re}(s) \leqslant 1$.
Conjecture 2: the non-trivial zeros of the Psi function are all in the strip $0 < \mathrm{Re}(s) \leqslant 0.73$.

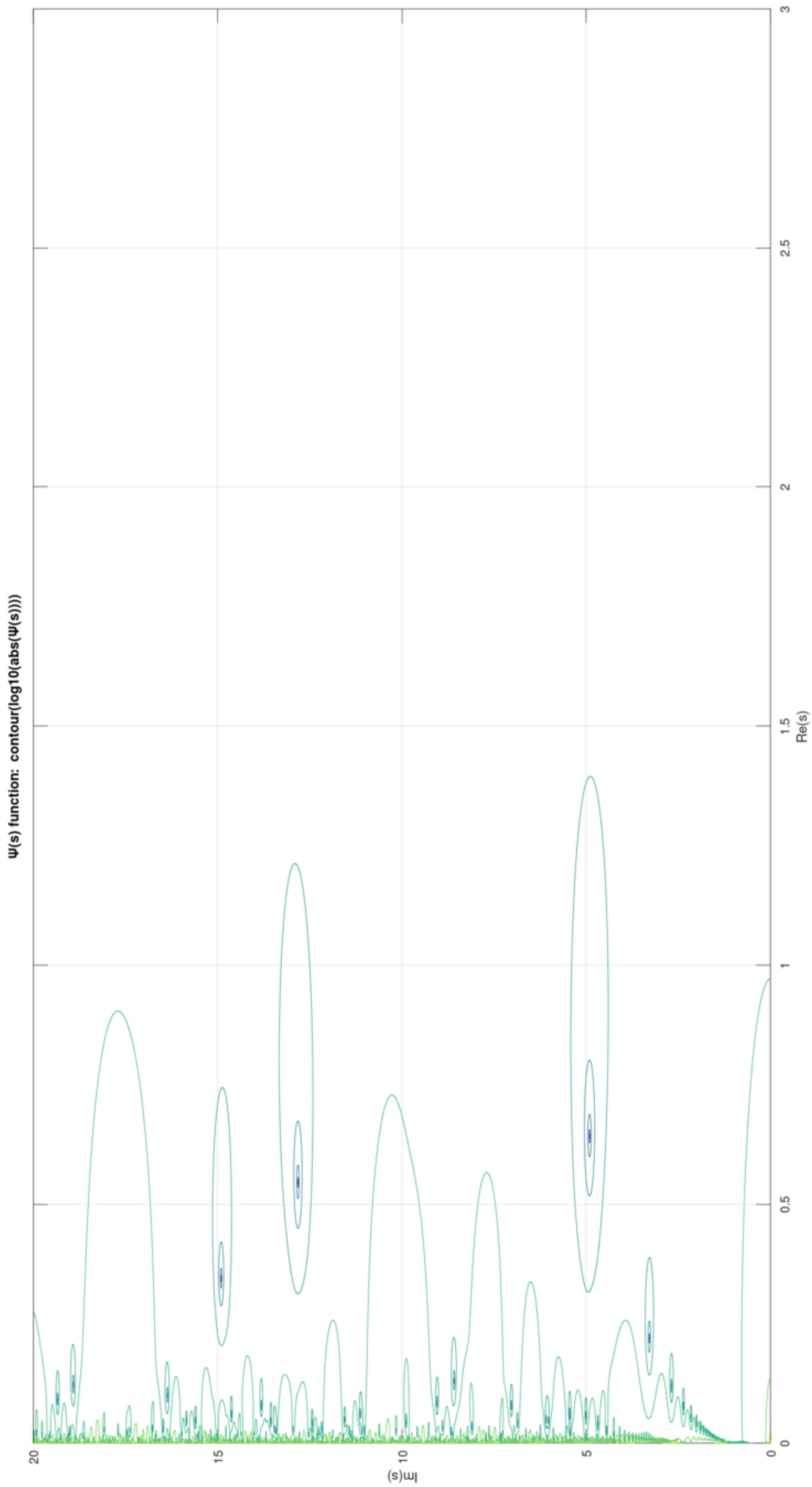
A proposal for a new mathematical constant

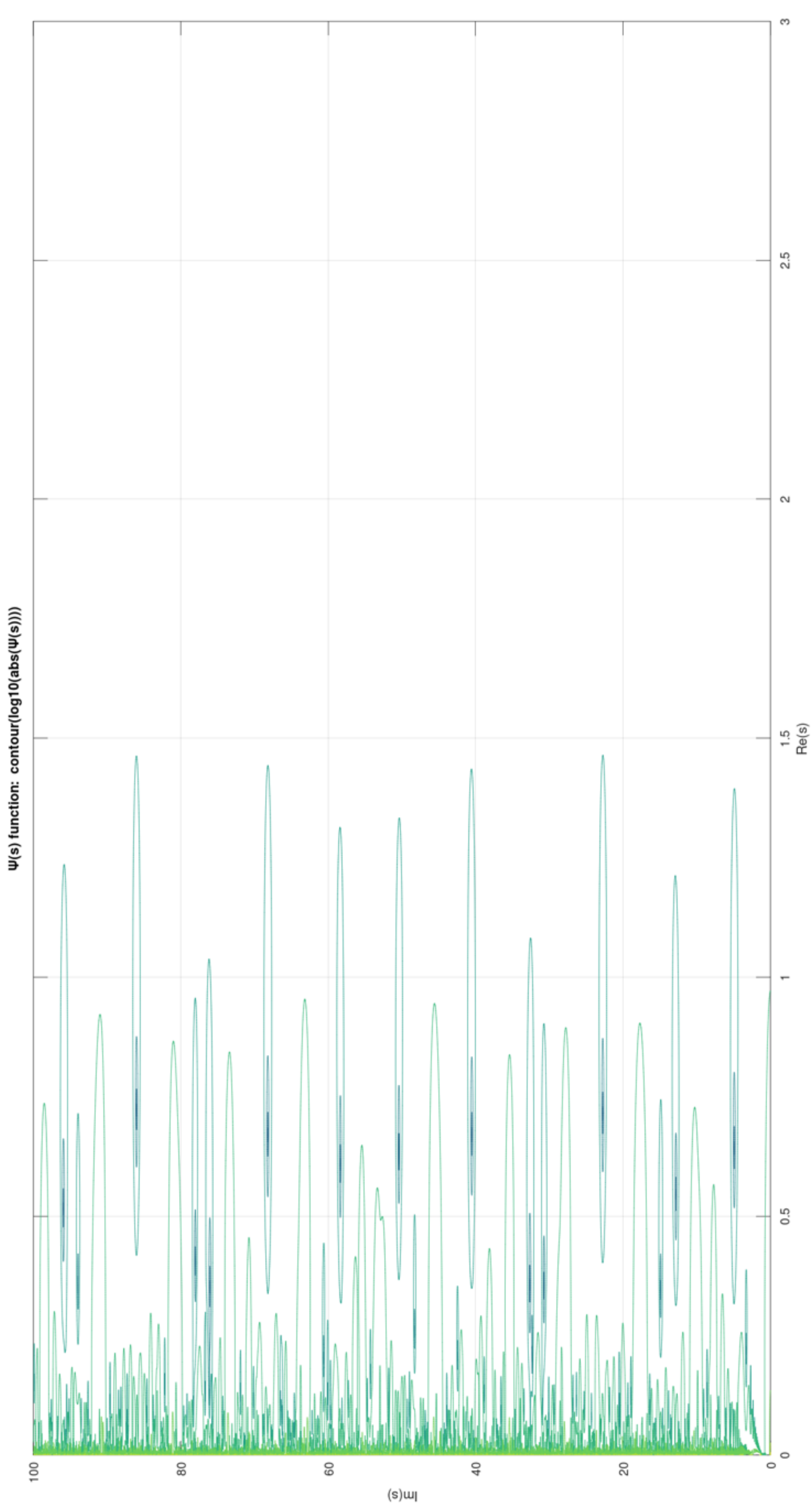
Below is a sequence of graphs of the levels (contour) of the function log10(abs(psi(s))) for s belonging to the complex plane.

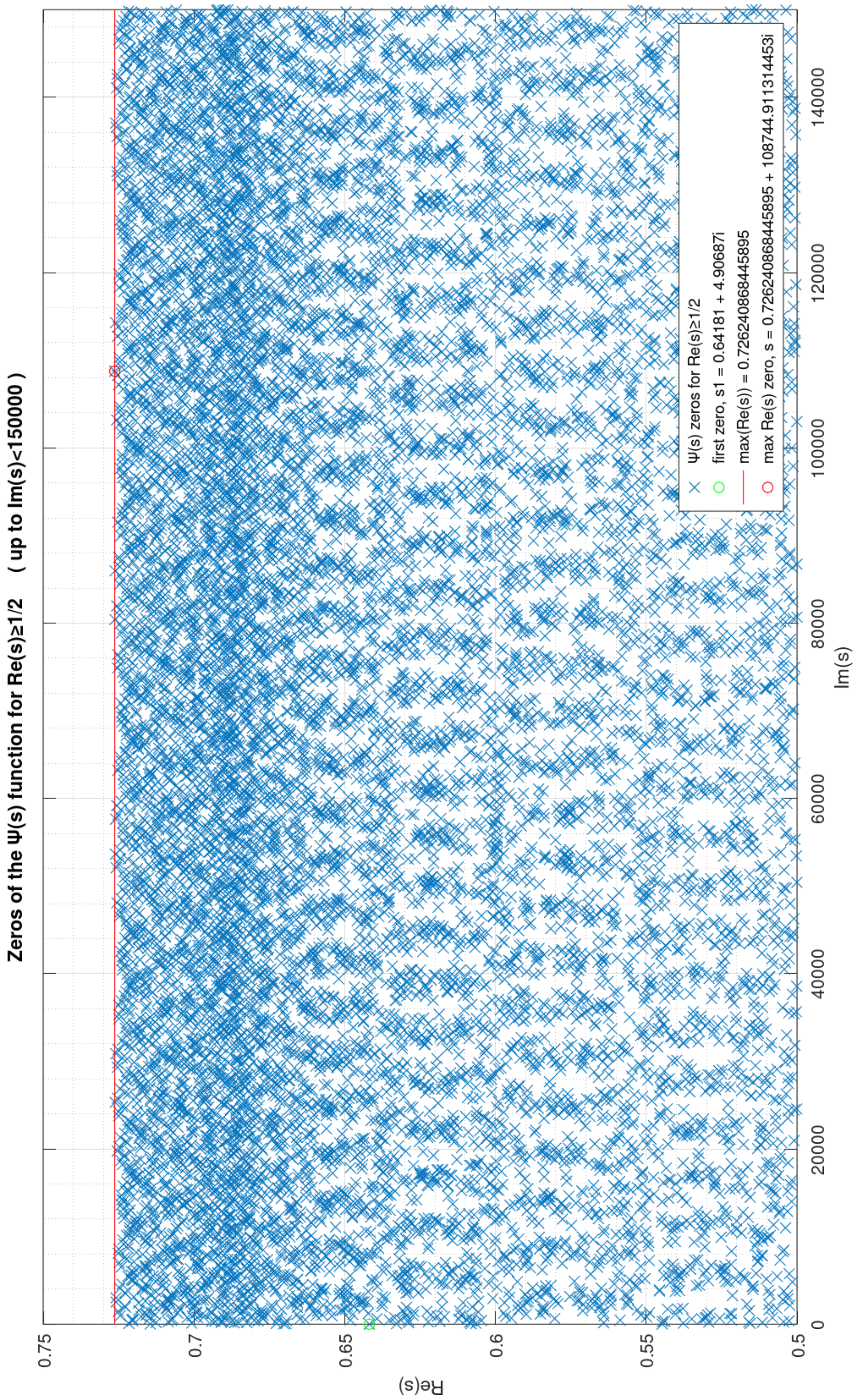
In the first graph, the **first zero in the semi-plane** $\mathbf{Re(s)\geqslant\frac{1}{2}}$ is in evidence.
His position is approximately $s_1 \simeq \mathbf{0.6418158643 + 4.9068764351i}$.

$\mathbf{Im(s_1) \simeq 4.9068764351}$ seems to be **a new mathematical constant**.
A search in the OEIS Foundation database https://oeis.org/ is negative.
Also unsuccessful was a search for a composed constant (see at the end).


Ψ(s) function: contour(log10(abs(Ψ(s))))
Re(s)
Im(s)
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5
6

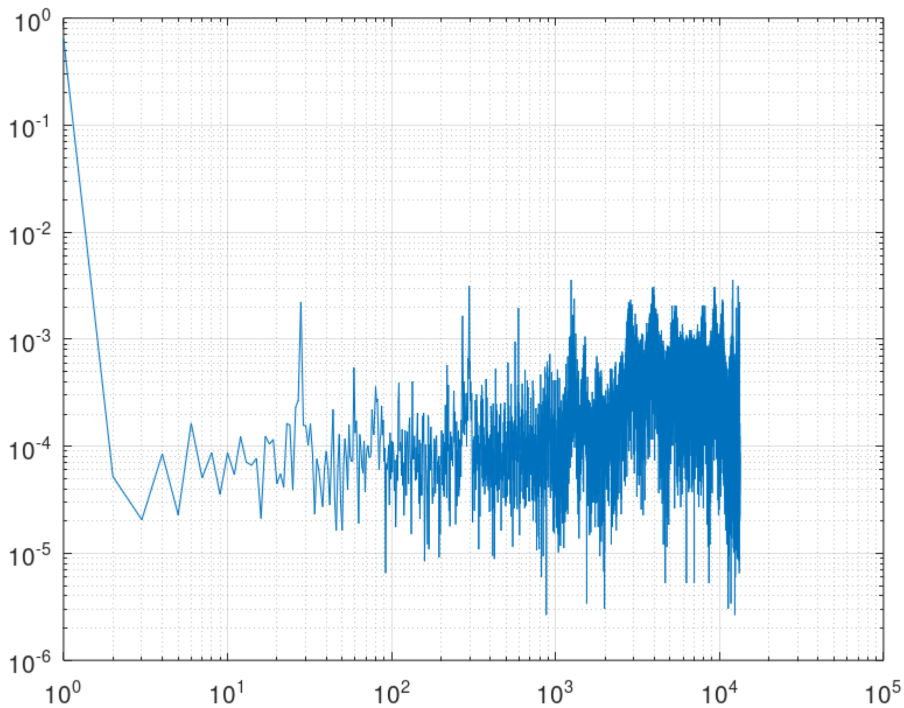

Ψ(s) function: contour(log10(abs(Ψ(s))))
Re(s)
Im(s)
0
0.5
1
1.5
2
2.5
3
5
10
15
20


Ψ(s) function: contour(log10(abs(Ψ(s))))
Re(s)
Im(s)
0
0.5
1
1.5
2
2.5
3
0
20
40
60
80
100

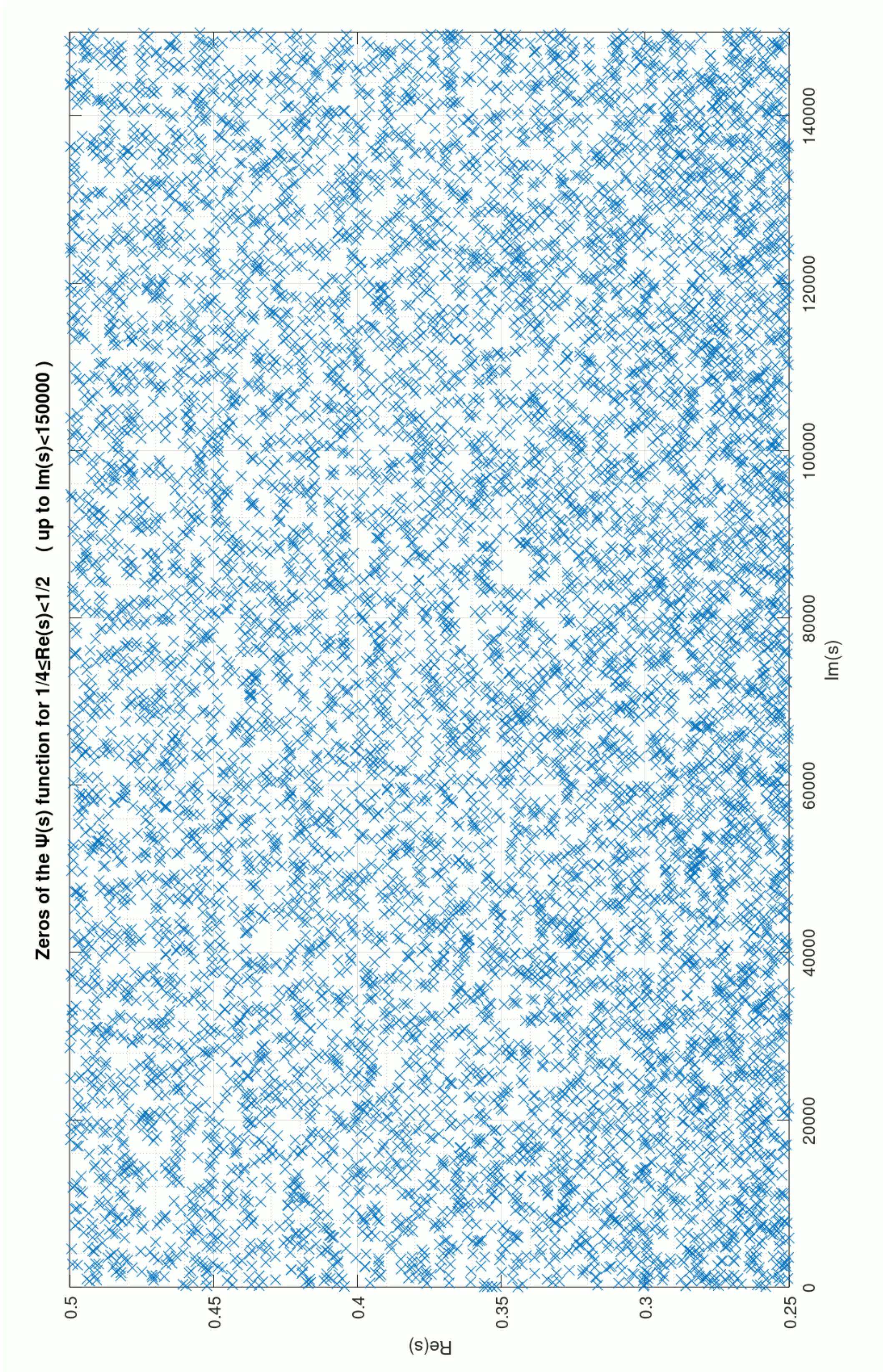

Zeros of the Ψ(s) function for Re(s)≥1/2 (up to Im(s)<150000)
Ψ(s) zeros for Re(s)≥1/2
first zero, s1 = 0.64181 + 4.90687i
max(Re(s)) = 0.726240868445895
max Re(s) zero, s = 0.726240868445895 + 108744.911314453i
Re(s)
Im(s)
0.75
0.7
0.65
0.6
0.55
0.5
0
20000
40000
60000
80000
100000
120000
140000

Seeing the previous graph of the first 13283 zeros (0⩽Im(s)⩽150000) for Re(*s*) ⩾ $\frac{1}{2}$, it seems that some kind of periodic patten is present in the positions of the zeros. An FFT analysis of the coordinates of the zeros reveals something similar.
Furthermore, both conjectures 1 and 2 seem true.

FFT analysis of the zeros real part (for Re(*s*) ⩾ 1/2)



In the successive graph there are the first 9486 zeros (0⩽Im(s)⩽150000) for the strip $\frac{1}{4}$⩽Re(*s*) < $\frac{1}{2}$.

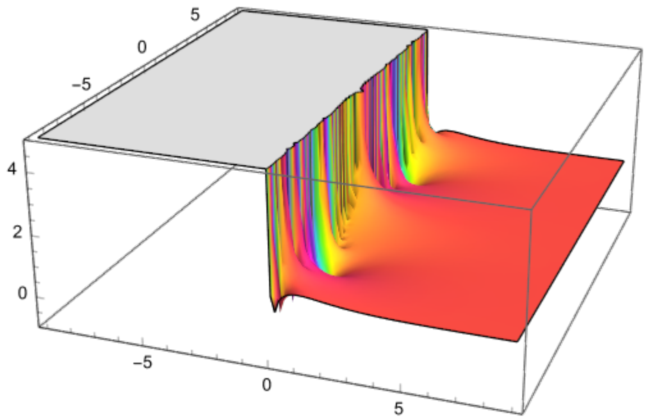

Zeros of the Ψ(s) function for 1/4≤Re(s)<1/2 (up to Im(s)<150000)
Re(s)
Im(s)
0.5
0.45
0.4
0.35
0.3
0.25
0
20000
40000
60000
80000
100000
120000
140000

In[∘]:=
```
(*   Approximate Psi(s) function   *)

Psi[s_, nmax_] := ParallelSum[1 / n ! ^s, {n, 1, nmax}]
```

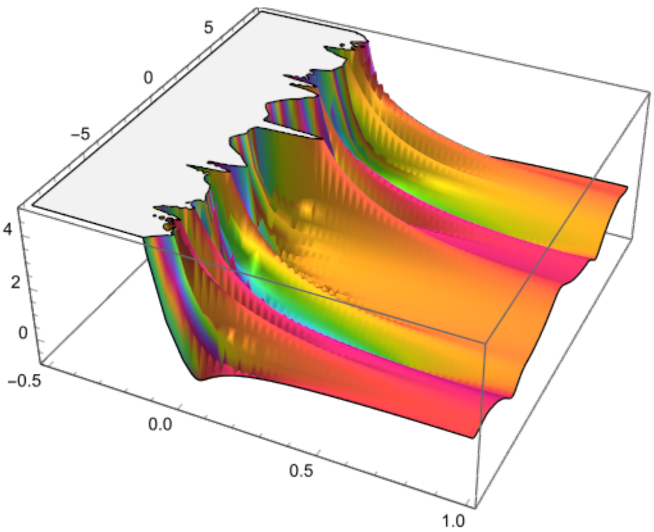
```
ComplexPlot3D[Psi[s, 10], {s, -9 - 9 I, 9 + 9 I}, PlotRange → {-1, 5}]
```

Out[∘]=




```
ComplexPlot3D[Psi[s, 10], {s, -1 / 2 - 9 I, 1 + 9 I}, PlotRange → {-1, 5}]
```

Out[∘]=



```
(*   Examples of real values   *)
```

```
N[Psi[1000, 200], 100]    (*   limit   Psi(s) --Re(s)→ ∞--> 1   *)
```

Out[ ]=

```
1.0000000000000000000000000000000000000000000000000000000000000000000000000000000000
 00000000000000000
```

```
N[Psi[1, 200], 100]     (*    Psi(1) = e - 1   *)
```

Out[ ]=

```
1.7182818284590452353602874713526624977572470936999595749669676277240766303535475945
 71382178525166427
```

In[ ]:=

```
N[Psi[1 / 2, 200], 100]
```

Out[ ]=

```
2.4695063145210475624756367446601502576897561839949691996779257019718354880034088340
 38662508412007135
```

In[ ]:=

```
N[Psi[1 / 4, 300], 100]
```

Out[ ]=

```
3.7063221734793910732444276016472558137744061380342803382068041093450018356719647449
 61441193290389664
```

In[ ]:=

```
N[Psi[5 / 1000, 20 000], 100]
```

Out[ ]=

```
58.356775821716875223061509326910407324145736718139894476755786989329630308133281342
 27727344209568700
```

In[ ]:=

```
N[Psi[1 / 1000, 20 000], 100]
```

Out[ ]=

```
212.89343444045988644216289367877454278023323209715002451741002064447015144892838658
 14643740128295603
```

```
(*   The s1 zero   *)

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
         {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 50]

Out[•]=
        {x → 0.64181586433314113248112139394951847372318338379012,
         y → 4.9068764351428513475351082583558535315328564648993}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
         {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 150]

Out[•]=
        {x →
          0.641815864333141132481121393949518473723183383790117174605186222762580517064969791⋱
           362762127145058179633534712208669467283714546798848631625316442136767, y →
          4.90687643514285134753510825835585353153285646489933763520288952487008096849160406⋱
           011106519694492018106732629379978481024986065008293255799891424877982}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
         {{x, 1 / 2}, {y, 5}}, WorkingPrecision → 200]

Out[•]=
        {x →
          0.64181586433314113248112139394951847372318338379011717460518622276258051706496979⋱
           136276212714505817963353471220866946728371454679884863162531644213676681386342214⋱
           014473677286328546523427080926215482659, y →
          4.90687643514285134753510825835585353153285646489933763520288952487008096849160406⋱
           011106519694492018106732629379978481024986065008293255799891424877981697719647867⋱
           89243286585094776786735969645818985011}
```

```
(*   Other examples of zeros   *)

In[•]:= FindRoot[{Re[Psi[x + y * I, 2000]], Im[Psi[x + y * I, 2000]]},
  {{x, 1336 / 1000000}, {y, 1 + 4705 / 1000000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.0013361234465891396568602622971086091779190211534074,
 y → 1.0047047214630754990772104173473400204889209870526}

In[•]:= FindRoot[{Re[Psi[x + y * I, 20000]], Im[Psi[x + y * I, 20000]]},
  {{x, 1336 / 1000000}, {y, 1 + 4705 / 1000000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.0013361234423282565356094057650499049578042780328027,
 y → 1.0047047214631007857421115748599516575637423334916}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 670 / 1000}, {y, 40 + 516 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.67022547274446564754942561921853627239573382162093,
 y → 40.516540228521088324200375633146033781651332005560}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 721 / 1000}, {y, 86 + 2 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.72159477459801208832092300307574122028151736384671,
 y → 86.002031717653971712641931481424395869298288479074}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 200 / 1000}, {y, 512 + 787 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.19977206033939305741111784335367081276729571786498,
 y → 512.78788254499022664698311303017535684680849482170}

In[•]:= FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 611 / 1000}, {y, 1019 + 180 / 1000}}, WorkingPrecision → 50]
Out[•]=
{x → 0.61144000424789254536789403748651182040962005969469,
 y → 1019.1796100394148909618609953176907836874492265833}
```

0.6919053669 + 149 998.1372808481 i

In[◦]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 692 / 1000}, {y, 149 998 + 137 / 1000}}, WorkingPrecision → 50]
```

Out[◦]=
```
{x → 0.69190536689323821234606604538539980100012880561469,
 y → 149 998.13728084803537276334254075480663772664388012}
```

0.726240868445895 + 108 744.911314453 i

```
(*   The zero with maximum Re(s)    ( Im(s)<150000 )   *)
```

In[◦]:=
```
FindRoot[{Re[Psi[x + y * I, 1000]], Im[Psi[x + y * I, 1000]]},
  {{x, 726 / 1000}, {y, 108 744 + 911 / 1000}}, WorkingPrecision → 50]
```

Out[◦]=
```
{x → 0.72624086844586790634634422045970079610134134053575,
 y → 108 744.91131445345894385570366150252207947684867118}
```

```
(*   Factorial   *)
Table[{n, n!}, {n, 1, 40}] // Column
```

Out[*]=

```
{1, 1}
{2, 2}
{3, 6}
{4, 24}
{5, 120}
{6, 720}
{7, 5040}
{8, 40 320}
{9, 362 880}
{10, 3 628 800}
{11, 39 916 800}
{12, 479 001 600}
{13, 6 227 020 800}
{14, 87 178 291 200}
{15, 1 307 674 368 000}
{16, 20 922 789 888 000}
{17, 355 687 428 096 000}
{18, 6 402 373 705 728 000}
{19, 121 645 100 408 832 000}
{20, 2 432 902 008 176 640 000}
{21, 51 090 942 171 709 440 000}
{22, 1 124 000 727 777 607 680 000}
{23, 25 852 016 738 884 976 640 000}
{24, 620 448 401 733 239 439 360 000}
{25, 15 511 210 043 330 985 984 000 000}
{26, 403 291 461 126 605 635 584 000 000}
{27, 10 888 869 450 418 352 160 768 000 000}
{28, 304 888 344 611 713 860 501 504 000 000}
{29, 8 841 761 993 739 701 954 543 616 000 000}
{30, 265 252 859 812 191 058 636 308 480 000 000}
{31, 8 222 838 654 177 922 817 725 562 880 000 000}
{32, 263 130 836 933 693 530 167 218 012 160 000 000}
{33, 8 683 317 618 811 886 495 518 194 401 280 000 000}
{34, 295 232 799 039 604 140 847 618 609 643 520 000 000}
{35, 10 333 147 966 386 144 929 666 651 337 523 200 000 000}
{36, 371 993 326 789 901 217 467 999 448 150 835 200 000 000}
{37, 13 763 753 091 226 345 046 315 979 581 580 902 400 000 000}
{38, 523 022 617 466 601 111 760 007 224 100 074 291 200 000 000}
{39, 20 397 882 081 197 443 358 640 281 739 902 897 356 800 000 000}
{40, 815 915 283 247 897 734 345 611 269 596 115 894 272 000 000 000}
```

```
(*   Some examples of contributions to the summation   *)

In[•]:= N[1 / (200 !) ^ (1 / 2)]
Out[•]=
```
3.56087×10^{-188}

```
In[•]:= N[1 / (200 !) ^ (1 / 4)]
Out[•]=
```
1.88703×10^{-94}

```
In[•]:= N[1 / (200 !) ^ (1 / 1000)]
Out[•]=
0.421797

In[•]:= N[1 / (2000 !) ^ (1 / 1000)]
Out[•]=
```
1.83857×10^{-6}

```
In[•]:= N[1 / (20 000 !) ^ (1 / 1000)]
Out[•]=
```
4.59981×10^{-78}

```
(*   Im(s1) :  Search for a composed constant   *)

zero = 4.9068764351428513475351082583558535315
lista = {Pi, Pi^3, Sqrt[Pi], E, E^2, E^3, Sqrt[E], Zeta[2], Zeta[3], Zeta[4], Zeta[5],
  1/2, 1/3, 1/4, 1/5, 1/7, 1/9, 1/11, 1/13, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, √2,
  √3, √5, √6, √7, √10, Erf[1], GoldenRatio, GoldenAngle, Sin[1], EulerGamma,
  -StieltjesGamma[1], Catalan, Glaisher, Khinchin, ChampernowneNumber[10]}
lista2 = (DeleteDuplicates[Subsets[#1, 6]] &)[lista]
lista3 = Exp[Total[Log[lista2], {2}]]
N[lista3, 10]
(*  N[Abs[lista3-zero],6]  *)
delta = lista3 - zero;
If[Min[Abs[delta]] < 0.0000001, 1000, 0]
If[Min[Abs[delta + 1]] < 0.0000001, 1000, 0]
If[Min[Abs[delta - 1]] < 0.0000001, 1000, 0]
If[Min[Abs[delta + 1/2]] < 0.0000001, 1000, 0]
If[Min[Abs[delta - 1/2]] < 0.0000001, 1000, 0]
```

Out[∘]=

```
4.9068764351428513475351082583558535315
```

Out[∘]=

```
{π, π^3, √π, e, e^2, e^3, √e, π^2/6, Zeta[3], π^4/90, Zeta[5], 1/2, 1/3,
 1/4, 1/5, 1/7, 1/9, 1/11, 1/13, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, √2, √3, √5,
 √6, √7, √10, Erf[1], GoldenRatio, GoldenAngle, Sin[1], EulerGamma,
 -StieltjesGamma[1], Catalan, Glaisher, Khinchin, ChampernowneNumber[10]}
```

Out[∘]=

```
{{}, {π}, {π^3}, {√π}, {e}, ··· 9531031 ···,
 {Sin[1], EulerGamma, -StieltjesGamma[1], Glaisher, Khinchin, ChampernowneNumber[10]},
 {Sin[1], EulerGamma, Catalan, Glaisher, Khinchin, ChampernowneNumber[10]},
 {Sin[1], -StieltjesGamma[1], Catalan, Glaisher, Khinchin, ChampernowneNumber[10]},
 {EulerGamma, -StieltjesGamma[1], Catalan, Glaisher, Khinchin, ChampernowneNumber[10]}}
Full expression not available (original memory size: 4.2 GB)
```

Out[∘]=

```
{1, π, π^3, √π, e, ··· 9531031 ···,
 -EulerGamma Glaisher Khinchin ChampernowneNumber[10] Sin[1] StieltjesGamma[1],
 Catalan EulerGamma Glaisher Khinchin ChampernowneNumber[10] Sin[1],
 -Catalan Glaisher Khinchin ChampernowneNumber[10] Sin[1] StieltjesGamma[1],
 -Catalan EulerGamma Glaisher Khinchin ChampernowneNumber[10] StieltjesGamma[1]}
Full expression not available (original memory size: 3 GB)
```

Out[∘]=

```
{1.000000000, 3.141592654, 31.00627668, 1.772453851, 2.718281828,
 ··· 9531031 ···, 0.01503724305, 0.1891565932, 0.02386213352, 0.01636847558}
Full expression not available (original memory size: 0.9 GB)
```

Out[ ]=
0

Out[ ]=
0

Out[ ]=
0

Out[ ]=
0

Out[ ]=
0

```
(*   Im(s1) :  Continued fraction analysis   *)

FindRoot[{Re[Psi[x + y * I, 2000]], Im[Psi[x + y * I, 2000]]},
 {{x, 642 / 1000}, {y, 4 + 907 / 1000}}, WorkingPrecision → 1300]
```

Out[∘]=

```
{x →
  0.641815864333141132481121393949518473723183383790117174605186222762580517064969798
   136276212714505817963353471220866946728371454679884863162531644213676681386342214
   014473677286328546523427080926215482659479846249916772492289981881602237283118520
   224441733305593875858538500963161583572755706478906179621021863582393239954271300
   503502042856687435114521357650184029138357836279597641018323977079283298040458696
   731233952852112935835391970426094563136419697930464740539224604774622139270410310
   883300773670887868723255714605758361472626859853516634028910492511180426434028687
   469000619271547376321921267183053622038356028968164050943866084904974890274292374
   447804960526525767810002287049574748726328086291789851695525285462162494026507574
   409428793197081474784375042756618935839178364120591994589823921547751413933949664
   928169905469287001947076676493780660278262779008511797583880293665325425591240943
   440197237970925898648481904212149184271654322901455568241021109578630241268664621
   478071706202434014264749702504989616415505963629366941652805857284605274284474455
   707409634355217714580370355168509142392733687679688854005664553058471638547030321
   463797883424477441208860865195679249633625851126577967203915388651776118255555097
   522165229613271427591423164133783708352684905277676601248274086947278355049865055
   38354, y →
  4.906876435142851347535108258355853531532856464899337635202889524870080968491604062
   011106519694492018106732629379978481024986065008293255799891424877981697719647867
   892432865850947767867359696458189850113066290730661163382959563711283636246554538
   625931100616959500717821440100831297284353752563899725222486732424080246881024436
   476822698977545287852117588467627983378497928237204133439200760220176455332413095
   927618670017106975125537969090901282059969946969312936208257707365556006004179229
   465530153123264287211246495825043838207728909226565103494263397562423968007420895
   872609475341080560619069066527838141275945106530265008503736348761000663072981081
   316512212681275478633034411846669135671790444853471372638060302960421000405484683
   877142583697078634966482414766363394485439385050048739051424110820935291533803587
   639688330694992681321634092378634336942330072999423307993381790609195131059278099
   426223854295990204032347432446554574335708979998882199510715783327002681548879243
   605424013369974190584732910136843718270760145419240880993313563108919689318688074
   699814138410071194343613796537519549618072151976698738858435327732348177365633853
   549679221828253366020757153655341617928906793558384781839689666936592212099246055
   045083010654624082674625659576589323831859522696385479439657550270486404280608434
   4431}
```

In[ ]:=
```
zero = %[[2]][[2]]
```

Out[ ]=
```
4.906876435142851347535108258355853531532856464899337635202889524870080968491604060 1⋱
 11065196944920181067326293799784810249860650082932557998914248779816977196478678924⋱
 32865850947767867359696458189850113066290730661163382959563711283636246554538625931⋱
 10061695950071782144010083129728435375256389972522248673242408024688102443647682269⋱
 89775452878521175884676279833784979282372041334392007602201764553324130959276186700⋱
 17106975125537969090901282059969946969312936208257707365556006004179229465530153123⋱
 26428721124649582504383820772890922656510349426339756242396800742089587260947534108⋱
 05606190690665278381412759451065302650085037363487610006630729810813165122126812754⋱
 78633034411846669135671790444853471372638060302960421000405484683877142583697078634⋱
 96648241476636339448543938505004873905142411082093529153380358763968833069499268132⋱
 16340923786343369423300729994233079933817906091951310592780994262238542959902040323⋱
 47432446554574335708979998882199510715783327002681548879243605424013369974190584732⋱
 91013684371827076014541924088099331356310891968931868807469981413841007119434361379⋱
 65375195496180721519766987388584353277323481773656338535496792218282533660207571536⋱
 55341617928906793558384781839689666936592212099246055045083010654624082674625659576⋱
 589323831859522696385479439657550270486404280608434443 1
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In[∘]:= **ContinuedFraction[zero, 1000]**

Out[∘]=

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(*   Conjecture 3: irrational and transcendental number ???   *)

Conclusions

By definition, in some mode, the **Psi function** is a “part” of the **Riemann’s Zeta function**.
The zeros of the Psi function seems to be in the **critical strip** $0 < \mathrm{Re}(s) \leqslant 1$.
I proposed as **new mathematical constant** the imaginary part of the first zero of the Psi function in the semi-strip $\mathrm{Re}(s) \geqslant \frac{1}{2}$: **$\mathrm{Im}(s_1) \simeq 4.9068764351$** .
I’m sorry, but I have not found reference materials really regarding a function like the Psi function and the study of its zeros.

References

1) https://oeis.org/A091131
2) https://oeis.org/A058303
3) https://mathworld.wolfram.com/FactorialSums.html
4) https://mathworld.wolfram.com/RiemannZetaFunction.html

Rev. B