On the total energy of open Friedmann-Robertson-Walker universes

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Abstract

The idea that the universe has zero total energy when one includes the contribution from the gravitational field is reconsidered. A Hamiltonian is proposed as an energy for the exact equations of FRW cosmology: it is then shown that this energy is constant. Thus open and critically open FRW universes have the energy of their asymptotic state of infinite dilution, which is Minkowski space with zero energy. It is then shown that de Sitter space, the inflationary attractor, also has zero energy, and the argument is generalized to Bianchi models converging to this attractor.

1 Introduction

Over the years, there have been many suggestions and assertions in the literature that the total energy of the universe, that which includes not only the material energy but also the contribution from the gravitational field, should be zero. The subject is non-trivial partially due to the fact that in general relativity there is no universally accepted prescription for the localization of energy including the contribution from gravity. In the case of a cosmological metric, a rationalized energy localization would provide an expression to integrate over three-dimensional spatial sections, yielding the total energy of the universe. In principle, this could be infinite for an open universe and this would likely be the instinctive response of most physicists.

In this paper we do not address the energy localization problem but rather we focus upon the *total* energy of the universe. The idea that the universe should have zero total energy dates back many years (a topological argument due to P. Bergmann is quoted in Tryon 1973) and returned to the fore with the works of Albrow (1973) and Tryon (1973) suggesting that the universe originated as a quantum fluctuation of the vacuum. The idea was developed in Guth's theory of inflation (Guth 1981) that has had a major impact on modern cosmology. Further, an approach to quantum cosmology proposed that the universe was born as the result of quantum tunneling from nothing (Vilenkin 1983).

The question as to whether or not the total energy of the universe is zero was re-opened by Rosen (1994), Cooperstock (1994), Cooperstock & Israelit (1995), Johri *et al.* (1995), Banerjee & Sen (1997), Radinschi (1999), and Xulu (2000), who studied both Friedmann-Robertson-Walker (hereafter referred to as FRW) universes and anisotropic Bianchi models. The approaches followed by these authors are varied. Apart from Cooperstock (1995) and Cooperstock & Israelit (1995), a common feature is the use of pseudotensors which brings these approaches into question. Pseudotensors have been shown to be useful tools in the study of bounded systems which are asymptotically Minkowskian (in Cartesian coordinates). However, for cosmological metrics, we are concerned with systems that are infinite or do not possess an asymptotically flat exterior. In spite of this, it is note-worthy that these authors invariably conclude that the total energy of open and closed FRW universes, as well as of Bianchi models, vanishes.

In this paper, open FRW cosmologies are discussed and it is shown that for many equations of state (both time-independent and time-dependent), the total energy is constant and, one could argue, has the value zero. The essential elements that are employed to make this deduction are the global conservation of energy and the vanishing of the energy of flat spacetime. Energy conservation, including the contribution from gravity, is built into the Einstein field equations by virtue of the vanishing of the covariant divergence of the energy-momentum tensor. That flat spacetime represents the absence of energy can be seen as follows. The mass parameter m of the Schwarzschild metric [1]

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \tag{1.1}$$

yields the total energy including the contribution from gravity. When m is set to zero, the metric becomes flat and hence flat spacetime has zero energy.

We assume that the energy density of the material content of the universe ρ is non-negative and we consider an open or critically open (the case favoured by inflationary theories and by current cosmological data (Liddle & Lyth 2000)) FRW universe described by the line element

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \right] \tag{1.2}$$

in comoving coordinates (t, r, θ, φ), where the curvature index K can assume the values -1 or 0.

2 An explicit energy definition

If the total energy of the universe is constant for a certain form of its material content, the same should be true when the same cosmological metric is generated by a different form of matter.

Following the dynamical systems approach to cosmology (Stabell & Refsdal 1966, Madsen & Ellis 1988, Madsen *et al.* 1992, Wainwright & Ellis 1997; Amendola *et al.* 1990, Foster 1988, Gunzig *et al.* 2000, Rocha-Filho *et al.* 2000, Gunzig *et al.* 2001*a*, 2001*b*), we consider a $K = 0$ FRW universe with a scalar field $\phi(t)$ as the only source of gravity; ϕ is allowed to couple nonminimally to the Ricci curvature as described by the action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \left(\frac{1}{\kappa} - \xi \, \phi^2 \right) R - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right] , \tag{2.1}$$

[1] We use $\kappa \equiv 8\pi G$, where G is Newton's constant; the metric signature is $- + + +$, the speed of light and Planck's constant assume the value unity. The components of the Ricci tensor are given in terms of the Christoffel symbols $\Gamma^\delta_{\alpha\beta}$ by $R_{\mu\rho} = \Gamma^\nu_{\mu\rho,\nu} - \Gamma^\nu_{\nu\rho,\mu} + \Gamma^\alpha_{\mu\rho}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\nu_{\alpha\mu}$, and an overdot denotes differentiation with respect to the comoving cosmic time t.

where ξ is a dimensionless coupling constant and $V(\phi)$ is the scalar field potential. $H \equiv \dot{a}/a$ and ϕ are chosen as dynamical variables and the relevant dynamical system is

$$6\left[1-\xi\left(1-6\xi\right)\kappa\phi^2\right]\left(\dot{H}+2H^2\right)-\kappa\left(6\xi-1\right)\dot{\phi}^2-4\kappa V+6\kappa\xi\phi\,\frac{dV}{d\phi}=0\;, \tag{2.2}$$

$$\frac{\kappa}{2}\,\dot{\phi}^2+6\xi\kappa H\phi\dot{\phi}-3H^2\left(1-\kappa\xi\phi^2\right)+\kappa V=0\,, \tag{2.3}$$

$$\ddot{\phi}+3H\dot{\phi}+\xi R\phi+\frac{dV}{d\phi}=0\;. \tag{2.4}$$

Let us restrict now to the case of a conformally coupled scalar, obtained by setting $\xi = 1/6$ in the action (2.1). This value of the coupling constant is dictated by physical requirements such as the Einstein equivalence principle (Sonego and Faraoni 1993, Grib and Poberii 1995, Grib and Rodrigues 1996), renormalizability (Callan, Coleman and Jackiw 1970), or the requirement that ξ, which is a running coupling at high energies, sit in a stable infrared fixed point of the renormalization group (Parker and Toms 1985, Buchbinder, Odintsov and Shapiro 1992). We allow the scalar field to acquire a mass and include in the picture a quartic self-interaction plus (motivated by the role of de Sitter attractors in phase space) the possibility of a cosmological constant, as described by the potential

$$V(\phi)=\frac{m^2\,\phi^2}{2}+\lambda\phi^4+V_0\;. \tag{2.5}$$

By introducing the new variables (Rocha Filho *et al.* 2000)

$$\psi\equiv\sqrt{\frac{\kappa}{6}}\;a\;, \tag{2.6}$$

$$\varphi\equiv a\,\phi\;, \tag{2.7}$$

and the conformal time η defined by $dt \equiv a\,d\eta$, the field equations (2.2) and (2.4) become

$$\psi''-\frac{\kappa m^2}{6}\,\varphi^2\psi-4V_0\psi^3=0\;, \tag{2.8}$$

$$\varphi''+\frac{6m^2}{\kappa}\,\varphi\,\psi^2+4\lambda\varphi^3=0\;, \tag{2.9}$$

where a prime denotes differentiation with respect to the conformal time η. Eqs. (2.8) and (2.9) can be derived from the Lagrangian

$$Ł=\frac{1}{2}\left(\varphi'\right)^2-\frac{18}{\kappa^2}\,\left(\psi'\right)^2-\frac{3m^2}{\kappa}\,\varphi^2\psi^2-\lambda\varphi^4-\frac{36}{\kappa^2}\,V_0\psi^4\;, \tag{2.10}$$

while the equation

$$\frac{1}{2}\left(\varphi'\right)^2-\frac{18}{\kappa^2}\left(\psi'\right)^2+\frac{3m^2}{\kappa}\varphi^2\psi^2+\lambda\varphi^4+\frac{36}{\kappa^2}V_0\psi^4=\text{const.} \tag{2.11}$$

is obtained by manipulating eqs. (2.8) and (2.9) and integrating once.

The momenta canonically conjugated to the variables φ and ψ are $p_\varphi\equiv\partial L/\partial(\varphi')=\varphi'$ and $p_\psi\equiv\partial L/\partial(\psi')=-36\psi'/\kappa^2$. It is straightforward to check that, by using the associated Hamiltonian

$$E=\frac{1}{2}\left(\varphi'\right)^2-\frac{18}{\kappa^2}\left(\psi'\right)^2+\frac{3m^2}{\kappa^2}\varphi^2\psi^2+\lambda\varphi^4+\frac{36}{\kappa^2}V_0\psi^4\;, \tag{2.12}$$

the Hamilton equations

$$p'_\varphi=-\frac{\partial E}{\partial\varphi}\;,\qquad p'_\psi=-\frac{\partial E}{\partial\psi}\;, \tag{2.13}$$

reproduce the field equations (2.8) and (2.9). Furthermore, eq. (2.11) states that

$$E=\text{constant}\;. \tag{2.14}$$

As customary in classical mechanics and in quantum physics, one identifies the Hamiltonian with the energy of the system, thus obtaining a total energy of the universe (2.12) that is conserved. Note that de Sitter space is among the possible solutions of the field equations (2.8), (2.9), and (2.4), and therefore it has constant energy as well. This concludes the first step of the deduction of zero energy for FRW universes, while the next section presents the second step.

3 The limit to Minkowski spacetime

Open or critically open FRW cosmologies have Minkowski space as their asymptotic state. In fact, the dynamical system (2.2)-(2.4) has the Minkowski space $(H,\phi)=(0,0)$ as a fixed point, with attractive behaviour in the half-plane $H>0$ and repulsive behaviour for $H<0$ (Gunzig *et al.* 2001*a*). It was speculated that, when $V_0=0$, the universe could have emerged from points arbitrarily close to the Minkowski fixed point, a classical analog of the proposals of Albrow (1973), Tryon (1973) and of Prigogine *et al.* (1988, 1989). The same Minkowski fixed point is an attractor at large times, in a certain basin, for solutions expanding into infinite dilution and asymptotically approaching $(H,\phi)=(0,0)$.

Consider first an open ($K = -1$) or critically open ($K = 0$) universe; its asymptotic state at large times is one of infinite dilution. In fact, the conservation equation (Liddle & Lyth 2000, Kolb & Turner 1994, Weinberg 1972, Landau & Lifshitz 1989)

$$\dot{\rho} + 3H\left(P + \rho\right) = 0 \tag{3.1}$$

yields $\dot{\rho} < 0$ when $P \geq -\rho/3$ and the universe expands. Since ρ is bounded from below by zero and is monotonically decreasing, one has $\rho(t) \rightarrow 0$ at large times: the asymptotic state is one of infinite dilution corresponding to Minkowski space. The Hubble radius is given by the Friedmann equation

$$H^2 = \frac{\kappa\,\rho}{3} - \frac{K}{a^2} \tag{3.2}$$

and is

$$H^{-1} = \left(\frac{\kappa\rho}{3} - \frac{K}{a^2}\right)^{-1/2} \;; \tag{3.3}$$

it is the only characteristic geometrical scale and diverges as $t \rightarrow +\infty$. This property, familiar in the particular case of a matter- or radiation-dominated era in a $K = 0$ FRW universe, is quite general.

In principle the situation could be different when $P < -\rho/3$ with the universe undergoing accelerated expansion, $\ddot{a} > 0$. Power-law solutions with scale factor $a(t) = a_0\, t^p$ and $p = 2/\left(3\gamma\right)$ are obtained for $K = 0$ by imposing the equation of state $P = \left(\gamma - 1\right)\rho$. These solutions exhibit divergent Hubble radius $H^{-1} = t/p$, have Minkowski spacetime as their limit at large times if $\gamma > 0$, and are attractors in phase space. The property of future convergence to Minkowski spacetime is shared by all solutions satisfying $P > -\rho$. In fact, expanding universes satisfying this assumption have $\dot{\rho} = -3H\left(P + \rho\right) < 0$.

Problems may in principle arise with the better-known inflationary attractor (for both $K = -1$ and $K = 0$ universes), de Sitter space

$$a(t) = a_0\, \mathrm{e}^{H\,t} \;, \qquad H = \text{const.} \;, \tag{3.4}$$

that corresponds to constant Hubble radius H^{-1} and density and pressure $\rho = \Lambda/\kappa = -P$, where Λ is the cosmological constant.

Being a fixed point in phase space, de Sitter space is forever removed from Minkowski space, but there are good reasons to believe that its total energy is the same as for Minkowski space. By considering a scalar field $\phi(t)$ nonminimally coupled to gravity, one can obtain a *superinflationary* regime with $\dot{H} > 0$; this is impossible when the scalar $\phi(t)$ couples minimally to gravity (Faraoni 2002). The existence of superinflationary solutions with nonminimally coupled scalars was demonstrated both exactly and with numerical methods in (Gunzig *et al.* 2001*a*, Rocha-Filho *et al.* 2000, Gunzig *et al.* 2001*b*). They find solutions emerging from points arbitrarily close to Minkowski space, and therefore with the same total energy, that expand and are attracted by de Sitter space. These solutions have constant energy, and hence *the energy E_{dS} of the asymptotic de Sitter space must coincide with that of Minkowski space* [2].

Since the only quantity characteristic of the de Sitter metric is the Hubble constant H, one might expect that the energy of de Sitter space E_{dS} could depend on the value of H, and therefore, in principle, $E_{dS} = E_{Minkowski}$ could be true only for special values of H. However this is not the case, as it was shown (Gunzig *et al.* 2001*a*, 2001*b*) that for *any* value H_0 of the de Sitter fixed point (H_0, ϕ_0), one can find solutions emerging from points arbitrarily close to Minkowski space and converging to de Sitter space. Since the total energy of these solutions is constant and initially $E_{Minkowski}$, it must be $E_{dS} = E_{Minkowski}$ for any value of H.

If Minkowski space has zero energy, the result holds true for all de Sitter spaces, whatever the source of gravity may be (a scalar field coupled minimally or nonminimally, a cosmological constant, exotic dark matter, supergravity fields, etc.), and also for all the solutions in the attraction basin of the de Sitter fixed point. This result is compatible with a recent paper (Kastor and Traschen 2002) introducing a non-negative energy for universes that are asymptotically de Sitter[3]. This is best seen by considering the Schwarzschild-de Sitter metric

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \ , \quad (3.5)$$

which is asymptotically de Sitter. The Kastor-Traschen conserved charge, constructed from conformal asymptotic Killing vectors, is $E_{KT} = M\, a(t)$. In the limit $M \rightarrow 0$, in which de Sitter space is recovered, the conserved charge vanishes.

The previous considerations are particularly relevant because de Sitter spaces are also attractors for anisotropic universes. It follows from the cosmic no-hair theorems that inflation is a generic phenomenon (see Goldswirth & Piran (1992) for a review). A wide variety of initial conditions including initial anisotropy leads to the same de Sitter-like expansion. Then, the energy of a Bianchi model converging to a de Sitter solution

[2] The homoclinic solutions mentioned above that emerge from initial points arbitrarily close to the Minkowski fixed point in the infinite past, and return to it in the far future, are those that escape the attraction basin of the de Sitter attractor considered here. In this subsection only heteroclinic solutions connecting different fixed points are considered.

[3] We acknowledge a referee for pointing out this reference.

with zero energy must also be constant, and equal to zero. This result is compatible with pseudotensor-based claims of zero total energy for Bianchi models (Radinschi 1999, Xulu 2000).

4 Discussion and conclusions

At this point, one could argue that, since the energy of the universe is constant and the zero level is arbitrary, one can *choose* as zero the energy of Minkowski space. This is the only possible choice compatible with the argument presented in the Introduction. Since the FRW metric $g_{ab}(t)$ approaches the Minkowski metric η_{ab} (which has zero energy) as $t \to +\infty$, *the energy E associated with g_{ab} must also be identically zero at all times.* This argument applies to any cosmological metric, be it FRW or not, that has Minkowski space as its asymptotic limit at large times or in the infinite past and has constant energy (provided that the time parameter can be given an unambiguous geometrical meaning, as in the case of an FRW metric).

There are now different arguments pointing to the result of constant and vanishing total energy for open and critically open FRW spacetimes. The arguments provided in this paper support previous pseudotensor-based claims. The pseudotensorial methods as well as those in Cooperstock & Israelit (1995) indicate a zero value for closed FRW universes. The idea that the universe has zero energy is embodied in the Wheeler-de Witt equation of quantum cosmology, which corresponds to the eigenvalue problem for the Hamiltonian of a quantum universe with zero energy eigenvalue (e.g. Kolb & Turner 1994).

One might question the notion that an infinite spacetime filled with matter should have zero energy. The instinctive answer might be that the energy is infinite. However, the open FRW universe dilutes its matter density to an infinite extent and is hence asymptotic to Minkowski spacetime. We know that the latter has zero energy because the Schwarzschild spacetime with energy m becomes the Minkowski spacetime when $m = 0$.

Even in an infinite spacetime such as Schwarzschild spacetime, there is always an *effective* boundary in the sense that, even for a Schwarzschild mass that expands forever to infinite dilution, there exists a sphere that encloses all the matter at any time t. This is what allows us to determine the non-vanishing global energy m of the spacetime. By contrast, in FRW there is no such boundary available at any t. Interestingly, this is the case for a finite closed ($K = +1$) model as well. The vanishing of total energy in the case of a closed universe might be expected by analogy with the necessarily vanishing of total charge for a closed universe, i.e. every closed surface in a finite space encloses a finite volume of space on either side. Thus the electric flux through the surface equals the total charge in the interior as well as to the same amount of total charge in the exterior with the necessarily opposite sign (Landau & Lifshitz 1989). There is the suggestion that it is the property of unboundedness that is the crucial element in rendering these universes, both open and closed, with zero energy.

The results developed in this paper that the energy of the universe is constant and zero for open or critically open FRW universes, and for Bianchi models evolving into de Sitter spacetimes, should not be regarded as merely technical. Indeed it is well to question why universes that are so different all have zero total energy. One could speculate that this fact might be related to the problem of the origin of the universe. Indeed, since the universe is by definition an isolated system, the zero energy result is compatible with the universe emerging from a "system" with zero energy, be it quantum vacuum (Albrow 1973, Tryon 1973, Guth 1981), "nothing" (Vilenkin 1983), flat empty space (Prigogine *et al.* 1988, 1989, Gunzig *et al.* 2001*a*, 2001*b*), or something else. In such a picture, matter particles would have to be created at the expense of the gravitational field energy (e.g. Prigogine *et al.* 1988, 1989). It seems inconceivable that the cosmos could emerge from any physical system that has nonvanishing total energy. This would require an exchange of energy between the universe and a third system, making a cosmological spacetime an open system from the thermodynamical point of view.

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REFERENCES

Albrow, M.G. 1973, *Nature* 241, 56

Amendola, L., Litterio, M. & Occhionero, F. 1990, *Int. J. Mod. Phys. A* 5, 3861

Banerjee, N. & Sen, S. 1997, *Pramana J. Phys.* 49, 609

Buchbinder, I., Odintsov, S.D. & Shapiro, I. 1992, *Effective Action in Quantum Gravity* (Bristol: IOP Publishing)

Callan, C.G. Jr., Coleman & Jackiw, R. 1970, *Ann. Phys. (NY)* **59**, 42

Cooperstock, F.I. 1994, *Gen. Rel. Grav.* 26, 323

Cooperstock, F.I. & Israelit, M.I. 1995, *Found. Phys.* 25, 631

Faraoni, V. 2002, *Int. J. Mod. Phys. D* 11, 471

Foster, S. 1988, *Class. Quant. Grav.* 15, 3485

Goldswirth, D. & Piran, T. 1992, *Phys. Rep.* 214, 223

Grib, A.A. & Poberii, E.A. 1995, *Helv. Phys. Acta* **68**, 380

Grib, A.A. & Rodrigues, W.A. 1995, *Gravit. Cosmol.* **1**, 273

Gunzig, E., Faraoni, V., Rocha Filho, T.M., Figueiredo, A. & Brenig L. 2000, *Class. Quant. Grav.* 17, 1783

Gunzig, E., Saa, A., Brenig, L., Faraoni, V., Rocha Filho, T.M. & Figueiredo, A. 2001*a*, *Phys. Rev. D* 63, 067301

Guth, A.H. 1981, *Phys. Rev. D* 23, 347

Johri, V.B., Kalligas, D, Singh, G.P. & Everitt, C.W.F. 1995, *Gen. Rel. Grav.* 27, 313

Kastor, D. & Traschen, J. 2002, preprint hep-th/0206105

Kolb, E.W. & Turner, M.S. 1994, *The Early Universe* (Reading, MA: Addison-Wesley)

Landau, L.D. & Lifshitz, E.M. 1989, *The Classical Theory of Fields* (4th revised edition, Oxford: Pergamon Press)

Liddle, A.R. & Lyth, D. 2000, *Cosmological Inflation and Large Scale Structure* (Cambridge: Cambridge University Press)

Madsen, M.S. & Ellis, G.F.R. 1988, *MNRAS* 234, 67

Madsen, M.S., Mimoso, J.P, Butcher, J.A. & Ellis, G.F.R. 1992, *Phys. Rev. D* 46, 1399

Parker, L. & Toms, D.J. 1985, *Phys. Rev. D* **32**, 1409

Prigogine, I., Geheniau, J., Gunzig, E. & Nardone, P. 1988, *Proc. Natl. Acad. Sci. USA* 85, 7428

Prigogine, I., Geheniau, J., Gunzig, E. & Nardone, P. 1989, *Gen. Rel. Grav.* 21, 767

Radinschi, I. 1999, *Acta Phys. Slov.* 49, 789 (preprint gr-qc/0008034)

Rocha Filho, T.M., Figueiredo, A., Brenig, L., Gunzig, E. & Faraoni, V. 2000, *Int. J. Theor. Phys.* 39, 1933

Rosen, N. 1994, *Gen. Rel. Grav* 26, 319

Saa, A., Gunzig, E., Brenig, L., Faraoni, V., Rocha Filho, T.M. & Figueiredo, A. 2001*b*, *Int. J. Theor. Phys.* 40, 2295

Stabell, R. & Refsdal, S. 1966, *Mon. Not. R. Astr. Soc.* 132, 379

Sonego, S. & Faraoni, V. 1993, *Class. Quant. Grav.* **10**, 1185

Tryon, E.P. 1973, *Nature* 246, 396

Vilenkin, A. 1983, *Phys. Rev. D* 27, 2848

Wainwright, J. & Ellis, G.F.R. 1997, *Dynamical Systems in Cosmology* (Cambridge: Cambridge University Press)

Weinberg, S. 1972, *Gravitation and Cosmology* (New York: Wiley)

Xulu, S. 2000, *Int. J. Theor. Phys.* 39, 1153 (preprint gr-qc/9910015)