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SOME MATHEMATICAL PROPOSITIONS ON THE SRAFFA MODEL

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I Introduction

Since the publication in 1960 of Piero Sraffa's book "Production of Commodities by Means of Commodities, Prelude to a Critique of Economic Theory", quite a lot of papers concerning the Sraffa theory and its relationship with classical and modern economics have appeared in print. This short work has quite a different scope. It is essentially a mathematical treatment of a system of equations describing a Sraffa real model.[1] Necessary and sufficient conditions for existence, uniqueness, positivity of the solution and corresponding analytical expressions will be given, and the relationship between the rate of profit and the wage level will be analytically defined. The existence of a linear inverse relationship between the rate of profit and the wage level in a type system is a well known fact. It will be proven that in a real system the inverse nature of the relationship is retained, while the linearity disappears.

As this paper is essentially a mathematical work, the theoretical consequences of the inverse relationship between the rate of profit and the wage level, and the connections between the Sraffa theory and Marx and Ricardo theories, fall beyond its scope and will not be dealt with.

While Sraffa's book describes in full the behaviour of the model, it lacks completely a mathematical formalization, thus rendering difficult a full comprehension of the less immediate consequences of the theory. I hope this paper will help those who are interested in developing this line of economic thought.

II Description of the Model

The model we will consider throughout this paper consists of a system of equations

1 Throughout this paper, the terms "real model" and "real system" refer to the model in part I, chapter II, paragraph 11 of reference (2). The term "type system" refers to part I, chapter V of reference (2).

based on n technological linear processes, each one producing a single commodity. Joint production and alternative processes will be excluded. For each process, a positive labour input coefficient will be defined; the total amount of labour will be set equal to one. The $(n \times n)$ matrix of the input coefficients will be normalized by setting the total production of each commodity equal to one.

The above normalization of the input coefficients may appear to be an arbitrary restriction of the Sraffa theory, which concerns an economy of a given productive structure, without dealing with the existence or non-existence of constant returns to scale. On the other hand, the use of a normalized input matrix, while necessary for developing a mathematical framework, does not restrict the model—since any matrix describing any possible productive configuration, can always be normalized without altering the natural proportion of the commodities used as the input. Therefore, the normalization in this model does not necessarily presume constant returns to scale, but is essentially only the choice of a unit of measure for the input coefficients.

The first n equations will set the price of each commodity equal to its production cost. According to Sraffa's approach, the prices and the wage level will be measured as a fraction of the value of the net product measured with the prices of the system. Therefore, in the last equation the linear combination of the net product of each commodity and the corresponding price will be set equal to one. The prices and the wage level will be considered endogenous variables, while the rate of profit is an exogenous one.

The mathematical system is the following:

$$(1) \quad \begin{cases} (1') & A'p(1+r)+fw=p \\ (1'') & l(I-A')p=1 \end{cases}$$ [2]

where:

- A — $(n \times n)$ matrix of input coefficients satisfying the following conditions: non-negative, indecomposable, vital.[3]
- f — column n-vector of quantity of labour ($f>0$, $lf=1$).
- p — column n-vector of prices.
- w — scalar wage per unit of labour.
- r — scalar rate of profit.

Whenever convenient, the variable $\rho=1+r$ will be used. The above system will be investigated in the set of variables defined by:

$p \geq 0, \neq 0,\ w \geq 0,\ r \geq 0.$

2 The symbol l refers to the row n-vector with every component equal to one and it will be used as a sum operator. I is the $(n \times n)$ unit matrix and the apostrophe indicates transposition.

3 Vital means satisfying the Solow conditions: $\sum_{j=1}^{n} a_{ij} \leq 1$, for every i and with strict inequality for at least one i. Note that from the above conditions: $l(I-A') \geq 0, \neq 0$.

III Some Propositions Concerning the Real System

Let us define:

λ^* the P. and F. eigenvalue of the matrix A'.[4] From the Solow conditions on A and corollary (1)[5]: $\lambda^* < 1$.

$R = (1 - \lambda^*)/\lambda^*$.

$\Gamma = 1 + R = 1/\lambda^*$.

$\lambda(\rho)$ the P. and F. eigenvalue of the matrix $\rho A'$.
The following two lemmas can be proven.

Lemma (1). If $\rho \geq 1$, then $\lambda(\rho) = \lambda^* \rho$.
Note. Therefore: $\lambda(\Gamma) = 1$.
Proof. From the P. and F. theorem: $A'u = \lambda^* u$. Then:

$$(2) \quad \rho A'u = \lambda^* \rho u.$$

Let E be the set of the real eigenvalues of the matrix $\rho A'$. Expression (2) implies that $\lambda^* \rho \epsilon E$. Let us suppose there exists δ such as: $\delta > \lambda^* \rho$ and $\delta \epsilon E$. Then $\rho A'\mathrm{g} = \delta \mathrm{g}$, where g is a column n-vector.
Hence:

$$(3) \quad A'g = (\delta/\rho)g, \text{ with } \delta/\rho > \lambda^*.$$

Expression (3) contradicts part (c) of the P. and F. theorem and is therefore impossible. Thus $\lambda^* \rho$ will be the greatest real eigenvalue of $\rho A'$. Q.E.D.
The following lemma concerns the behaviour of the equations (1') alone, without the condition (1'').

Lemma (2). A necessary and sufficient condition for a solution p ($p \geq 0, \neq 0$) to the equations (1') to exist for any $w > 0$ is that $r < R$. In this case there is only one solution p, which is strictly positive and given by:

$$(4) \quad p = (\mathrm{I} - \rho A')^{-1} f w.$$

Proof. From theorem (1)[6] it follows that a necessary and sufficient condition for a solution p ($p \geq 0, \neq 0$) to the equations (1') to exist for any $w > 0$ is that $\lambda(\rho) < 1$. In this case there is only one solution p, which is strictly positive and given by (4). From the above definitions and lemma (1), the following double implications are proven:

$$\lambda(\rho) < 1, \leftrightarrow \lambda^* \rho < 1, \leftrightarrow \rho < \Gamma, \leftrightarrow r < R. \text{ Q.E.D.}$$

Let us now examine the behaviour of the whole system (1). The symbols ***p***, ***w***, ***r*** refer

4 See the Perron and Frobenius theorem, paragraph IV.
5 See corollary (1), paragraph IV.
6 See theorem (1), paragraph IV.

to given values of the variables p, w, r. Whenever convenient, the symbol $\boldsymbol{\rho}=1+\boldsymbol{r}$ will be used. The term "solution" will only refer to solutions $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ to the system (1) such as: $\boldsymbol{p}\geq 0, \neq 0$, $\boldsymbol{w}\geq 0$, $\boldsymbol{r}\geq 0$.

Proposition (1). If $r>R$, then system (1) will not have any solution.
Proof. Directly from lemma (2). Q.E.D.

Proposition (2). The following three implications are true.
(i). If $\boldsymbol{r}=R$ and $\boldsymbol{w}=0$, then system (1) has only one solution and $\boldsymbol{p}>0$.
(ii). If system (1) has a solution and $\boldsymbol{r}=R$, then system (1) has only one solution and $\boldsymbol{p}>0$, $\boldsymbol{w}=0$.
(iii). If system (1) has a solution and $\boldsymbol{w}=0$, then system (1) has only one solution and $\boldsymbol{p}>0$, $\boldsymbol{r}=R$.
In all the above cases the solution will be given by:

$$(5)\quad \begin{cases} p=u/l(I-A')u \\ w=0 \end{cases}$$

where u is the P. and F. right eigenvector of A'.
Proof of the implication (i). If $\boldsymbol{r}=R$ and $\boldsymbol{w}=0$, then system (1) becomes:

$$(6)\quad \begin{cases} (6')\quad A'p=\lambda^* p \\ (6'')\quad l(I-A')p=1. \end{cases}$$

The P. and F. theorem provides a solution $p=\alpha u$, with $u>0$ and $\alpha\neq 0$, to the equations (6')—since λ^* is the P. and F. eigenvalue of A'.
Condition (6'') will define $\alpha=(1/l(I-A')u)>0$. Thus solution (5). Q.E.D.
Proof of the implication (ii). The following system:

$$\begin{cases} (I-\Gamma A')p=fw \\ l(I-A')p=1 \end{cases}$$

cannot have a solution for $w>0$—since from lemma (1) $\lambda(\Gamma)=1$, and theorem (1) requires $\lambda(\Gamma)<1$. Hence it must be true that $\boldsymbol{w}=0$ and the rest follows as in the proof of the implication (i). Q.E.D.
Proof of the implication (iii). Since $\boldsymbol{p}\geq 0, \neq 0$, from part (g) of the P. and F. theorem the equality $A'\boldsymbol{p}=(1/\boldsymbol{\rho})\boldsymbol{p}$ implies $\boldsymbol{\rho}=\Gamma$ and theorefore $\boldsymbol{r}=R$. The rest follows as in the proof of the implication (i). Q.E.D.

Proposition (3). The following three implications are true.
(i). If $\boldsymbol{r}=0$ and $\boldsymbol{w}=1$, then system (1) has only one solution and $\boldsymbol{p}>0$.
(ii). If system (1) has a solution and $\boldsymbol{r}=0$, then system (1) has only one solution and $\boldsymbol{p}>0$, $\boldsymbol{w}=1$.
(iii). If system (1) has a solution and $\boldsymbol{w}=1$, then system (1) has only one solution and $\boldsymbol{p}>0$, $\boldsymbol{r}=0$.

In all the above cases the solution will be given by:

$$(7)\quad \begin{cases} p=(I-A')^{-1}f \\ w=1. \end{cases}$$

Proof of the implication (i). If $r=0$ and $w=1$, then system (1) becomes:

$$(8)\quad \begin{cases} (8') \quad (I-A')p=f \\ (8'') \quad l(I-A')p=1. \end{cases}$$

Since $lf=1$, condition (8'') is implicit in (8'). Since from the Solow conditions on A and corollary (1) $\lambda^*<1$, then theorem (1) provides solution (7). Q.E.D.
Proof of the implication (ii). System (1) becomes:

$$(9)\quad \begin{cases} (9') \quad (I-A')p=fw \\ (9'') \quad l(I-A')p=1. \end{cases}$$

Adding the components of (9'):

$$(10)\qquad l(I-A')p=w.$$

Since (9'') and (10) must be compatible, then $w=1$. The rest follows as in the proof of the implication (i). Q.E.D.
Proof of the implication (iii). System (1) becomes:

$$(11)\quad \begin{cases} (11') \quad (I-\rho A')p=f \\ (11'') \quad l(I-A')p=1. \end{cases}$$

Adding the components of (11'):

$$(12)\qquad l(I-\rho A')p=1.$$

Since (11'') and (12) must be compatible, then $\rho=1$ and therefore $r=0$.
The rest follows as in the proof of the implication (i). Q.E.D.

Proposition (4). The following two implications are true.

(i). If system (1) has a solution and $0<w<1$, then system (1) has only one solution and $p>0$, $0<r<R$.

(ii). If $0<r<R$, then system (1) has only one solution and $p>0$, $0<w<1$.
In both the above cases the solution will be given by:

$$(13)\quad \begin{cases} p=(I-\rho A')^{-1}f/l(I-A')(I-\rho A')^{-1}f \\ w=1/l(I-A')(I-\rho A')^{-1}f. \end{cases}$$

Proof of the implication (i). If the set of values (p, w, r) is a solution to the system (1), then it must also be a solution to the equations (1'). Therefore from lemma (2) there is only one solution and:

$$(14)\qquad p=(I-\rho A')^{-1}fw>0,\ r<R.$$

Substituting (14) into condition (1''):

$$(15)\qquad [l(I-A')(I-\boldsymbol{\rho}A')^{-1}f]\boldsymbol{w}=1.$$

Let us examine the scalar function:

$$(16)\qquad m(\rho)=l(I-A')(I-\rho A')^{-1}f.$$

According to the Solow conditions: $l(I-A')\geq 0, \neq 0$; from theorem (2)[7],—since $r<R\rightarrow\lambda(\rho)<1$ from the proof of lemma (2)—the following limit is proven:

$$(17)\qquad \lim_{k\rightarrow\infty}(\rho A')^k=0.$$

Hence theorem (3)[8] can be applied:

$$(18)\qquad (I-\rho A')^{-1}=\sum_{k=0}^{\infty}\rho^k A'^k>0.$$

Since the matrix (18) is strictly positive and strictly increasing for ρ, the function $m(\rho)$ will therefore be positive and strictly increasing for ρ. Since $m(1)=1$, the following double implication is true:

$$(19)\qquad m(\rho)>1\leftrightarrow r>0.$$

The following implications are also true:
(hypothesis) $0<\boldsymbol{w}<1$, $\rightarrow$(from expressions (15) and (16)) $0<(1/m(\boldsymbol{\rho}))<1$,$\rightarrow m(\boldsymbol{\rho})>1$, $\rightarrow$(from implication (19)) $\boldsymbol{r}>0$.
Expressions (14) and (15) prove (13). Q.E.D.
Proof of the implication (ii). Since from the hypothesis $\boldsymbol{r}<R$, from lemma (2):

$$(20)\qquad p=(I-\boldsymbol{\rho}A')^{-1}fw>0 \text{ for any } w>0$$

will be the only solution to the equations (1').
Substituting (20) into condition (1''):

$$(21)\qquad \boldsymbol{w}=1/l(I-A')(I-\boldsymbol{\rho}A')^{-1}f.$$

Moreover, the following implications are true:
(hypothesis) $0<\boldsymbol{r}<R$, $\rightarrow$(from expressions (18) and (19)) $m(\boldsymbol{\rho})>1$, $\rightarrow 0<(1/m(\boldsymbol{\rho}))<1$, $\rightarrow$(from expression (21)) $0<\boldsymbol{w}<1$.
Expressions (20) and (21) prove (13). Q.E.D.

Proposition (5). If $0<r<R$, then the relationship between r and w in system (1) will be a strictly decreasing monotonic function given by the following expressions:

$$w=1/l(I-A')(I-\rho A')^{-1}f$$

or:

$$w=1/l(I-A')(\sum_{k=0}^{\infty}\rho^k A'^k)f$$

where $A'^0=I$.

7 See theorem (2), paragraph IV.
8 See theorem (3), paragraph IV.

Proof. Directly from expressions (21) and (18). Q.E.D.

IV List of the Theorems Used.

The Perron and Frobenius Theorem. Suppose T is an $(n \times n)$, non-negative, indecomposable matrix. Then there exists an eigenvalue e such that:

(a) e real and positive;
(b) with e can be associated strictly positive left and right eigenvectors;
(c) $e \geq |d|$, where d is any other eigenvalue of T different from e;
(d) the eigenvectors associated with e are unique to constant multiples;
(e) if $0 \leq B \leq T$ and b is an eigenvalue of B, then $|b| \leq e$, where $|b| = e \rightarrow B = T$;
(f) e is a simple root of the characteristic equation of T;
(g) there exists no other eigenvalue of T with which non-negative, non-zero eigenvectors are associated.

Corollary (1). In the above case: $\min_i \sum_{j=1}^{n} t_{ij} \leq e \leq \max_i \sum_{j=1}^{n} t_{ij}$, where equality on either side implies equality throughout. A similar proposition holds for column sums.
Theorem (1). A necessary and sufficient condition for the solution x $(x \geq 0, \neq 0)$ to the system: $(sI - T)x = c$ (T: $(n \times n)$, non-negative, indecomposable) to exist for any $c \geq 0, \neq 0$, is $s > e$. In this case there is only one solution which is strictly positive and given by: $x = (sI - T)^{-1}c$. If $s = 1$, then the condition becomes: $e < 1$.
Theorem (2). For a non-negative, primitive matrix, as $k \rightarrow \infty$, $T^k = e^k uv' + 0(k^z |\lambda_2|^k)$, where u, v' right and left P. and F. eigenvectors of T normalized so that: $v'u = 1$; λ_2 the next greater (in module) eigenvalue of T after e; $z = m_2 - 1$, where m_2 is the multiplicity of λ_2 in the characteristic equation.

An irreducible acyclic matrix is primitive and conversely. The powers of an irreducible cyclic matrix may be studied in terms of powers of primitive matrices.
Theorem (3). If C, real or complex matrix, is such that: $\lim_{k \rightarrow \infty} C^k = 0$, then: $(I - C)^{-1} = \sum_{k=0}^{\infty} C^k$, where $C^0 = I$.

All the proofs can be found in reference (1), except the proof of part (g) of the P. and F. theorem, which follows.

Proof of part (g). Let us suppose that the following equality is true:

$$(22) \qquad Ty = gy,$$

with:

g eigenvalue of T;
y corresponding right eigenvector of T such as $y \geq 0, \neq 0$.

From the above it follows that g must be real. From the P. and F. theorem, parts (a) and (b):

$$(23) \qquad v'T = ev'$$

with:

e the P. and F. eigenvalue of T, $(e>0)$;

v' corresponding left eigenvector of T, $(v'>0)$.

Multiplying (22) by v' and from (23):

(24) $$gv'y=v'Ty=ev'y;$$

since $v'y>0$, then from (24):

(25) $$g=e.$$ Q.E.D.

References

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