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Mathematical Note

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1. The model

Let's take into consideration the following model (F):

(1f)
$$\begin{aligned} P_{a1} &= l_a W + a_a P_{a0} + b_a P_{b0} \\ P_{b1} &= l_b W + a_b P_{a0} + b_b P_{b0} \end{aligned}$$

(2f)
$$P_{b1} = 1$$

(3f)
$$\begin{aligned} A_0^d &\geq D_{a0} + a_a A_1 + a_b B_1 \\ B_0^d &\geq D_{b0} + b_a A_1 + b_b B_1 \\ L &\geq l_a A_1 + l_b B_1 \\ D_{a1} &= A_1 \\ D_{b1} &= B_1 \end{aligned}$$

If in the first relation of (3f) the inequality sign holds, then: $P_{a0} = 0$; if it holds in the second one, then: $P_{b0} = 0$. However, as it is specified in paragraph 15, assumption (ii), it should be noted that such cases will not be taken into consideration during the economic dissertation. The inclusion in the present demonstration is due to the need of clarifying the mathematical passages.

If the inequality sign holds in the third relation of (3f), then: $W = 0$.

(4f) $$I = (a_a A_1 + a_b B_1)P_{a0} + (b_a A_1 + b_b B_1)P_{b0}$$

(5f) $$S = (A_0^s - D_{a0})P_{a0} + (B_0^s - D_{b0})P_{b0}$$

(6f) $$P_{b1}(r_b + 1) = P_{b0}$$

(7f) $$\frac{A_0^d}{B_0^d} = \frac{A_0^s}{B_0^s}$$

The meaning of the symbols is the following: P_{a0}, P_{b0}, P_{a1} and P_{b1} refer to prices; W indicates wage; S and I represent savings and investments; A_1 and B_1 correspond to quantities of produced commodities; A_0^d and B_0^d specify demands, while A_0^s, B_0^s and L refer to endowments. The single-valued mappings D_{a0}, D_{b0}, D_{a1} and D_{b1} designate standard Walrasian demand functions, which are characterised by the following assumptions:

1. The set of the independent variables includes the first five among the above-specified symbols.
2. In the non-negative orthant of the independent variables, each demand function is continuous and positive. The last assumption, more

demanding than the usual non-negativity, in set for technical reasons, in order to ensure continuity of the following function (9).

For the sake of simplicity, we suppose that the technical coefficients are positive. Moreover, the usual vitality conditions will hold. Finally, r_b specifies the profit rate. The first eleven among the above-listed symbols constitute the unknowns of the model, while r_b is exogenously defined.

Let's substitute (6f) and (2f) into the second equation of (1f), thus obtaining:

(8) $$1 = l_b W + a_b P_{a0} + b_b (r_b + 1)$$

Throughout easy passages, the following propositions can be derived from (1f), (2f), (6f) and (8).

(a) The quantities W e P_{a1} can be defined as functions of the unique variable P_{a0}.

(b). Let's consider the interval: $H_{r_b} = \{-1 \le r_b \le \max_{r_b}\}$, with: $\max_{r_b} = \frac{1-b_b}{b_b}$. Let's define: $\beta_{r_b} = \frac{1-b_b(r_b+1)}{a_b}$. For every $r_b \in H_{r_b}$, we can determine an interval: $H_{P_{a0}} = \{0 \le P_{a0} \le \beta_{r_b}\}$ such that, for every $P_{a0} \in H_{P_{a0}}$, it results: $W \ge 0$, $P_{a1} > 0$.

(c). In the interval $H_{P_{a0}}$ the functions $W(P_{a0})$ e $P_{a1}(P_{a0})$ are continuous.

(d). If: $-1 < r_b \le \max_{r_b}$, then: $P_{b0} > 0$; if: $r = -1$, then: $P_{b0} = 0$.

Due to the income correction, which is specified in Paragraph 9, the third of (3f) is always satisfied with the equality sign, except in border solutions, which are examined in Assumption (iii), Paragraph 16 and in Paragraphs 21, 24, 26, where there will be a continuos set of solutions characterised by $W = 0$.

Let’s assume that the equality sign holds in the first two relations of (3f). Substituting the last two equations of (3f) into the expressions at the right of the equality sign in the first two relations of (3f) and in equation (4f), we are able to define the variables A_0^d, B_0^d and I. Moreover, equation (5f) defines the variable S. In the interval $H_{P_{a0}}$, the four above-quoted expressions are continuous functions of the unique variable P_{a0}.

Let's define:

(9) $$\delta(P_{a0}) = \frac{D_{a0} + a_a D_{a1} + a_b D_{b1}}{D_{b0} + b_a D_{a1} + b_b D_{b1}}$$

(10) $$\gamma = \frac{A_0^s}{B_0^s} = \text{Constant}$$

In the interval $H_{P_{a0}}$, the function $\delta(P_{a0})$ is continuous.

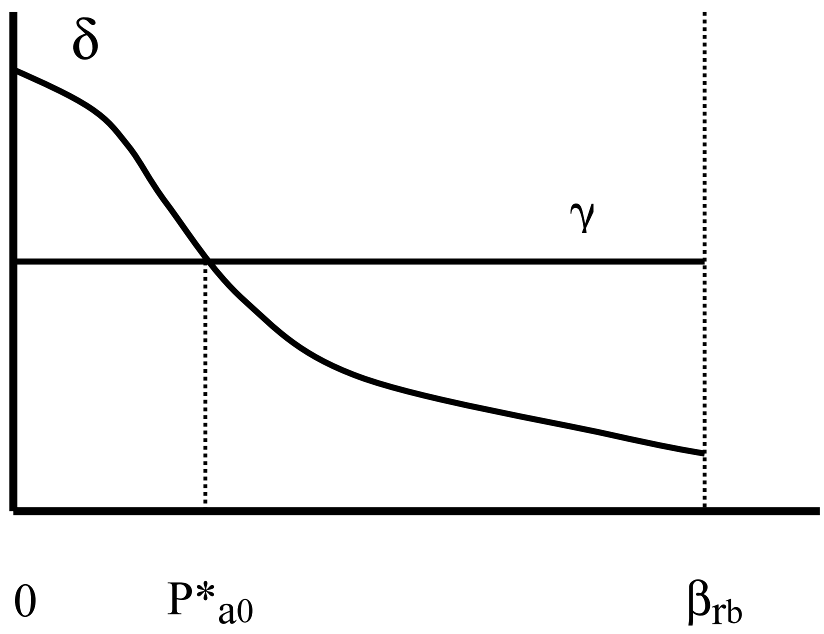
Taking into consideration equation (7f), for every $r_b \in H_{r_b}$ one, and only one, of the following three sentences is necessarily true.

(I). There exists $P_{a0}^* \in H_{P_{a0}}$ such as: $\delta(P_{a0}^*) = \gamma$.

(II). For every $P_{a0} \in H_{P_{a0}}$ it results: $\delta(P_{a0}) < \gamma$.

(III). For every $P_{a0} \in H_{P_{a0}}$ it results: $\delta(P_{a0}) > \gamma$.

The following graph shows an example of case (I).



In case (I), it results: $P_{a0} \geq 0$ and the first two relations of (3f) are satisfied with the equality sign. Assumption (ii) in Paragraph 15 of the text confirms the main argument there for case (I) above. Therefore, the remaining two cases will be examined only for the sake of completeness.

In case (II), the proportion between total demands of commodities a_0 and b_0 is unable to reach the proportion set by the endowments. Therefore, it follows: $P_{a0} = 0$. As a consequence, the first relation of (3f) is satisfied with the inequality sign.

In case (III) there are two possibilities. If: $r_b = -1$, the proportion between total demands of commodities a_0 and b_0 is always above the proportion set by the endowments and hence it ensues: $P_{b0} = 0$; the second relation of (3f) is satisfied with the inequality sign and P_{a0} is suitable to assume any value in the interval $H_{P_{a0}}$. If: $-1 < r_b \leq \max_{r_b}$, model (F), together with model (E), does not admit any solution, since case (III) requires: $P_{b0} = 0$, while such an occurrence is prevented by equations (2f) e (6f).

Let's define:

$$N_{r_b} = \{-1 < r_b \leq \max_{r_b} / \text{case (III) is true}\}$$

$$Z_{r_b} = H_{r_b} - N_{r_b}$$

For every $r_b \in Z_{r_b}$, we can determine the mappings $S(r_b)$ and $I(r_b)$.