UNIVERSITÀ DEGLI STUDI DI FERRARA

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
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VSP
Varieties of Sums of Powers

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Introduction

... the end of all our exploring
will be to arrive where we started
and know the place for the first time.
T.S. Eliot, "Little Gidding"

A well know theorem of linear algebra asserts that if F_2 is a nondegenerate quadratic form over a k-vector space of dimension $n+1$, with k algebraically closed, then F_2 can be written has $n+1$ squares of linear forms

$$F_2 = L_1^2 + ... + L_{n+1}^2.$$

The linear forms L_i considered as vectors in the dual space V^* are mutually orthogonal with respect to the dual quadratic form F_2^*.
For more than hundred years algebraists and geometers have searched for a generalization of this construction to homogeneous forms F_d on V of arbitrary degree. This problem is known as the *Waring problem* for homogeneous form.
The more important object of the study is the *variety of sums of powers* $\mathbf{VSP}(F_d, h)^o$ parametrizing all representations of F_d as a sum of powers of h linear forms. A decomposition $\{L_1, ..., L_h\}$ in h linear forms of F_d is called an *h-polar polyhedron* of F_d. The variety $\mathbf{VSP}(F_d, h)^o$ can be viewed as the subvariety of the symmetric power $\mathbb{P}V^{*(h)}$ of $\mathbb{P}V^*$ parametrizing the polar polyhedra of F_d.
The *Waring problem* for homogeneous form was only recently solved by *J.Alexander* and *A.Hirschowitz*. Their result also yields, via *Terraccini's lemma*, the dimension of $\mathbf{VSP}(F_d, h)^o$. The varieties $\mathbf{VSP}(F_d, h)^o$ were studied in the classical algebraic geometry by *A. Dixon, F. Palatini, T Reye, H. Richmond, J. Rosanes, G. Scorza, A. Terracini*, and others.
The lack of techniques of higher dimensional algebraic geometry did not allow them to give any explicit construction of the varieties $\mathbf{VSP}(F_d, h)^o$ or to study a possible compactification $\mathbf{VSP}(F_d, h)$ of $\mathbf{VSP}(F_d, h)^o$.
The interest in varieties of power sums theory has been reawaken in *1992* by a work of *S.Mukai*, who gave a construction of $\mathbf{VSP}(F_d, h)^o$ in the cases

(n,d,h)=(2, 2, 3), (2, 4, 6), (2, 6, 10)

for a general polynomial F_d and also constructed a smooth compactication $\mathbf{VSP}(F_d, h)$ which turned out to be a *Fano threefold* in the first two cases and a *K3* surface in the third case. The construction of Mukai employs a generalization of the concept of the dual quadratic form to forms of arbitrary even degree $d = 2k$.

Other smooth compactifications of $\mathbf{VSP}(F_d, h)$ are known for general cubic polynomials. If n =*2*, $\mathbf{VSP}(F_3, \mathit{4})$ is isomorphic to the projective plane, if n =*3*, $\mathbf{VSP}(F_3, \mathit{5})$ is one point (this is a classical result of *Sylvester*), if n =*4*, $\mathbf{VSP}(F_3, \mathit{8})$ is a smooth Fano variety of dimension *5* and, if n =*5*, $\mathbf{VSP}(F_3, \mathit{10})$ is a holomorphic symplectic *4*-fold.
The state of art in varieties of power sums classification is resumed in the following table.

d	n	h	$\mathbf{VSP}(F_d, h)$	*Reference*
$2h-1$	1	h	1 *point*	*Sylvester*
2	2	3	*quintic Fano threefold*	*Mukai* [*Muk*92]
3	2	4	$\mathbb{P}^2$	*Dolgachev and Kanev* [*DK*93]
4	2	6	*Fano threefold of genus twelve*	*Mukai* [*Muk*92]
5	2	7	1 *point*	*Hilbert, Richmond, Palatini*
6	2	10	*K*3 *surface of genus* 20	*Mukai* [*Muk*92]
7	2	12	5 *points*	*Dixon and Stuart*
8	2	15	16 *points*	*Mukai* [*Muk*92]
2	3	4	$\mathbf{G}(1,4)$	*Ranestad and Schreier* [*RS*00]
3	3	5	1 *point*	*Sylvester's Pentahedral Theorem*
3	4	8	$\mathcal{W}$	*Ranestad and Schreier* [*RS*00]
3	5	10	$\mathcal{S}$	*Iliev and Ranestad* [*IR*01*b*]

Where $\mathcal{W}$ is a fivefold and $\mathcal{S}$ is is a smooth fourfold.

In the first chapter we describe some classical objects of Algebraic Geometry. In particular we state some properties of Grassmannians, Hilbert Schemes and Secant Varieties that will be very important in the study of Varieties of Power Sums.
In the second chapter we define the concept of variety of power sums and we prove some general facts about these varieties.
In the third chapter we report Mukai's construction and we prove Mukai's theorem.
The last chapter is the most important, we give some new proof about well known theorems and we state some new results. We prove by geometrical methods Hilbert's and Sylvester's theorems. Then we give an alternative proof of Dolgachev - Kanev's theorem and using the same idea we will find that ***VSP***$(F_2, \mathit{4})$ is a Grassmannian, moreover we give a method to reconstruct all 4-polar polyhedra of quadric and cubic polynomials. Finally we state some original results about varieties of power sums rationality, in particular we prove rationality of varieties of power sums of quadrics by arguments from linear algebra.

A.M. - 5 July 2009

Chapter 1

General Results

In this first chapter we describe some classical objects of Algebraic Geometry. In particular we state some properties of Grassmannians, Hilbert Schemes and Secant Varieties that will be very important in the study of Varieties of Power Sums.

1.1 Grassmannians

Let V be a k-vector space of dimension n and let $W \subseteq V$ be a subspace of dimension h. Let $\{v_1,\dots,v_h\}$ be a basis of W and consider the h-multivector $v_1\wedge\dots\wedge v_h$ in the h-wedge product $\bigwedge^h V$. If $\{u_1,\dots,u_h\}$ is another basis of W and B is the matrix of change of basis we have $v_1\wedge\dots\wedge v_h = det(B)(u_1\wedge\dots\wedge u_h)$. The matrix B is invertible so $det(B) \neq 0$ and the two multivectors $v_1\wedge\dots\wedge v_h$ and $u_1\wedge\dots\wedge u_h$ identifies the same point in the projective space $\mathbb{P}(\bigwedge^h V)$. If we denote with $\boldsymbol{G}(h,n)$ the set of the subspaces of dimension h of V we have a well defined map

$$Pk{:}\boldsymbol{G}(h,n) \rightarrow \mathbb{P}(\textstyle\bigwedge^h V),\ \textit{defined by}\ W \mapsto [v_1\wedge\dots\wedge v_h]$$

If $\{e_1,\dots,e_n\}$ is a basis of V then $\{e_{i_1}\wedge\dots\wedge e_{i_h}\}$ with $i_1 < i_2 < \dots < i_h$ is a basis of $\bigwedge^h V$. So $dim(\bigwedge^h V) = \binom{n}{h}$ and $\mathbb{P}(\bigwedge^h V) \cong \mathbb{P}^N$ with $N = \binom{n}{h}$ - 1.
We can write the vector $v_1\wedge\dots\wedge v_h$ in the basis $\{e_{i_1}\wedge\dots\wedge e_{i_h}\}$ as

$$v_1\wedge\dots\wedge v_h = \textstyle\sum_{i_1<\dots<i_h} p_{i_1,\dots,i_h}\, e_{i_1}\wedge\dots\wedge e_{i_h}$$

The elements $p_{i_1,\dots,i_h}$ are called the *Plücker coordinates* of W.
Given a multivector $w \in \bigwedge^h V$ and a vector $v \in V$ we say that v divides w if there exist a multivector $u \in \bigwedge^{h-1} V$ such that $w = v \wedge u$. A multivector $w \in \bigwedge^h V$ is totally decomposable if and only if the space of vectors dividing w has dimension h.
For any $[w] = Pk(W)$ we can recover W as the space of vectors v such that $v \wedge w = 0$ in $\bigwedge^{h+1} V$. So the map Pk is injective and it is called the *Plücker embedding*. Now we give a more explicit description of this embedding. If $H = <v_1,\dots,v_h>$ and $\{e_1,\dots,e_n\}$ is a basis of V we can write $v_i = v_i^1 e_1 + \dots + v_i^n e_n$. We consider the $h\times n$ matrix

$$\mathcal{M}=\begin{pmatrix} v_1^1 & \dots & \dots & v_1^n \\ \vdots & \ddots & \ddots & \vdots \\ v_h^1 & \dots & \dots & v_h^n \end{pmatrix}$$

If $\Delta_{i_1,\ldots,i_h}$ is the determinant of the matrix $h\times h$ whose columns are the columns $i_1,\ldots,i_h$ of $\mathcal{M}$ with $i_1 <\ldots< i_h$ then the *Plücker embedding* can be write in the following way

$$Pk{:}\boldsymbol{G}(h,n) \to \mathbb{P}(\bigwedge^h V), \text{ defined by } W \mapsto [\Delta_{1,..,h}:\ldots:\Delta_{n-h,..,n}].$$

Now we fix a multivector $w \in \bigwedge^h V$ and consider the map

$$\varphi_w{:}\ V\to \bigwedge^{h+1} V,\ v\mapsto w \wedge v.$$

Then w is totally decomposable if and only if $dim(Ker(\varphi_w)) = h$ if and only if

$$rank(\varphi_w) = n\text{-}h.$$

We note that the rank of φ_w is never strictly less than n-h and we conclude that

$$[w] \in \boldsymbol{G}(h,n) \Leftrightarrow rank(\varphi_w) \leq n\text{-}h$$

Now the map $L{:}\bigwedge^h V \to Hom(V,\bigwedge^{h+1} V)$ defined by $w\mapsto\varphi_w$ is linear and $\boldsymbol{G}(h,n)\subseteq\mathbb{P}(\bigwedge^h V)$ is the subset defined by the vanishing of $(n\text{-}h+1)\times(n\text{-}h+1)$ minors of the matrix of L. We see that $\boldsymbol{G}(h,n)$ is an algebraic variety called the *Grassmannian of the h-planes of V*.

REMARK 1. *Any h-plane $W \subseteq V$ determine a (n-h)-plane $\frac{V}{W}$, and we have an exact sequence*

$$0 \mapsto W \to V \to \tfrac{V}{W} \mapsto 0$$

By dualization we obtain another exact sequence

$$0 \mapsto (\tfrac{V}{W})^* \to V^* \to W^* \mapsto 0$$

Considering the canonical isomorphism of a vector space of finite dimension with its bidual, if we dualize the second sequence we recover the first sequence. So we have a bijective correspondence between the h-plane in V and the (n-h)-plane in V then*

$$\boldsymbol{G}(h,V) \cong \boldsymbol{G}(n\text{-}h,V^*).$$

PROPOSITION 1. *The Grassmannian $\boldsymbol{G}(h,n)$ parameterizing the h-planes in $\mathbb{P}^n$ is a smooth variety of dimension $(h+1)(n\text{-}h)$.*

Proof: We denote by $P_H \in \boldsymbol{G}(h,n)$ the point corresponding to the $(h+1)$-plane H of V^{n+1}. Let $\{v_0,\ldots,v_h\}$ be a basis of H. If $\{e_0,\ldots,e_n\}$ is a basis of V^{n+1} then we can write $v_i = v_i^0 e_0+\ldots+v_i^n e_n$. We consider the matrix

$$\mathcal{M}=\begin{pmatrix} v_0^0 & \ldots & \ldots & v_0^n \\ \vdots & \ddots & \ddots & \vdots \\ v_h^0 & \ldots & \ldots & v_h^n \end{pmatrix}$$

Let $\mathcal{M}_h$ a $(h+1)\times(h+1)$ minor of $\mathcal{M}$ obtained extracting h-columns in $\mathcal{M}$, say the first h, we consider the set

$$\mathcal{U}_I = \{P_H \in \boldsymbol{G}(h+1,n+1) \mid det(\mathcal{M}_I) \neq 0\}$$

The sets $\mathcal{U}_I$ are open sets in $\boldsymbol{G}(h+1,n+1)$ and on $\mathcal{U}_I$ the matrix $\mathcal{M}_I$ is invertible and we have

$$\mathcal{M}_I^{-1}\mathcal{M}=\begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{0,1} & \dots & \lambda_{0,n-h} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \lambda_{h,1} & \dots & \lambda_{h,n-h} \end{pmatrix}$$

We note that any point $P_H \in \mathcal{U}_I$ determines uniquely a matrix of this form. So we have a bijective correspondence

$$\psi_I\colon \mathcal{U}_I \to k^{(h+1)(n-h)},\ P_H \mapsto (\lambda_{0,1},\dots,\lambda_{0,n-h},\dots,\lambda_{h,n-h})$$

So $\mathcal{U}_I \cong k^{(h+1)(n-h)}$. Now the open sets of the form $\mathcal{U}_I$ cover $\boldsymbol{G}(h+1,n+1)$ and we conclude that $\boldsymbol{G}(h+1,n+1)$ is smooth. □

The first non trivial example of Grassmannian is the case of the lines in $\mathbb{P}^3$.

EXAMPLE 1. *A line $L = <x,y>$ in $\mathbb{P}^3$ corresponds to a plane H in V^4. If $\{e_0,\dots,e_3\}$ is a basis of V^4 we can write $x = x_0 e_0 + \dots + x_3 e_3$ and $y = y_0 e_0 + \dots + y_3 e_3$. In this case we have $N = 5$ and the Plücker embedding is*

$$Pk\colon \boldsymbol{G}(1,3) \to \mathbb{P}^5,\ L \mapsto [\Delta_{0,1}:\Delta_{0,2}:\Delta_{0,3}:\Delta_{1,2}:\Delta_{1,3}:\Delta_{2,3}]$$

where $\Delta_{i,j} = x_i y_j - x_j y_i$.
The $\Delta_{i,j}$ satisfy the equation $\Delta_{0,1}\Delta_{2,3} - \Delta_{0,2}\Delta_{1,3} + \Delta_{0,3}\Delta_{1,2} = 0$.
If $[X_0:\dots:X_5]$ are the homogeneous coordinates on $\mathbb{P}^5$ then $\boldsymbol{G}(1,3)$ is contained in the quadric

$$K = \boldsymbol{V}(X_0X_5 - X_1X_4 + X_2X_3)$$

But we know that $\boldsymbol{G}(1,3)$ is a projective variety of dimension 4 so it must be equal to K. We conclude that the Grassmannian $\boldsymbol{G}(1,3)$ is a smooth quadric hypersurface in $\mathbb{P}^5$.

Now we enunciate the following proposition on the degree of $\boldsymbol{G}(h,n)$ in its *Plücker embedding* without proving it.

PROPOSITION 2. *The Grassmannian $\boldsymbol{G}(h,n)$, embedded in $\mathbb{P}^N$ via the Plücker embedding, is a variety of degree*

$$deg(\boldsymbol{G}(h,n)) = (h(n-h))!\ \prod_{j=1}^{h} \frac{(j-1)!}{(n-h+j-1)!}$$

Proof: Harris - Algebraic Geometry a first Course [Lecture. 19 p.247].

Finally we define two important vector bundles on $\boldsymbol{G}(h,n)$. Consider the map

$$\pi\colon \boldsymbol{G}(h,n)\times V \to \boldsymbol{G}(h,n),\ (x,v) \mapsto x.$$

On each $x \in \boldsymbol{G}(h,n)$ the fibre $\pi^{-1}(x)$ is isomorphic to the vector space V so we have defined a vector bundle of rank $n = dim(V)$ on $\boldsymbol{G}(h,n)$ called the *trivial bundle* and denoted by $\mathcal{E}_G$. Now we consider the subvariety $\mathcal{I} \subseteq \boldsymbol{G}(h,n)\times V$ defined by

$$\mathcal{I} = \{(x,v) \in \boldsymbol{G}(h,n)\times V \ /\ v \in W_x\}$$

where W_x is the h-subspace of V corresponding to $x \in$ ***G****(h,n)*. Then for each $x \in$ ***G****(h,n)* the fibre of the map

$$\varphi\colon \mathcal{I} \times V \to \boldsymbol{G}(h,n),\ (x,v) \mapsto x,$$

is isomorphic to k^h. In this way we get a vector bundle of rank h on ***G****(h,n)* called the *universal bundle* and denoted by $\mathcal{S}_G$. We note that for any $x \in$ ***G****(h,n)* the fibre of φ is a subvector space of the fibre of π and $\mathcal{S}_G$ is a subbundle of $\mathcal{E}_G$. We have the exact sequence

$$0 \mapsto \mathcal{S}_G \longrightarrow \mathcal{E}_G \longrightarrow \mathcal{Q}_G \mapsto 0$$

where $\mathcal{Q}_G \cong \frac{\mathcal{E}_G}{\mathcal{S}_G}$ is a quotient vector bundle of rank *n-h* on ***G****(h,n)*.

1.1.1 The Plücker embedding

In this section we prove directly that the *Plücker map* is indeed an embedding. We consider the map

$$Pk\colon \boldsymbol{G}(h,n) \to \mathbb{P}(\textstyle\bigwedge^h V), \text{ defined by } W \mapsto [\Delta^W_{1,..,h}:\ldots:\Delta^W_{n-h,..,n}].$$

Suppose that $W = < w_1, ..., w_h >$ and $Z = < z_1, ..., z_h >$ are two h-subspaces of V such that $pk(W) = pk(Z)$, then there exists a non zero $\lambda \in k$ such that $\Delta^W_{j_1,..,j_h} = \lambda \Delta^Z_{j_1,..,j_h}$ for any $j_1, ..., j_h$. We write

$$w_i = w^i_0 e_0 + ... + w^i_n e_n \text{ and } z_i = z^i_0 e_0 + ... + z^i_n e_n.$$

Then we consider the matrix

$$\begin{pmatrix} w^1_0 & \ldots & w^1_n \\ \vdots & \ddots & \vdots \\ w^h_0 & \ldots & w^h_n \\ z^1_0 & \ldots & z^1_n \\ \vdots & \ddots & \vdots \\ z^h_0 & \ldots & z^h_n \end{pmatrix}$$

It is clear from the relations $\Delta^W_{j_1,..,j_h} = \lambda \Delta^Z_{j_1,..,j_h}$ that this matrix has rank h, so $W = Z$ and the *Plücker map* is injective.
We saw that the Grassmannian $\mathbf{G}(h,n)$ is covered by the affine sets

$$\mathcal{U}_{i_1,\ldots,i_h} = \{[p_{1,\ldots,h}, ..., p_{n-h,\ldots,h}] | p_{i_1,\ldots,i_h} \neq 0\}.$$

Now we consider the *Plücker map* on this open affine subset to prove that its differential is injective. It not restrictive to consider $\mathcal{U}_{1,\ldots,h}$, since $p_{1,\ldots,h} = \Delta_{1,\ldots,h} \neq 0$ the points in $\mathcal{U}_{1,\ldots,h}$ can be represented by a matrix in the form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & x^0_{h+1} & \ldots & x^0_n \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & x^h_{h+1} & \ldots & x^h_n \end{pmatrix}$$

The $(h+1)\times(h+1)$ minor

$$\begin{pmatrix} 1 & 0 & \dots & 0 & x^0_{h+1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & x^h_{h+1} \end{pmatrix}$$

has determinant equal to x^h_{h+1}. Taking all the minors we can interpret the *Plücker map* on $\mathcal{U}_{1,\dots,h}$ as a morphism on $\mathbb{A}^{(h+1)(n-h)}$ in the form

$$pk_{\mathcal{U}_{1,\dots,h}} : \mathcal{U}_{1,\dots,h} \rightarrow \mathbb{A}^N,\ pk_{\mathcal{U}_{1,\dots,h}}([x^0_{h+1} : ... : x^h_n]) = [x^0_{h+1} : ... : x^h_n : P_0 : ... : P_t].$$

Where $P_0, ..., P_t$ are polynomial function in the $x^0_{h+1}, ..., x^h_n$. So modulo a change of basis the Jacobian matrix of $pk_{\mathcal{U}_{1,\dots,h}}$ is

$$J(pk_{\mathcal{U}_{1,\dots,h}}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ \frac{\partial P_0}{\partial x^0_{h+1}} & \dots & \dots & \frac{\partial P_0}{\partial x^0_{h+1}} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial P_t}{\partial x^0_{h+1}} & \dots & \dots & \frac{\partial P_t}{\partial x^0_{h+1}} \end{pmatrix}$$

and it this clear that $rank(J(pk_{\mathcal{U}_{1,\dots,h}})) = h+1$. So the the differential of $pk_{\mathcal{U}_{1,\dots,h}}$ is injective and since the situation is similar on the other sets of the covering we conclude that the *Plücker map* is an embedding.

EXAMPLE 2. *We consider again $\boldsymbol{G}(1,3)$ and the map*

$$Pk : \boldsymbol{G}(1,3) \longrightarrow \mathbb{P}^5,$$
$$W \mapsto [x_0y_1 - x_1y_0 : x_0y_2 - x_2y_0 : x_0y_3 - x_3y_0 : x_1y_2 - x_2y_1 : x_1y_3 - x_3y_1 : x_2y_3 - x_3y_2].$$

On $\mathcal{U}_{0,1}$ we can assume $x_0 = y_1 = 1$ and $x_1 = y_0 = 0$, so

$$pk_{\mathcal{U}_{0,1}}([x_2 : x_3 : y_2 : y_3]) = [y_2 : y_3 : x_2 : x_3 : x_2y_3 - x_3y_2]$$

The Jacobian matrix is

$$J(pk_{\mathcal{U}_{0,1}}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y_3 & -y_2 & -x_3 & x_2 \end{pmatrix}$$

and clearly its rank is 4.

1.1.2 Tangent space to Grassmannians

Since we have covered $\mathbf{G}(h,n)$ by affine open subsets its is immediate to describe its tangent space at each point Λ; it is just the underlying vector space of any affine piece in

which Λ lies in. Now we want to describe the tangent space in a intrinsic way to get the tangent bundle of $\mathbf{G}(h,n)$.
Let $B = \{e_0, ..., e_n\}$ be a basis of V and look for simplicity at the affine piece $\mathcal{U}_{0,\dots,h}$ represented by the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & x^0_{h+1} & \dots & x^0_n \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & x^h_{h+1} & \dots & x^h_n \end{pmatrix}$$

Let Λ be a point in $\mathcal{U}_{0,\dots,h}$, this is the same of fixing a basis $B^{'} = \{v_0, ..., v_h\}$ of Λ and whose coordinates with respect to the basis B are the rows of the given matrix. Since the zero vector in the tangent space in Λ must correspond to Λ the right way to interpret a tangent vector to $\mathbf{G}(h,n)$ in Λ is as a matrix of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & x^0_{h+1} + t^0_{h+1} & \dots & x^0_n + t^0_n \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & x^h_{h+1} + t^h_{h+1} & \dots & x^h_n + t^h_n \end{pmatrix}$$

where $(t^0_{h+1}, ..., t^h_n)$ represents a tangent direction. It is natural to interpret this matrix as the matrix of a linear map $\Lambda \to V$ with respect to the basis $B^{'}$ and B. Note that this morphism maps each vectors v_i to itself plus a linear combination depending only on the t^i_j. So it is natural to compose our map with the projection map $V \to V/\Lambda$ for finally getting a linear map $\Lambda \to V/\Lambda$ whose matrix is precisely

$$\begin{pmatrix} t^0_{h+1} & \dots & t^0_n \\ \vdots & \ddots & \vdots \\ t^h_{h+1} & \dots & t^h_n \end{pmatrix}$$

when taking $B^{'}$ as a basis of Λ and the classes of $e_{h+1}, ..., e_n$ as a basis of V/Λ. The important fact is that this map is independent on the affine chart chosen an so we can canonically identify the tangent space of $\mathbf{G}(h,n)$ in Λ with the vector space $Hom(\Lambda, V/\Lambda)$. Then we have

$$T_\Lambda \mathbf{G}(h,n) = Hom(\Lambda, V/\Lambda).$$

Now recalling our description of the universal bundle $\mathcal{S}_G$ and of the quotient bundle $\mathcal{Q}_G$ we have that the tangent sheaf of $\mathbf{G}(h,n)$ is naturally isomorphic to $\mathcal{H}om(\mathcal{S}_G, \mathcal{Q}_G)$,

$$\mathcal{T}\mathbf{G}(h,n) \cong \mathcal{H}om(\mathcal{S}_G, \mathcal{Q}_G) \cong \check{\mathcal{S}_G} \otimes \mathcal{Q}_G.$$

1.2 The Hilbert Scheme

The Grassmannians parametrize the subspace of a given dimension of a projective space. The Hilbert schemes are a sort of generalization of the Grassmannias, in some sense they parametrize the subvarieties of $\mathbb{P}^n$ with a given degree and dimension.

1.2.1 The Hilbert Polynomial

We begin this section defining the Hilbert polynomial of a projective scheme X in $\mathbb{P}^n$. The idea is to associate to the homogeneous coordinate ring $S(X)$ of X a polynomial $P_X \in \mathbb{Q}[x]$ that codifies some numerical invariants of X as the dimension and the degree.

Definition 1. *A polynomial $P(x) \in \mathbb{Q}[x]$ is called a numerical polynomial if $P(n) \in \mathbb{Z}$ for all integers $n \gg 0$.*

Proposition 3. *If $P \in \mathbb{Q}[x]$ is a numerical polynomial then there are integers $k_0,\ldots,k_r$ such that*

$$P(x) = k_0\binom{x}{r} + k_1\binom{x}{r-1} + \ldots + k_r.$$

If $F:\mathbb{Z} \to \mathbb{Z}$ is a function and there exists a numerical polynomial $Q(x)$ such that the difference function $\Delta F = F(n+1) - F(n)$ for all $n \gg 0$, then there exists a numerical polynomial $P(x)$ such that $F(n) = P(n)$ for all $n \gg 0$.

Proof: We proceed by induction on the degree of P. If $deg(P) = 0$ we take $k_0 = \ldots = k_r = 0$. Now $\binom{x}{r} = \frac{x^r}{r!} + \ldots$ so we can express a polynomial $P \in \mathbb{Q}[x]$ with $deg(P) = r$ in the above form with $k_0,\ldots,k_r \in \mathbb{Q}$. We define the difference polynomial as $\Delta P(x) = P(x+1) - P(x)$ so

$$\Delta\binom{x}{r} = \binom{x+1}{r} - \binom{x}{r} = \frac{1}{r!}x(x-1)\ldots(x-r+2)r = \binom{x}{r-1} \text{ and}$$
$$\Delta P = k_0\binom{x}{r-1} + k_1\binom{x}{r-2} + \ldots + k_{r-1}.$$

Now $deg(\Delta P) = r-1$ and by induction $k_0,\ldots,k_{r-1} \in \mathbb{Z}$, $P(n) \in \mathbb{Z}$ for $n \gg 0$ implies $k_r \in \mathbb{Z}$. Let $F:\mathbb{Z} \to \mathbb{Z}$ be a function, by the preceding part we can write

$$Q = k_0\binom{x}{r} + k_1\binom{x}{r-1} + \ldots + k_r, \text{ with } k_0,\ldots,k_r \in \mathbb{Z} \text{ and let}$$
$$P = k_0\binom{x}{r+1} + k_1\binom{x}{r} + \ldots + k_r\binom{x}{1}.$$

Then $\Delta P = k_0\Delta\binom{x}{r+1} + k_1\Delta\binom{x}{r} + \ldots + k_r\Delta\binom{x}{1} = Q$. But $\Delta F(n) = Q(n)$ for $n \gg 0$ implies that $\Delta(F-P)(n) = 0$ for $n \gg 0$ so $(F-P)(n) = k_{r+1}$ constant for $n \gg 0$, with $k_{r+1} \in \mathbb{Z}$. We have

$$F(n) = P(n) + k_{r+1} = k_0\binom{x}{r+1} + k_1\binom{x}{r} + \ldots + k_r\binom{x}{1} + k_{r+1} \text{ for all } n \gg 0.$$

□

Let $S = \bigoplus_{k\in\mathbb{Z}} S_k$ be a graded ring. A graded S-module is a S-module M with a decomposition $M = \bigoplus_{h\in\mathbb{Z}} M_h$ such that $S_k M_h \subseteq M_{k+h}$. We define the twisted module $M(l)$ by $M(l)_h = M_{h+l}$ for any $l \in \mathbb{Z}$. The annihilator of M is

$$Ann(M) = \{s \in S \text{ such that } s\cdot m = 0 \;\forall\; m \in M\}.$$

It is a homogeneous ideal in S. The *Hilbert function* of M is defined by

$$h_M(l) = dim_k M_l \text{ for each } l \in \mathbb{Z}.$$

THEOREM 1. *(Hilbert-Serre) Let M be a finitely generated graded S-module where*

$$S = k[x_0,\dots,x_n].$$

Then there exists a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $h_M(l) = P_M(l)$ for all sufficiently large integers l. Furthermore $deg(P_M(z)) = dim(\boldsymbol{V}(Ann(M)))$ where $\boldsymbol{V}(Ann(M))$ denotes the zero set in $\mathbb{P}^n$ defined by the homogeneous ideal $Ann(M)$.

Proof: We note that if $0 \mapsto M' \xrightarrow{f} M \xrightarrow{g} M'' \mapsto 0$ is a short exact sequence, then $h_M(l) = dim_k M_l = dim_k M'_l + dim_k M''_l = h_{M'}(l) + h_{M''}(l)$. Now we prove that $\boldsymbol{V}(Ann(M)) = \boldsymbol{V}(Ann(M')) \cup \boldsymbol{V}(Ann(M''))$. Let $s \in Ann(M)$ then $s \cdot m = 0$ for any $m \in M$. We consider $m' \in M'$ and $m'' \in M''$. Then $f(s \cdot m') = s \cdot f(m') = 0$ but f is injective so $s \cdot m' = 0$. Now there exists $m \in M$ such that $g(m) = m''$ and $s \cdot m'' = g(s \cdot m) = 0$. Now $Ann(M) \subseteq (Ann(M')) \cap (Ann(M''))$ implies $\boldsymbol{V}(Ann(M')) \cup \boldsymbol{V}(Ann(M'')) \subseteq \boldsymbol{V}(Ann(M))$. Let $x \notin \boldsymbol{V}(Ann(M))$ then there exists $P \in Ann(M)$ such that $P(x) \neq 0$. From $Ann(M) \subseteq (Ann(M')) \cap (Ann(M''))$ we have that $P \in (Ann(M')) \cap (Ann(M''))$ and so $x \notin \boldsymbol{V}(Ann(M')) \cup \boldsymbol{V}(Ann(M''))$.
Now M is a finitely generated graded module over the noetherian ring S so M admits a filtration with quotients of the form $\frac{S}{\mathcal{P}}(l)$ with $\mathcal{P}$ a homogeneous prime ideal and we have $M \cong \frac{S}{\mathcal{P}}(l)$. The shift l corresponds to a change of variables so we can consider $M = \frac{S}{\mathcal{P}}$. If $\mathcal{P} = (x_0,\dots,x_n)$ we note that $Ann(M) = \mathcal{P}$. Then $h_M(l) = 0$ for any $l>0$ and so $P_M(l) = 0$ for any $l>0$ and $deg(P_M) = dim(V(\mathcal{P})) = -1$ with the convention that the zero polynomial has degree -1 and the empty set has dimension -1. If $\mathcal{P} \neq (x_0,\dots,x_n)$ we choose $x_i \notin (x_0,\dots,x_n)$. Then we have the exact sequence

$$0 \mapsto M \xrightarrow{F} M \xrightarrow{G} \frac{M}{x_iM} \mapsto 0$$

where $F(Q) = x_iQ$. Let $Q \in M$ such that $x_iQ = 0$ in M, $x_i \notin \mathcal{P}$ implies $Q \in \mathcal{P}$ because $\mathcal{P}$ is a prime ideal so $Q = 0$ in M and F is injective. The projection G is clearly surjective. Let $Q \in Ker(G)$ then $Q \in x_iM$ and there exists $H \in M$ such that $Q = x_iH = F(H)$. Let $Q \in Im(F)$ then there exists $H \in M$ such that $F(H) = x_iH = Q$ so $G(Q) = 0$ and $Q \in Ker(G)$. We conclude that the sequence is exact.
Then $h_{\frac{M}{x_iM}}(l) = h_M(l) - h_M(l-1) = (\Delta h_M)(l-1)$. Moreover $\boldsymbol{V}(Ann(\frac{M}{x_iM})) = \boldsymbol{V}(\mathcal{P}) \cap H$, where H is the hyperplane $x_i = 0$ and $\boldsymbol{V}(\mathcal{P})$ is not contained in H because $x_i \notin \mathcal{P}$ so $dim(\boldsymbol{V}(Ann(\frac{M}{x_iM}))) = dim(\boldsymbol{V}(\mathcal{P})) - 1$. Now by induction on $dim(\boldsymbol{V}(Ann(M)))$ we can assume that $h_{\frac{M}{x_iM}}$ coincides with a polynomial $P_{\frac{M}{x_iM}}$ for any $l \gg 0$ with $deg(P_{\frac{M}{x_iM}}) = dim(\boldsymbol{V}(Ann(\frac{M}{x_iM})))$. By proposition 3 we have that h_M is a polynomial function corresponding to a polynomial P_M of degree $dim(\boldsymbol{V}(\mathcal{P}))$. Clearly P_M is unique.

□

Let $X \subseteq \mathbb{P}^n$ be a scheme of dimension r then its homogeneous coordinate ring

$$S(X) = \frac{k[x_0,\dots,x_n]}{I_X}$$

is a finitely generated graded $k[x_0,\dots,x_n]$-module.

DEFINITION 2. *The polynomial P_M is called the Hilbert polynomial of the module M. The polynomial P_X associated to the ring S(X) is called the Hilbert polynomial of the scheme X and by Hilbert-Serre theorem we have $deg(P_X) = r = dim(X)$. We define the degree of X to be r! times the leading coefficient of P_X and the arithmetic genus of X to be*

$$p_a(X) = (-1)^{dim(X)}(p_a(0) - 1).$$

EXAMPLE 3. *We consider the case $X = \mathbb{P}^n$ then $S = k[x_0,...,x_n]$ and*

$$P_X(z) = h_X(z) = \binom{z+n}{n} = \frac{1}{n!}z^n + ...,$$

so $dim(X) = n$, $deg(X) = 1$ and $p_a(X) = (-1)^n(\binom{n}{n} - 1) = 0$.
If $X = \boldsymbol{V}(P)$ is a hypersurface in $\mathbb{P}^n$ with P a homogeneous polynomial of degree d then we have the exact sequence

$$0 \mapsto S(-d) \xrightarrow{F} S \xrightarrow{G} \frac{S}{(P)} \mapsto 0$$

where $F(Q) = P \cdot Q$. So $h_{\frac{S}{(P)}}(z) = h_S(z) - h_S(z-d)$ and

$$P_X(z) = \binom{z+n}{n} - \binom{z-d+n}{n} = \frac{d}{(n-1)!}z^{n-1} + ...,$$

so $deg(X) = d$ and $dim(X) = n-1$. In particular if $C \subseteq \mathbb{P}^2$ is a curve of degree d then $P_X(z) = \binom{0+2}{2} - \binom{0-d+2}{2} = 1 - \frac{1}{2}(d-2)(d-1)$ so $p_a(C) = \frac{1}{2}(d-1)(d-2)$.
More generally if $X \subseteq \mathbb{P}^n$ is a hypersurface of degree d we have $P_X(0) = 1 - \binom{-d+n}{n}$ and

$$p_a(X) = (-1)^n \frac{-d+n}{n} = (-1)^n \frac{(-d+n)(-d+n-1)...(-d+1)}{n!} = \frac{(d-n)(d-n+1)...(d-1)}{n!} = \binom{d-1}{n}.$$

For example for a cubic surface $X \subseteq \mathbb{P}^3$ we have $p_a(X) = 0$.
Finally let X be a complete intersection of two surfaces of degree a,b in $\mathbb{P}^3$. We write $\mathbb{I}(Y)=(f,g)$ with f homogeneous of degree a and g homogeneous of degree b. We consider the exact sequence

$$0 \mapsto \frac{S}{(f)}(-b) \xrightarrow{\cdot g} \frac{S}{(f)} \xrightarrow{\pi} \frac{S}{(f,g)} \mapsto 0.$$

Then

$$h_{\frac{S}{(f)}} = h_{\frac{S}{(f)}(-b)} + h_{\frac{S}{(f,g)}} \text{ and } h_{\frac{S}{(f,g)}}(z) = h_{\frac{S}{(f)}}(z) - h_{\frac{S}{(f)}(-b)}(z) \text{ so } h_{\frac{S}{(f,g)}}(z) = h_{\frac{S}{(f)}}(z) - h_{\frac{S}{(f)}}(z-b).$$

We have

$$h_Y(z)=\binom{z+3}{3}-\binom{z-a+3}{3}-\binom{z-b+3}{3}+\binom{z-a-b+3}{3},\ P_Y(0)=1-\binom{-a+3}{3}-\binom{-b+3}{3}+\binom{-a-b+3}{3}.$$

Then

$$(P_Y(0)-1)=-\frac{(3a^2b+3ab^2-12ab)}{-6} \text{ and } p_a(Y)=\frac{1}{6}(3a^2b+3ab^2-12ab+6)=\frac{1}{2}(a^2b+ab^2-4ab)+1.$$

We conclude that the arithmetic genus of a curve that is scheme theoretic complete intersection of surfaces of degree a,b in $\mathbb{P}^3$ is given by

$$p_a(Y)=\frac{1}{2}ab(a+b-4)+1.$$

EXAMPLE 4. *Let*

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N,\ ([x_0 : ... : x_n], [y_0 : ... : y_m]) \mapsto [x_0y_0 : ... : x_iy_j : ... : x_ny_m],$$

with $N = nm + n + m$, *be the Segre embedding, and let* $\Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ *be the Segre variety.*
A homogeneous polynomial of degree d *on* $\Sigma_{n,m}$ *corresponds to a bihomogeneous polynomial of bidegree* (d, d) *on* $\mathbb{P}^n \times \mathbb{P}^m$. *Then*

$$h_{\Sigma_{n,m}}(d) = \binom{d+n}{d}\binom{d+m}{d} = \frac{(d+n)...(d+1)}{n!}\frac{(d+m)...(d+1)}{m!} = \frac{1}{n!m!}d^{n+m} + ...$$

We have $dim(\Sigma_{n,m}) = n + m$ *and* $deg(\Sigma_{n,m}) = \frac{1}{n!m!}(n + m) = \binom{n+m}{n}$.
In particular for the smooth quadric surface $\Sigma_{1,1} = Q \subseteq \mathbb{P}^3$ *we have*

$$h_{\Sigma_{1,1}}(d) = (d + 1)^2.$$

If we compute the Hilbert polynomial of Q *using the formula for an hypersurface in* $\mathbb{P}^3$ *we obtain* $h_Q(d) = \binom{d+3}{3} - \binom{d+1}{3} = (d + 1)^2$.

1.2.2 Flat families and Hilbert Scheme

In this section we define the Hilbert scheme an we state some of its property without proves. For a complete treatment of Hilbert schemes theory see, for example, *E.Sernesi, Deformations of Algebraic schemes, Springer.*
The notion of representable functor has several applications in Algebraic Geometry, the Hilbert scheme is an example. Let $\mathcal{C}$ be a category and let $X \in Ob(\mathcal{C})$. We have the covariant functor $Hom_{\mathcal{C}}(X, -)$ and the controvariant functor $Hom_{\mathcal{C}}(-, X)$.

DEFINITION 3. *A covariant functor* $F : \mathcal{C} \longrightarrow \mathfrak{Sets}$ *is representable if there exists an object* X *in* $Ob(\mathcal{C})$, *such that* F *is isomorphic to* $Hom_{\mathcal{C}}(X, -)$.
A controvariant functor $F : \mathcal{C} \longrightarrow \mathfrak{Sets}$ *is representable if there exists an object* X *in* $Ob(\mathcal{C})$, *such that* F *is isomorphic to* $Hom_{\mathcal{C}}(-, X)$.

In this case the object $X \in Ob(\mathcal{C})$ represents the functor F and this object is unique up to isomorphism.
In this section we denote by

- $\mathfrak{Sch}(k)$ the category of schemes over k,
- $\mathfrak{Sets}$ the category of sets.

Let X be a quasi projective scheme over the algebraically closed field k. A flat family of proper subscheme of X parametrized by a scheme S is a closed subscheme $Z \subseteq S \times X$, such that the projection $\pi : Z \to S$ is flat and proper. If $s \in S$ is a closed point we note $Z_s = \pi^{-1}(s)$. We denote by $\mathfrak{Flat}(S)$ the set of all flat families of proper subschemes of X parametrized by S.
Given a flat family and a morphism of schemes $f : S' \to S$, we have a morphism

$$f \times Id_X : S' \times X \to S \times X,$$

and the family $Z' = (f \times Id_X)^{-1}(Z)$ is again flat. In this way we obtain a morphism

$$\widehat{f} : \mathfrak{Flat}(S) \longrightarrow \mathfrak{Flat}(S'), \quad Z \mapsto (f \times Id_X)^{-1}(Z).$$

We get a controvariant functor

$$\underline{Hilb}(X) : \mathfrak{Sch}(k) \longrightarrow \mathfrak{Sets}, \text{ defined by } S \mapsto \mathfrak{Flat}(S),$$

that we can consider as a covariant functor

$$\underline{Hilb}(X) : \mathfrak{Sch}(k)^{op} \longrightarrow \mathfrak{Sets}, \text{ defined by } S \mapsto \mathfrak{Flat}(S).$$

Let $P \in \mathbb{Q}[T]$ be a polynomial. We denote by $\mathfrak{H}\mathfrak{P}_P(S)$ the set of $Z \subseteq S \times X$ such that Z is proper and flat over S and Z_s has Hilbert polynomial P for any $s \in S$. For a flat family $Z \subseteq S \times X$ the map

$$\mathcal{HP} : S \longrightarrow \mathbb{Q}[T], s \mapsto P_{Z_s},$$

where P_{Z_s} is the Hilbert polynomial of Z_s, is a locally constant function. This implies that given a polynomial $P \in \mathbb{Q}[T]$, the functor

$$\underline{Hilb}_P(X) : \mathfrak{Sch}(k)^{op} \longrightarrow \mathfrak{Sets}, \text{ defined by } S \mapsto \mathfrak{H}\mathfrak{P}_P(S),$$

is a subfunctor of $\underline{Hilb}(X)$.

THEOREM 2. *(Grothendieck) The functor $\underline{Hilb}_P(X)$ is representable by a quasi projective scheme $Hilb_P(X)$. If X is projective then $Hilb_P(X)$ is also projective.*

The theorem implies that there exists a scheme $Hilb_P(X)$, whose points parametrize the subschemes of X with a given Hilbert polynomial P. The scheme $Hilb_P(X)$ is called the *Hilbert scheme*.
Let X be a projective schemes. We consider the constant polynomial $P = h$, with $h \in \mathbb{Z}$. The subschemes of X with Hilbert polynomial P have dimension zero and degree h, i.e, these subschemes are the sets of h points counted with multiplicity. We denote by $Hilb_h(X)$ the corresponding Hilbert scheme.

THEOREM 3. *(Grothendieck) Let $[Z] \in Hilb_h(X)$ be a closed point, representing a subscheme Z of a scheme X. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then there is a canonical isomorphism*

$$T_{[Z]}Hilb_h(X) \cong Hom_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z).$$

THEOREM 4. *(Fogarty) Let X be a smooth connected quasi projective surface. Then for each $h \in \mathbb{N}$ the Hilbert scheme $Hilb_h(X)$ is connected and smooth of dimension $2h$.*

In chapter 2 the fact that $Hilb_h(\mathbb{P}^2)$ is connected and smooth will be very important. Fogarty's theorem is fundamental for several properties of $Hilb_h(X)$. For higher dimensional schemes much less is true.

COROLLARY 1. *Let X be a quasi projective scheme of dimension n and let $[Z] \in Hilb_h(X)$ be a closed point such that $dim(T_xZ) \leq 2$ for any $x \in Z$. Then $Hilb_h(X)$ is smooth of dimension hn at $[Z]$. In particular $Hilb_h(X)$ is smooth for all n if $h \leq 3$.*

1.3 Secant Varieties

Let $X \subseteq \mathbb{P}^n$ be an irreducible variety. For any p, q in X we con consider the line $<p,q>$ in $\mathbb{P}^n$. In this way we get a rational map

$$\varphi\colon X\times X \dashrightarrow \boldsymbol{G}(1,n), \text{ defined by } (p,q) \mapsto <p,q>.$$

The map φ is defined in the complement of the diagonal $\Delta \subseteq X\times X$. It is called the secant lines map and the closure of its image is called the *variety of secant lines to* X and denoted by $\mathcal{S}(X)$.
Now let $p_1,\dots,p_h \in X$ be h points in general position. If X is irreducible and not contained in any $(h\text{-}1)$-plane we can define the map

$$\varphi_h\colon \underbrace{X\times \dots \times X}_{h-times} \dashrightarrow \boldsymbol{G}(h\text{-}1,n), \text{ defined by } (p_1,\dots,p_h) \mapsto <p_1,\dots,p_h>.$$

The map φ_h is called the secant h-planes map of X and the closure of its image $\mathcal{S}_h(X)$ is called the variety of secant h-planes of X.
The union

$$Sec_2(X) = \bigcup_{L\in\mathcal{S}(X)} L \subseteq \mathbb{P}^n$$

is a subvariety of $\mathbb{P}^n$ called the secant lines variety of X. More in general the variety

$$Sec_h(X) = \bigcup_{H\in\mathcal{S}_h(X)} H \subseteq \mathbb{P}^n$$

is a subvariety of $\mathbb{P}^n$ called the secant h-planes variety of X.

EXAMPLE 5. *Let $C \subseteq \mathbb{P}^3$ be the twisted cubic curve and let $p \in \mathbb{P}^3$ be a generic point such that $p \notin C$. There exists a line L such that $p \in L$ and L is secant to C. If a such line will not exists then the projection of C in $\mathbb{P}^2$ from p is a smooth plane cubic $\overline{C}$ isomorphic to C, but $g(C) = 0$ and $g(\overline{C}) = 1$, a contradiction. If there exists two distinct lines L,R secant to C and containing p then the plane $H = <L,R>$ is such that $H\cdot C \geq 4$, a contradiction because $deg(C) = 3$ and C is not contained in a plane.*
So the generic point $p \notin C$ lies on a unique secant line to C, we conclude that $Sec(C)$ is the space $\mathbb{P}^3$.

Let $X \subseteq \mathbb{P}^n$ be an irreducible variety and let $\Delta \subseteq X\times X\times X$ be the locus of triples with two ore more points equal. The locus $V_{1,3}(X)$ of the triples of distinct points $(p,q,r) \in X\times X\times X$ such that p,q,r are collinear is a subvariety of $X\times X\times X\backslash\Delta$ and so its closure $\overline{V_{1,3}(X)}$ is a subvariety of $X\times X\times X$.
More generally for any integers h,l we define the variety $\overline{V_{l,h}(X)} \subseteq X^h$ to be the closure of the locus in X^h of the h-uples of distinct points contained in a l-plane.
The variety $Sec_{1,3}(X) \subseteq \mathbb{P}^n$ is the closure of the locus of lines $<p,q,r> \in \overline{V_{1,3}(X)}$. We define the variety $Sec_{h,l}(X)$ of h-secant l-planes to be the closure of the locus of l-planes containing and spanned by h distinct points of X.

We note that the map

$$\varphi\colon X\times X \dashrightarrow \boldsymbol{G}(1,n)$$

is generically finite because the fibre over a point $L \in \boldsymbol{G}(1,n)$ will be positive dimensional if and only if $L \subseteq X$. Then the dimension of $\mathcal{S}(X)$ as a subvariety of the Grassmannian $\boldsymbol{G}(1,n)$ is equal to $dim(X\times X) = 2\cdot dim(X)$.
Now we consider the incidence correspondence

$$\mathcal{I} = \{(p,L) \mid p \in L\} \subseteq \mathbb{P}^n \times \mathcal{S}(X) \subseteq \mathbb{P}^n \times \boldsymbol{G}(1,n).$$

The image $\Pi_1(\mathcal{I}) = \{p \in \mathbb{P}^n \mid p \in L \text{ for some } L \in \mathcal{S}(X)\} = Sec_2(X)$ is the secant variety of X. The map $\Pi_2 : \mathcal{I} \to \mathcal{S}(X)$ is surjective and all its fibres have dimension one. If the fibre of Π_1 is finite (i.e. if $p \in \mathbb{P}^n$ is a generic point there are a finite number of secant lines of X such that $p \in L$) we have $dim(\mathcal{I}) = dim(Sec_2(X))$. Furthermore $dim(\mathcal{I}) = dim(\mathcal{S}(X))+1 = 2 \cdot dim(X)+1$. We conclude that

$$dim(Sec_2(X)) = 2 \cdot dim(X)+1.$$

Proposition 4. *Let X be an irreducible variety in $\mathbb{P}^n$.*
The variety $\mathcal{S}_h(X) \subseteq \boldsymbol{G}(h\text{-}1,n)$ of secant (h-1)-planes of X is an irreducible variety of dimension $h \cdot dim(X)$. The secant variety $Sec_h(X) \subseteq \mathbb{P}^n$ is irreducible of dimension at most $h \cdot dim(X)+(h\text{-}1)$ with equality holding if and only if the generic point lying on a secant (h-1)-planes of X lies on only a finite number of secant (h-1)-planes of X.

Proof: The map $\varphi_h : \underbrace{X \times ... \times X}_{h-times} \dashrightarrow \boldsymbol{G}(h\text{-}1,n)$ is generically finite because the fibre over a point $H \in \boldsymbol{G}(h\text{-}1,n)$ has positive dimension if and only if $H \subseteq X$. So

$$dim(\mathcal{S}_h(X)) = dim(\underbrace{X \times ... \times X}_{h-times}) = h \cdot dim(X).$$

It is the image of an irreducible variety via a rational map so it is irreducible.
Now we consider the incidence correspondence

$$\mathcal{I} = \{(p,H) \mid p \in H\} \subseteq \mathbb{P}^n \times \mathcal{S}_h(X) \subseteq \mathbb{P}^n \times \boldsymbol{G}(k,n).$$

The map $\Pi_2 : \mathcal{I} \to \mathcal{S}_h(X)$ is surjective and its fibres have dimension $h\text{-}1$. The image of the first projection $\Pi_1 : \mathcal{I} \to \mathbb{P}^n$ is the variety $Sec_h(X)$. We have

$$dim(\mathcal{I}) - dim(\Pi_1^{-1}(p)) = dim(Sec_h(X)) \text{ and } dim(Sec_h(X)) \leq dim(\mathcal{I}).$$

On the other hand we have

$$dim(\mathcal{I})\text{-}dim(\Pi_2^{-1}(H)) = dim(Sec_h(X)) \text{ and}$$
$$dim(\mathcal{I}) = h \cdot dim(X)+(h\text{-}1).$$

We conclude that

$$dim(Sec_h(X)) \leq h \cdot dim(X)+(h\text{-}1).$$

The equality $dim(Sec_h(X)) = h \cdot dim(X)+(h\text{-}1)$ holds if and only if the fibre of the first projection is finite. In other words if and only if the generic point lying on a secant *(h-1)*-plane lies on only a finite number of secant *(h-1)*-planes of X. Finally Π_2 is surjective with all fibres irreducible so $\mathcal{I}$ is irreducible and via the first projection Π_1 the variety $Sec_h(X)$ is also irreducible. □

Now we give some examples.

Example 6. *If $X \subseteq \mathbb{P}^n$ is an irreducible curve not contained in any plane then*

$$dim(Sec_2(X)) = 3.$$

We can project X in $\mathbb{P}^3$. Now the projection of X is $\mathbb{P}^2$ from a generic point of $\mathbb{P}^3$ is an irreducible curve with a finite number of nodal singularities. So there is a finite number of secant lines of X passing through p. By the proposition we conclude that

$$dim(Sec_2(X)) = 2 \cdot dim(X) + 1 = 3.$$

EXAMPLE 7. *Let $X = \nu(\mathbb{P}^2) \subseteq \mathbb{P}^5$ be the Veronese surface. Let $u \in \mathbb{P}^5$ be a point lying on a secant line to X. We write the secant line as $<\nu(p),\nu(q)>$ with $p,q \in \mathbb{P}^2$. The line $L = <p,q> \subseteq \mathbb{P}^2$ is carried under the Veronese embedding ν in a conic $C \subseteq X$. Since $u \in <\nu(p),\nu(q)>$ and $\nu(p),\nu(q) \in C$ the point u lies on the plane H spanned by C. All lines passing through u and contained in H intersect C in two points and so are secant lines of X. We see that the generic point of $\mathbb{P}^5$ lying on a secant line of X lies on a 1-dimensional family of secant lines of X, so $dim(Sec_2(X)) \leq 4$. If it will be $dim(Sec_2(X)) \leq 3$ then the cones $<v,Sec_2(X)>$ with vertex a point $v \in Sec_2(X)$ will coincide, a contradiction. We conclude that $dim(Sec_2(X)) = 4$.*
There is another way to see this fact. The points of $Sec_2(X)$ are the conics which can be written as sum of two squares, i.e. the conics of rank equal to 1 or 2. So we can describe $Sec_2(X) \subseteq \mathbb{P}^5$ as the determinantal variety defined by

$$det\begin{pmatrix} X_0 & X_3 & X_4 \\ X_3 & X_1 & X_5 \\ X_4 & X_5 & X_2 \end{pmatrix} = 0$$

That is a cubic hypersurface in $\mathbb{P}^5$.

DEFINITION 4. *Let $X \subseteq \mathbb{P}^n$ be an irreducible nondegenerate variety. We say that X has defective secant variety if $dim(Sec_h(X)) < min\{h \cdot dim(X) + (h-1), n\}$.*
The difference

$$\delta(X) = h \cdot dim(X) + (h-1) - dim(Sec_h(X))$$

is called the defectivity of X.

EXAMPLE 8. *Let $\boldsymbol{G} = \boldsymbol{G}(1,n) \subseteq \mathbb{P}^{n(n+1)}$ be the Grassmannians of lines of $\mathbb{P}^n$ and let $p \in Sec_2(\boldsymbol{G})$ be a point. Then $p \in <u,v> = L \in \mathbb{P}^{n(n+1)}$ secant line of $\boldsymbol{G}$. The points u,v represent two lines R_1,R_2 in $\mathbb{P}^n$. Now two general lines span a 3-plane H. The lines contained in H are parametrized by the Grassmannian $\boldsymbol{G}(1,3) \subseteq \boldsymbol{G}(1,n)$.*
Now $dim(\boldsymbol{G}(1,3)) = 4$ and $\boldsymbol{G}(1,3)$ spans a 5-plane $E \subseteq \mathbb{P}^{n(n+1)}$. All lines $L \subseteq E$ and passing through p intersect $\boldsymbol{G}(1,3)$ in two points because $deg(\boldsymbol{G}(1,3)) = 2$. We see that any point $p \in Sec_2(\boldsymbol{G})$ lies on a 4-dimensional family of secant lines of $\boldsymbol{G}$. We conclude that

$$dim(Sec_2(\boldsymbol{G})) = 2 \cdot dim(\boldsymbol{G}) + 1 - 4 = 2 \cdot 2(n-1) + 1 - 4 = 4n - 7 \text{ and}$$
$$\delta(\boldsymbol{G}) = 2 \cdot dim(\boldsymbol{G}) + 1 - 4n + 7 = 4.$$

1.3.1 Veronese Varieties

The sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$, whose sections $H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d)) \cong k[x_0,\dots,x_n]_d$ are the homogeneous polynomials of degree d on $\mathbb{P}^n$, is a very ample sheaf on $\mathbb{P}^n$. A basis of the k-vector space $H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d))$ is given by the monomials of degree d in the $n+1$ variables $x_0,\dots,x_n$. This monomials are $\binom{n+d}{n}$ and we denote it by $\mathcal{M}_0,\dots,\mathcal{M}_N$, where

$$N = \binom{n+d}{n} - 1.$$

So the sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$ induces an embedding

$$\nu_d\colon \mathbb{P}^n \to \mathbb{P}^N, \text{ defined by } P \mapsto [\mathcal{M}_0(P):\dots:\mathcal{M}_N(P)]$$

called the *d-Veronese embedding.* Its image $V^n_{d^n} = \nu_d(\mathbb{P}^n)$ is a irreducible nonsingular variety in $\mathbb{P}^N$. A hyperplane section of $V^n_{d^n}$ corresponds via the embedding ν_d to a hypersurface of degree d in $\mathbb{P}^n$. In order to determine the degree of $V^n_{d^n}$ we have to intersect it whit n hyperplanes. In $\mathbb{P}^n$ we are intersecting n hypersurfaces of degree d and by Bezout's theorem the hypersurfaces intersect in d^n points counted with multiplicity. Via ν_d we find d^n points in $\mathbb{P}^N$. We conclude that

$$deg(V^n_{d^n}) = d^n.$$

The variety $V^n_{d^n}$ is called the *Veronese variety* of dimension n in $\mathbb{P}^N$.
The expected dimension of the h-secant variety of the Veronese variety $V^n_{d^n}$ is

$$dim(Sec_h(V^n_{d^n})) = h\cdot dim(V^n_{d^n})+(h-1) = h\cdot n+(h-1).$$

Note that a polynomial of degree r on $V^n_{d^n}$ corresponds to a polynomial of degree dr on $\mathbb{P}^n$. Then the Hilbert polynomial of $V^n_{d^n}$ is given by

$$h_{V^n_{d^n}}(r) = \binom{dr+n}{n} = \frac{(dr+n)\dots(dr+1)}{n!} = \frac{d^n}{n!}r^n + \dots$$

and we have again $dim(V^n_{d^n}) = n$ and $deg(V^n_{d^n}) = n!\frac{d^n}{n!} = d^n$.

REMARK 2. *Combining the Segre and the Veronese embeddings we can define the Segre-Veronese embedding*

$$\mathcal{SV} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N,$$

with $N = \binom{d+n}{n}\binom{h+m}{m} - 1$, *using the sheaf* $\mathcal{O}_{\mathbb{P}^n}(d)$ *on* $\mathbb{P}^n$ *and the sheaf* $\mathcal{O}_{\mathbb{P}^m}(h)$ *on* $\mathbb{P}^m$. *Let* $X = \mathcal{SV}(\mathbb{P}^n \times \mathbb{P}^m)$ *be the Segre-Veronese variety.*
A homogeneous polynomial of degree r *on* X *corresponds to a bihomogeneous polynomial of bidegree* (dr, hr) *on* $\mathbb{P}^n \times \mathbb{P}^m$. *Then the Hilbert polynomial of* X *is given by*

$$h_X(r) = \binom{dr+n}{n}\binom{hr+m}{m} = \frac{d^n h^m}{n!m!}r^{n+m} + \dots$$

We have that $dim(X) = n + m$ *and* $deg(X) = \frac{(n+m)!}{n!m!}d^n h^m = \binom{n+m}{n}d^n h^m$.

1.4 The Canonical Sheaf

Let A be a commutative ring, let B be an A-algebra and let M be a B-module.

Definition 5. *An A-derivation of B into M is a map δ:$B \to M$ such that*

1. *$\delta(b+b') = \delta(b) + \delta(b')$,*
2. *$\delta(b \cdot b') = b \cdot \delta(b') + b' \cdot \delta(b)$,*
3. *$\delta(a) = 0$ for all $a \in A$.*

The module of relative differential forms of B over A is a B-module $\Omega_{B/A}$ with an A-derivation δ:$B \to \Omega_{B/A}$ such that: for any B-module M and for any A-derivation δ':$B \to M$, the exists a unique B-module homomorphism f:$\Omega_{B/A} \to M$ such that $\delta' = f \circ \delta$.

Let f:$X \to Y$ be a morphism of schemes and let Δ:$X \to X \times_Y X$ be the diagonal morphism. The image $\Delta(X)$ is a locally closed subscheme in $X \times_Y X$ i.e. $\Delta(X)$ is a closed subscheme of an open subset $\mathcal{U}$ of $X \times_Y X$. So we can consider the sheaf of ideals $\mathcal{I}$ of $\Delta(X)$ in $\mathcal{U}$.
The sheaf $\mathcal{I}/\mathcal{I}^2$ has a structure of $\mathcal{O}_{\Delta(X)}$-module. Now we consider the sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ obtained by pull-back of $\mathcal{I}/\mathcal{I}^2$ via Δ. Since Δ induces an isomorphism of X to $\Delta(X)$, $\Omega_{X/Y}$ has a structure of $\mathcal{O}_X$-module.
Furthermore $\Delta(X)$ is a closed subscheme of $\mathcal{U}$ and so the sheaf $\mathcal{I}$ is a quasi-coherent sheaf of ideals on $\mathcal{U}$. Then also $\mathcal{I}/\mathcal{I}^2$ is a quasi-coherent sheaf of ideals on $\mathcal{U}$ and $\Omega_{X/Y}$ is a quasi-coherent sheaf on X.
Finally if Y is noetherian and f is a morphism of finite type then $X \times_Y X$ is also noetherian, so $\mathcal{I}$ and $\mathcal{I}/\mathcal{I}^2$ are coherent on $\mathcal{U}$ and $\Omega_{X/Y}$ is coherent on X.

Definition 6. *The sheaf of $\mathcal{O}_X$-module $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ defined above is the the sheaf of relative differentials of X over Y.*

If $\mathcal{U} = Spec(A)$ is an open affine subset of Y and $\mathcal{V} = Spec(B)$ is an open affine subset of X such that $f(\mathcal{V}) \subseteq \mathcal{U}$, then $\mathcal{V} \times_{\mathcal{U}} \mathcal{V}$ is an open affine subset of $X \times_Y X$ isomorphic to $Spec(B \otimes_A B)$ and $\Delta(X) \cap (\mathcal{V} \times_{\mathcal{U}} \mathcal{V})$ is the closed subscheme defined by the kernel of the diagonal morphism $B \otimes_A B \to B$. So $\mathcal{I}/\mathcal{I}^2$ is the sheaf associated to the module I/I^2 and $\Omega_{X/Y}$ is the sheaf associated to the module $\Omega_{B/A}$ of relative differential forms of B over A. This gives the connections between the sheaf $\Omega_{X/Y}$ and the sheaf associated to the module $\Omega_{B/A}$. Now we will use this connection to prove some propositions.

Proposition 5. *Let f:$X \to Y$ be a morphism, let g:$Y' \to Y$ be another morphism, and let f':$X' = X \times_Y Y' \to Y'$ be obtained by base extension. Then $\Omega_{X'/Y'} \cong g'^*(\Omega_{X/Y})$, where g':$X' \to X$ is the first projection.*

Proof: We can assume that the schemes are affine. Let $X = Spec(A')$, $Y = Spec(A)$ and $Y' = Spec(B)$, then $X' = X \times_Y Y' = Spec(B \otimes_A A')$. We have two morphism of rings $\overline{f} : A \to A'$, $\overline{g} : A \to B$, so A' and B are two A-algebras. Then $\Omega_{B \otimes_A A'/A'} \cong \Omega_{B/A} \otimes_B (B \otimes_A A')$ (Matsumura [2, p.186]). Passing to the shaves of differentials we have $\Omega_{X'/Y'} \cong g'^*(\Omega_{X/Y})$. □

PROPOSITION 6. *Let f:X→Y and g:Y→Z be morphisms of schemes. Then there is an exact sequence of shaves on X,*

$$f^*\Omega_{Y/Z} \rightarrow \Omega_{X/Z} \rightarrow \Omega_{X/Y} \mapsto 0.$$

Proof: We can assume that the schemes are affine. Let $X = Spec(A)$, $Y = Spec(B)$ and $Z = Spec(C)$. We have the morphisms of rings $\overline{f} : B \rightarrow A$, $\overline{g} : C \rightarrow B$ and $C \xrightarrow{\overline{g}} B \xrightarrow{\overline{f}} A$. So we have an exact sequence

$$\Omega_{B/C} \otimes_B A \rightarrow \Omega_{A/C} \rightarrow \Omega_{A/B} \mapsto 0$$

(Matsumura 2, [Th.57 p.186]). Passing at the shaves of differentials we have the exact sequence of the proposition. □

PROPOSITION 7. *Let f:X→Y be a morphism of schemes and let Z be a closed subscheme of X, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of shaves on Z,*

$$\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/Y} \otimes \mathcal{O}_Z \rightarrow \Omega_{Z/Y} \mapsto 0.$$

Proof: We can assume that the schemes are affine. Let $X = Spec(A)$, $Y = Spec(B)$ and $Z = Spec(A/I)$, where I is an ideal of A. The morphism of rings $\overline{f} : B \rightarrow A$ induces on A a structure of B-algebra. We have an exact sequence

$$I/I^2 \rightarrow \Omega_{A/B} \otimes_A \frac{A}{I} \rightarrow \Omega_{\frac{A}{I}/A} \mapsto 0$$

(Matsumura 2, [Th.58 p.187]). Passing at the shaves of differentials we have the exact sequence of the proposition. □

In what follow we use the notion of sheaf of differential on an abstract nonsingular variety to define the canonical sheaf and the shaves related to this.

DEFINITION 7. *An abstract variety X over an algebraically closed field k is nonsingular if all its local rings are regular local rings.*

The following theorem connects the concept of nonsingularity to the sheaf of differentials

THEOREM 5. *Let X be an irreducible separated scheme of finite type over an algebraically closed field k. Then $\Omega_{X/k}$ is a locally free sheaf of rank $n = dim(X)$ if and only if X is a nonsingular variety over k.*

Proof: Hartshorne [Th. 8.15 p.177].

THEOREM 6. *Let X be a nonsingular variety over k. Let $Y \subseteq X$ be an irreducible subvariety defined by the sheaf of ideals $\mathcal{I}$. Then Y is nonsingular if and only if*

1. *$\Omega_{Y/k}$ is locally free,*

2. *the sequence of proposition 7 is exact on the left also:*

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \mapsto 0.$$

Furthermore, in this case, $\mathcal{I}$ is locally generated by codim(Y,X) elements, and $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank codim(Y,X) on X.

Proof: Hartshorne [Th. 8.17 p.178].

THEOREM 7. *Let A be a ring, let $Y = Spec(A)$, and let $X = \mathbb{P}^n_A$. Then there is an exact sequence of shaves on X,*

$$0 \mapsto \Omega_{X/Y} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \mapsto 0$$

Proof: Hartshorne [Th. 8.13 p.176].

DEFINITION 8. *Let X be a nonsingular variety of dimension n over k. The tangent sheaf of X is the dual of the sheaf of differentials $\Omega_{X/k}$,*

$$\mathcal{T}_X = \mathcal{H}om(\Omega_{X/k},\mathcal{O}_X)$$

We have seen that $\Omega_{X/k}$ is a locally free of rank n so $\mathcal{T}_X$ is also locally free of rank n. The canonical sheaf of X is defined to be the n-th wedge product of the sheaf of differentials

$$\omega_X = \bigwedge^n \Omega_{X/k}$$

The canonical sheaf has rank $\binom{n}{n} = 1$ so it is an invertible sheaf. The associated divisor on X is called the canonical divisor of X and denoted by K_X.

After this definition we observe that $\Omega_{X/k}$ is the dual of the tangent sheaf and it is also called the *cotangent sheaf.* The sheaf $\mathcal{T}_X$ is locally free of rank n and so we can consider the associated vector bundle T_X, that is the *tangent bundle of* X, the fibre of T_X in a point $x \in X$ is the tangent space T_xX of X in x. In the same way we have a vector bundle of rank n associated to the sheaf $\Omega_{X/k}$, that is the *cotangent bundle* denoted by $(T_X\check{)}$.
Finally we observe that the dual of the canonical sheaf $\omega_X\check{} = \bigwedge^n \mathcal{T}_X$ is an invertible sheaf, called the *anticanonical sheaf* of X. The associated divisor is the *anticanonical divisor* of X and denoted by -K_X.
Since all these shaves are defined intrinsically, any numbers defined from them, are invariants of X up to isomorphism.

DEFINITION 9. *If X is a projective, nonsingular variety of dimension n, we define the geometric genus of X as the dimension of the k-vector space of sections of the canonical sheaf*

$$p_g = dim_k H^0(X,\omega_X)$$

By Serre duality theorem we have $p_g = dim_k H^0(X,\omega_X) = dim_k H^n(X,\mathcal{O}_X)$.

We study the tangent and the canonical sheaf for a nonsingular subvariety Y of X.

DEFINITION 10. *Let Y be a nonsingular subvariety of a nonsingular variety X over k, and let $\mathcal{I}$ its ideal sheaf. The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf of Y in X. Its dual $\mathcal{N}_{Y/X} = \mathcal{H}om(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_Y)$ is called the normal sheaf of Y in X. It is a locally free sheaf of rank codim(Y,X).*

Now we consider the exact sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/k}\otimes\mathcal{O}_Y \to \Omega_{Y/k} \mapsto 0.$$

If $dim(X) = n$ and $dim(Y) = r$ we have $rank(\Omega_{X/k}\otimes\mathcal{O}_Y) = n$ and $rank(\Omega_{Y/k}) = r$, so $rank(\mathcal{I}/\mathcal{I}^2) = n\text{-}r$ and $rank(\mathcal{N}_{Y/X}) = n\text{-}r = codim(Y,X)$.
Dualizing the exact sequence above we obtain

$$0 \mapsto \mathcal{T}_Y \to \mathcal{T}_X\otimes\mathcal{O}_Y \to \mathcal{N}_{Y/X} \mapsto 0.$$

We see that $\mathcal{N}_{Y/X} = \frac{\mathcal{T}_X\otimes\mathcal{O}_Y}{\mathcal{T}_Y}$, and we recover the usual geometrical interpretation of the normal sheaf as the sheaf of elements in the tangent of X modulo the elements in the tangent of Y.

Proposition 8. *Let Y be a nonsingular subvariety of a nonsingular variety X, with $codim(Y,X) = h$. The $\omega_Y = \omega_X\otimes\bigwedge^h\mathcal{N}_{Y/X}$. In the case $h = 1$ we can consider Y as a divisor on X and let $\mathcal{O}_X(Y)$ the associated invertible sheaf on X. Then $\omega_Y = \omega_X\otimes\mathcal{O}_X(Y)\otimes\mathcal{O}_Y$.*

Proof: We have $rank(\mathcal{I}/\mathcal{I}^2) = h$, $rank(\Omega_{X/k}\otimes\mathcal{O}_Y) = n$ and $rank(\Omega_{Y/k}) = n\text{-}h$. From the exact sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/k}\otimes\mathcal{O}_Y \to \Omega_{Y/k} \mapsto 0$$

taking the highest exterior powers we obtain

$$\textstyle\bigwedge^n\Omega_{X/k}\otimes\mathcal{O}_Y \cong \bigwedge^h\mathcal{I}/\mathcal{I}^2\otimes\bigwedge^{n-h}\Omega_{Y/k}.$$

Daualizing and considering the fact that the formation of highest exterior powers commutes with taking the dual sheaf we find $\omega_X\check{} \cong \omega_Y\check{}\otimes\bigwedge^h\mathcal{N}_{Y/X}$. Tensorizing by $\omega_X\otimes\omega_Y$ we have $\omega_Y = \omega_X\otimes\bigwedge^h\mathcal{N}_{Y/X}$.
If $h = 1$ we have $\mathcal{I}_Y \cong \mathcal{O}_X(Y)\check{}$ so $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X(Y)\check{}\otimes\mathcal{O}_Y$ and $\mathcal{N}_{Y/X} \cong \mathcal{O}_X(Y)\otimes\mathcal{O}_Y$. For the previous result with $r = 1$ we have $\omega_Y = \omega_X\otimes\mathcal{O}_X(Y)\otimes\mathcal{O}_Y$.

□

As a special case we will prove the adjunction formula for a nonsingular curve on a surface.

Proposition 9. *(Adjunction Formula) Let C be a nonsingular curve of genus g on a surface X and let K_X be the canonical divisor of X, then*

$$2g - 2 = C\cdot(C + K_X)$$

Proof: We have $\omega_C = \omega_X\otimes\mathcal{O}_X(C)\otimes\mathcal{O}_C$ and $deg(\omega_C) = 2g - 2$. But we also have $deg(\omega_X\otimes\mathcal{O}_X(C)\otimes\mathcal{O}_C) = C\cdot(C + K_X)$. □

Let $X = \mathbb{P}^n_k$. Dualizing the exact sequence

$$0 \mapsto \Omega_{X/k} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \mapsto 0$$

we have

$$0 \mapsto \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to \mathcal{T}_X \mapsto 0.$$

Now $rank(\Omega_{X/k}) = n$, $rank(\mathcal{O}_X(-1)^{n+1}) = n+1$ and $rank(\mathcal{O}_X) = 1$ so taking the highest exterior power in the first sequence we find

$$\bigwedge^{n+1}\mathcal{O}_X(-1)^{n+1} \cong \bigwedge^{n}\Omega_{X/k} \otimes \mathcal{O}_X$$

then we have $K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

REMARK 3. *We can compute the canonical sheaf of $\mathbb{P}^n$ directly. We consider the differential forms on $\mathbb{P}^n$.*
Let $[x_0, ..., x_n]$ be the homogeneous coordinated on $\mathbb{P}^n$, and let

$$\mathcal{U}_0 = \{[x_0, ..., x_n] | x_0 \neq 0\} \cong \mathbb{A}^n.$$

On $\mathcal{U}_0$ we have the coordinates $(y_1, ..., y_n)$ where $y_i = x_i/x_0$, and a basis of the differential forms is $dy_1 \wedge ... \wedge dy_n$.
Now we consider the open subset $\mathcal{U}_1$, whit coordinates $(z_1, ..., z_n)$ where $z_i = x_i/x_1$. We note that $y_1 = x_1/x_0 = 1/z_1$ and $y_k = \frac{x_k}{x_1}\frac{x_1}{x_0} = z_k y_1$ for any $k \geq 2$. By differentiation we have

$$dy_1 = -\tfrac{1}{z_1^2}dz_1 \text{ and } dy_k = z_k dy_1 + dz_k y_1 = -\tfrac{z_k}{z_1^2}dz_1 + \tfrac{1}{z_1}dz_k \text{ for any } k \geq 2.$$

Then

$$dy_1 \wedge ... \wedge dy_n = -\tfrac{1}{z_1^2}dz_1 \wedge (-\tfrac{z_2}{z_1^2}dz_1 + \tfrac{1}{z_1}dz_2) ... \wedge (-\tfrac{z_n}{z_1^2}dz_1 + \tfrac{1}{z_1}dz_n).$$

Since $dz_1 \wedge dz_1 = 0$ we have

$$dy_1 \wedge ... \wedge dy_n = -\tfrac{1}{z_1^2}dz_1 \wedge \tfrac{1}{z_1}dz_2 \wedge ... \wedge \tfrac{1}{z_1}dz_n = -\tfrac{1}{z_1^{n+1}}dz_1 \wedge dz_2 \wedge ... \wedge dz_n.$$

Since $z_1 = x_0/x_1$ we see that the canonical divisor of $\mathbb{P}^n$ is given by $K_{\mathbb{P}^n} = -(n+1)H_0$, where H_0 is the hyperplane defined by $x_0 = 0$. Then we have again that the canonical sheaf of $\mathbb{P}^n$ is $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Now prove a proposition that will be very useful.

PROPOSITION 10. *Let Y be a closed subscheme of $\mathbb{P}^n_k$.*

1. *If Y is a nonsingular hypersurface of degree d then $\omega_Y \cong \mathcal{O}_Y(d\text{-}n\text{-}1)$.*

2. *If $Y = H_1 \cap ... \cap H_r$ is a nonsingular complete intersection of hypersurfaces H_i of degree $deg(H_i) = d_i$ then $\omega_Y \cong \mathcal{O}_Y(\sum d_i\text{-}n\text{-}1)$.*

3. *If Y is a nonsingular hypersurface of degree d then $p_g(Y) = \binom{d-1}{n}$. In particular, if Y is a nonsingular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d\text{-}1)(d\text{-}2)$.*

4. *If Y is a nonsingular curve in $\mathbb{P}^3_k$, which is a complete intersection of nonsingular surfaces of degree d,e, then $p_g(Y) = \frac{1}{2}d{\cdot}e(d+e\text{-}4)+1$.*

Proof: 1) We know that $\omega_Y = \omega_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{O}_Y$. If Y is a hypersurface of degree d we have $\omega_X \cong \mathcal{O}_X(-n-1)$ and $\mathcal{O}_X(Y) \cong \mathcal{O}_X(d)$. We find $\omega_Y \cong \mathcal{O}_X(-n-1+d) \otimes \mathcal{O}_Y \cong \mathcal{O}_Y(d-n-1)$.
2) We proceed by induction on r. For $Y = H_1$ we have $\omega_Y \cong \mathcal{O}_Y(d_1-n-1)$ by 1).
For the complete intersection $Z = H_1 \cap \ldots \cap H_{r-1}$ we have, by induction hypothesis $\omega_Z \cong \mathcal{O}_Z(d_1+\ldots+d_{r-1}-n-1)$.
Now Y is a divisor of Z and it is nonsingular, we have $\omega_Y = \omega_Z \otimes \mathcal{O}_Z(Y) \otimes \mathcal{O}_Y$. We note that Z is determined on Y by a hypersurface H_r of degree d_r so $\mathcal{O}_Z(Y) \cong \mathcal{O}_Z(d_r)$. We conclude that

$$\omega_Y \cong \mathcal{O}_Z(d_1+\ldots+d_{r-1}-n-1) \otimes \mathcal{O}_Z(d_r) \otimes \mathcal{O}_Y \cong \mathcal{O}_Y(d_1+\ldots+d_r-n-1).$$

3) If $Y \subseteq \mathbb{P}^n_k$ is an hypersurface of degree d then the natural map

$$H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(Y, \mathcal{O}_Y(d))$$

is a bijection. From $\omega_Y \cong \mathcal{O}_Y(d-n-1)$ we have

$$p_g(Y) = dim_k H^0(Y, \omega_Y) = dim_k\ H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) = \binom{d-1}{n}.$$

4) We have $\omega_Y \cong \mathcal{O}_Y(d+e-3-1) \cong \mathcal{O}_Y(d+e-4)$. The degree of the canonical divisor is $deg(K_Y) = deg(Y)(d+e-4) = d \cdot e(d+e-4)$, but we also have $deg(K_Y) = 2g - 2$. Equaling the two expressions we obtain

$$p_g(Y) = g = \frac{1}{2} d \cdot e(d+e-4)+1.$$

□

We consider the special case of the Grassmannian $\mathbf{G}(h,n)$ parametrizing the h-planes of $\mathbb{P}^n$. We have the universal exact sequence

$$0 \mapsto \mathcal{S}_G \longrightarrow \mathcal{E}_G \longrightarrow \mathcal{Q}_G \mapsto 0$$

We recall that the tangent sheaf of $\mathbf{G}(h,n)$ is $\mathcal{T}\mathbf{G}(h,n) \cong \check{\mathcal{S}_G} \otimes \mathcal{Q}_G$, and we take the dual of the exact sequence tensorized by $\mathcal{Q}_G$,

$$0 \mapsto \check{\mathcal{Q}_G} \otimes \mathcal{Q}_G \longrightarrow \check{\mathcal{E}_G} \otimes \mathcal{Q}_G \longrightarrow \mathcal{T}\,\boldsymbol{G}(h,n) \mapsto 0$$

We recall that if $\mathcal{F}$ is a locally free sheaf of rank r the multiplication map

$$\bigwedge^t \mathcal{F} \otimes \bigwedge^{r-t} \mathcal{F} \longrightarrow \bigwedge^r \mathcal{F}$$

is a perfect pairing for any t, i.e. it induces an isomorphism of $\bigwedge^t \mathcal{F}$ with $(\bigwedge^{r-t} \mathcal{F})\check{} \otimes \bigwedge^r \mathcal{F}$.
Now $rank(\check{\mathcal{Q}_G} \otimes \mathcal{Q}_G) = (n-h)^2$, $rank(\check{\mathcal{E}_G} \otimes \mathcal{Q}_G) = (n+1)(n-h)$, $rank(\mathcal{T}\mathbf{G}(h,n)) = (h+1)(n-h)$, and taking the highest exterior powers we have

$$\bigwedge^{(n+1)(n-h)} \check{\mathcal{E}_G} \otimes \mathcal{Q}_G \cong \bigwedge^{(n-h)^2} (\check{\mathcal{Q}_G} \otimes \mathcal{Q}_G) \otimes \bigwedge^{(h+1)(n-h)} \mathcal{T}\mathbf{G}(h,n).$$

Taking the highest exterior powers in the universal exact sequence we have $\bigwedge^{n+1} \mathcal{E}_G \cong \bigwedge^{h+1} \mathcal{S}_G \otimes \bigwedge^{n-h} \mathcal{Q}_G$, and since the determinant of $\mathcal{S}_G$ is the invertible sheaf giving the *Plücker embedding* we write

$$\bigwedge^{h+1}\mathcal{S}_G \cong \bigwedge^{n-h}\check{\mathcal{Q}_G} = \mathcal{O}_G(1).$$

Then we get $\mathcal{O}_G(n+1) \cong \mathcal{O}_G(1) \otimes \mathcal{O}_G(-1) \otimes \check{\omega_G}$. We conclude that the anticanonical and the canonical shaves of $\mathbf{G}(h,n)$ are respectively

$$\check{\omega_G} \cong \mathcal{O}_G(n+1) \text{ and } \omega_G \cong \mathcal{O}_G(-n-1).$$

REMARK 4. *We think to $\mathbb{P}^n$ as the Grassmannian $\boldsymbol{G}(0,n)$. Then the universal bundle $\mathcal{S}_G$ becomes the tautological bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$. Then the universal sequence becomes*

$$0 \mapsto \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n}(-1) \mapsto 0$$

tensorizing by $\mathcal{O}_{\mathbb{P}^n}(1)$ and taking the dual we recover the Euler sequence

$$0 \mapsto \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \mapsto 0.$$

In particular from $\omega_G \cong \mathcal{O}_G(-n-1)$ we recover $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

1.5 Surfaces

In this section we briefly describe the *Enriques-Kodaira* classification of compact complex surfaces. We begin listing the most important objects for the classification. Let X be a projective variety, we consider

- The canonical sheaf ω_X of holomorphic *2*-forms.
- The plurigenera $P_n = dim_k H^0(nK)$ for $n \geq 1$ that are invariant under blowing-up.
- The Hodge invariants $h^{ij} = dim_k H^j(\Omega^i)$ where Ω^i is the sheaf of regular i-forms. Since $dim(X) = 2$ on a surfaces X we have only

$$h^{0,0}, h^{0,1}, h^{0,2}, h^{1,0}, h^{1,1}, h^{1,2}, h^{2,0}, h^{2,1}, h^{2,2}.$$

The Hodge invariants are arranged in the Hodge diamond

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

By Serre duality we have

$$h^{ij} = dim_k H^j(\Omega^i) = dim_k H^{2-i}(\Omega^{2-j}) = h_{2-i,2-j} \text{ and}$$
$$h^{0,0} = h^{2,2} = dim_k H^0(\Omega^0) = 1.$$

If the surface is algebraic we have $h^{ji} = h^{ij}$ and we have only three independent Hodge invariants.
The invariant $q = h^{0,1}$ is called the irregularity of X, $p_a = h^{0,2} - h^{0,1}$ is the arithmetic genus of X and $p_g = h^{0,2} = h^{2,0}$ is the geometric genus of X. We note that $q = p_g - p_a$.

1.5.1 Kodaira Dimension

Let X be a projective variety over a field k. We consider the canonical divisor K of X and the linear systems $|nK|$ for any $n \geq 1$.
The *Kodaira dimension* $\mathcal{K}(X)$ of X is the largest dimension of the image of X in $\mathbb{P}^N$ under the rational map determined by the linear system $|nK|$ for some $n \geq 1$ or $\mathcal{K}(X) = -1$ if $|nK| = \varnothing$ for all $n \geq 1$.
It is known that $-1 \leq \mathcal{K} \leq n$ for a variety of dimension n.

1.5.2 Surfaces Classification

Any surface is birational to a nonsingular surface. A nonsingular surface is called minimal if it cannot be obtained from another nonsingular surface by blowing up a point. Every surface X is birational to a minimal nonsingular surface, and this minimal nonsingular surface is unique if X has Kodaira dimension at least 0 or is not algebraic. Now we classify the nonsingular surfaces using Kodaira dimension.
One can prove the following three results

1. $\mathcal{K} = -1 \Leftrightarrow |12K| = \varnothing \Leftrightarrow X$ is either *rational* or *ruled.* And *Castelnuovo* proved that X is rational if and only if $p_a = P_2 = 0$.

2. A surface with $\mathcal{K} = 1$ is an *elliptic surface*, which is a surface X with a morphism $\pi : X \to C$ to a curve C such that almost all fibres of π are nonsingular elliptic curves (here we are assuming $char(k) \neq 2,3$).

3. $\mathcal{K} = 2$ if and only if $|nK|$ determines a birational map of X onto its image in $\mathbb{P}^N$ for some $n > 0$. These are called *surfaces of general type.*

It remains the case $\mathcal{K} = 0$. One can prove that $\mathcal{K} = 0 \Leftrightarrow 12K = 0$. A surface in this class must be one of the following (assume $char(k) \neq 2,3$).

- A $K3$ surfaces, which is defined as a surface with $K = 0$ and irregularity $q = 0$. These have $p_a = p_g = 1$.

- An *Enriques surface*, which has $p_a = p_g = 0$ and $2K = 0$.

- A *two-dimensional abelian variety*, which has $p_a = -1$ and $p_g = 1$.

- A *hyperelliptic surface*, which is a surface fibred over $\mathbb{P}^1$ by a pencil of hyperelliptic curves.

We resume these facts in the following table

$h^{1,0}$	$h^{2,0} = p_g$	$h^{1,1}$	p_a	q	**Type**
0	0	10	0	0	*Enriques*
1	0	2	-1	1	*Hyperelliptic*
0	1	20	1	0	*K3*
2	1	4	-1	2	*2-dimensional Abelian Variety*

Now we consider $K3$ surfaces. The Hodge diamond of a $K3$ surface is in the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

An example of *K3* surface are the smooth quartic surfaces in $\mathbb{P}^3$. Let X $\subseteq \mathbb{P}^3$ a smooth surface with $deg(X) = 4$. For the canonical sheaf we have

$$\omega_X = \mathcal{O}_X(4\text{-}3\text{-}1) = \mathcal{O}_X \text{ so } K = 0 \text{ and } p_g = dim_k H^0(\omega_X) = 1.$$

We compute

$$h^{0,2} = h^{2,0} = dim_k H^0(\Omega^2) = dim_k H^0(\omega_X) = 1$$
$$h^{0,1} = h^{1,0} = dim_k H^1(\Omega^0) = dim_k H^1(\mathcal{O}_X) = 0.$$

Then $p_a = h^{0,2} - h^{0,1} = 1$ and $q = p_g - p_a = 0$. We see that X is a *K3* surface.

1.5.3 Fano Varieties

We give the definition of Fano variety and state some property of these varieties omitting the proofs. For a deeper understanding of Fano varieties see *Parshin-Shafarevich Algebraic Geometry V.*

DEFINITION 11. *A Fano variety is a projective variety X whose anticanonical sheaf ω_X is ample.*

Fano varieties of dimension *1* and *2* are all rational and Fano varieties of dimension *2* are called *Del Pezzo surfaces.* Any Del Pezzo surface can be obtained by blowing-up $\mathbb{P}^2$ with the linear system of plane cubics passing through $r = 0,1,\ldots,6$. By blowing-up r points we obtain a Del Pezzo surface of degree $9\text{-}r$ in $\mathbb{P}^{9-3}$. For example if $r = 6$ we get a smooth cubic surface in $\mathbb{P}^3$.
Fano varieties have all Kodaira dimension *-1*.

1.6 Determinantal Varieties

A matrix $A \in M_{m,n}(k)$ defines a vector in the k-vector space k^{nm} and a point in the associated projective space $\mathbb{P}^{mn-1}$. For each positive integer h let M_h be the subset of matrices of rank h or less. This is just the common zero locus of $(h+1) \times (h+1)$ minor determinants, which are homogeneous polynomials of degree $h+1$ on the projective space $\mathbb{P}^{mn-1}$. Then this subset of matrices is a projective variety. We introduce the incidence correspondence

$$\mathcal{I} = \{(A, \Lambda) | \Lambda \subseteq Ker(A)\} \subseteq \mathbb{P}^{mn-1} \times \mathbf{G}(n-h, n).$$

We fix $\Lambda \in \mathbf{G}(n-h, n)$, the space of linear maps $A : k^n \to k^m$ such that $\Lambda \subseteq Ker(A)$ is just $Hom(k^n/\Lambda, k^m)$. Then the fibres of the second projection

$$\pi_2 : \mathcal{I} \longrightarrow \mathbf{G}(n-h, n)$$

are projective spaces of dimension $hm-1$. Clearly π_2 is surjective and we conclude that $\mathcal{I}$ is an irreducible variety of dimension

$$dim(\mathcal{I}) = dim(\mathbf{G}(n-h,n)) + hm - 1 = h(m+n-h) - 1.$$

The first projection

$$\pi_1 : \mathcal{I} \longrightarrow \mathbb{P}^{mn-1}$$

is generically injective on M_h. Then we have proved the following

PROPOSITION 11. *The variety $M_h \subseteq \mathbb{P}^{mn-1}$ of $m \times n$ matrices of rank at most h is an irreducible projective variety of codimension $(m-h)(n-h)$ in $\mathbb{P}^{mn-1}$.*

Let $A \in M_h \setminus M_{h-1}$ a matrix of rank h. We choose bases for k^m and k^n so that A is represented by the matrix

$$\begin{pmatrix} I_h & 0 \\ 0 & 0 \end{pmatrix}$$

where I_h is the $h \times h$ identity matrix. We consider the affine neighborhood $\mathcal{U}$ of A given by $\{X_{1,1} \neq 0\}$ and fix the euclidean coordinates $x_{i,j} = X_{i,j}/X_{1,1}$. Now we write a general element of $\mathcal{U}$ as

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & \dots & \dots & x_{1,m} \\ x_{2,1} & 1+x_{2,2} & x_{2,3} & \dots & \dots & \dots & x_{2,m} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ x_{h,1} & \dots & \dots & 1+x_{h,h} & x_{h,h+1} & \dots & x_{h,m} \\ x_{h+1,1} & \dots & \dots & 1+x_{h+1,h} & x_{h+1,h+1} & \dots & x_{h+1,m} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & x_{n,3} & \dots & \dots & \dots & x_{n,m} \end{pmatrix}$$

where A corresponds to the origin in this coordinate system. We note that the only $(h+1) \times (h+1)$ minors of this matrix with nonzero differential at the origin A are those involving the first h rows and columns. Their linear terms are exactly the coordinates $x_{i,j}$ with $i, j > h$. Since there are exactly $(m-h)(n-h)$ of these, we conclude that M_h is smooth at any point of $M_h \setminus M_{h-1}$.

REMARK 5. *We consider the case of symmetric matrices with $n = m = 3$ and $h = 2$. Let $\mathcal{U} = \{X_{1,1} \neq 0\} \subseteq \mathbb{P}^5$. We write the generic matrix $A \in \mathcal{U}$ in $M_2 \setminus M_1$ in the form*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ x_{1,2} & 1+x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{pmatrix}$$

Let $F(x_{1,2}, ..., x_{3,3}) = det(A)$. We see that $\frac{\partial F}{\partial x_{3,3}}(A) \neq 0$. Then the points in $M_2 \setminus M_1$ are smooth for M_2. Note that M_2 is the secant variety $Sec_2(V_4^2)$ of the Veronese surface $V_4^2 \subseteq \mathbb{P}^5$. The variety $Sec_2(V_4^2)$ is smooth outside V_4^2 and $Sing(Sec_2(V_4^2)) = V_4^2$.

Chapter 2

Varieties of Sums of Powers

In this chapter we define the concept of variety of power sums and we prove the main properties of these varieties.
Let V be a k-vector space of dimension $n+1$ over the algebraically closed field k. For any $F \in S^d V$ we denote by $\boldsymbol{V}(F)$ the hypersurface defined by F in the projective space $\mathbb{P}V \cong \mathbb{P}^n$. A linear form $l{:}V \to k$ defines a point $[l]$ in $\mathbb{P}V^*$ and a hyperplane $\boldsymbol{V}(l)$ in $\mathbb{P}V$.

2.1 Tensor Algebra and homogeneous polynomials

We consider the n-th tensorial product $V^{\otimes n}$ of V and write

$$V^{\otimes n} = \underbrace{V \otimes ... \otimes V}_{n-times},$$

We take the direct sum of the $V^{\otimes n}$ for $n=0,1,2...$,

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

We define on $T(V)$ a multiplication using the canonical isomorphism

$$\varphi{:}V^{\otimes k} \otimes V^{\otimes h} \to V^{\otimes k+h},\ (x_1 \otimes ... \otimes x_k) \otimes (y_1 \otimes ... \otimes y_h) \mapsto x_1 \otimes ... \otimes x_k \otimes y_1 \otimes ... \otimes y_h.$$

With this multiplication $T(V)$ is a graduate k-algebra and $V^{\otimes n}$ is the n-th graduate component.
The k-Algebra $T(V)$ is the *Tensor Algebra of V*.
Let J be the ideal of $T(V)$ generated by the elements of the form $v \otimes w - w \otimes v$ *with* $v,w \in T(V)$.
The quotient algebra $S(V) = \frac{T(V)}{J}$ is the *Symmetric Algebra of V*. We observe that $S(V)$ is a graduate k-algebra as quotient of $T(V)$, we denote by $S^k(V)$ the k-th graduated component of $S(V)$.

2.2 Polar Forms

We fix a basis $\{t_0,...,t_n\}$ of V and the dual basis $\{\xi_0,...,\xi_n\}$ of V^*. The ring morphism

$$S^k(V) \to Pol^k(V),\ t_{i_1} \otimes \ldots \otimes t_{i_k} \mapsto t_{i_1} \ldots t_{i_k}$$

allows us to identify $S^d(V)$ with the ring of homogeneous polynomials of degree d on V. The multilinear form $plz(P)$ is symmetric and P can be reconstructed from $plz(P)$.

PROPOSITION 12. *If $plz(P)$ is the symmetric multilinear form associated to a homogeneous polynomial of degree k on V we have*

$$k!P(v) = plz(P)(v,\ldots,v).$$

<u>*Proof*</u>: We have $plz(P)(v,\ldots,v) = \sum_{I \subseteq \underline{k}} (-1)^{k-Card(I)} P(Card(I)v)$. Now for any $i \leq k$ we have $\binom{k}{i}$ subsets of $\underline{k}$ of cardinality i. So
$plz(P)(v,\ldots,v) = (-1)^{k-1}\binom{k}{1}P(v) + (-1)^{k-2}\binom{k}{2}P(2v) + \ldots + (-1)^1\binom{k}{k-1}P((k-1)v) + (-1)^0\binom{k}{k}P(kv) =$
$(-1)^{k-1}\binom{k}{1}P(v) + (-1)^{k-2}\binom{k}{2}2^k P(v) + \ldots + (-1)^1\binom{k}{k-1}(k-1)^k P(v) + (-1)^0\binom{k}{k}k^k P(v) =$
$((-1)^{k-1}\binom{k}{1} + (-1)^{k-2}\binom{k}{2}2^k + \ldots + (-1)^1\binom{k}{k-1}(k-1)^k + (-1)^0\binom{k}{k}k^k)P(v)$.
Finally $plz(P)(v,\ldots,v) = P(v)\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^k = k!P(v)$. □

Let $F: V^k \to k$ be a symmetric multilinear form. We consider the map

$$Res(F): V \to k,\ Res(F)(v) = F(v,\ldots v).$$

We observe that $Res(F) \in S^k(V)$, moreover $plz(Res(F)) = k!F$. From $char(k)=0$ we have that any polynomial $P \in S^k(V)$ can be obtained by Res from a unique symmetric multilinear form.

DEFINITION 12. *The symmetric multilinear form $plz(P)$ is called the polarization of P and the map $Res(F)$ is called the restitution of F.*

EXAMPLE 9. *Let Q be a quadratic form on V.*

$$plz(Q)(v,w) = (-1)^{2-1}Q(v) + (-1)^{2-1}Q(w) + (-1)^{2-2}Q(v+w) = Q(v+w) - Q(v) - Q(w).$$
$$Res(plz(Q)(v,w)) = Res(Q(v+w) - Q(v) - Q(w)) = Q(2v) - Q(v) - Q(v) = 2Q(v).$$

2.3 Apolar Forms

Let V be a k-vector space of dimension $n+1$ and let V^* be the dual vector space. We have the map

$$V \times V^* \to k,\ (v,L) \mapsto L(v).$$

We want to generalize this fact constructing a map $S^k(V) \times S^d(V^*) \mapsto S^{d-k}(V^*)$. To do this we fix a system of coordinates $\{t_0,\ldots,t_n\}$ on V and the dual coordinates $\{\xi_0,\ldots,\xi_n\}$ on V^*.
Let $\varphi = \varphi(t_0,\ldots,t_n)$ be a homogeneous polynomial of degree k on V. We consider the differential operator

$$D_\varphi = \varphi(\partial_0,\ldots,\partial_n),\ with\ \partial_i = \frac{\partial}{\partial \xi_i}.$$

This operator acts on φ substituting the variable t_i with the partial derivative $\partial_i = \frac{\partial}{\partial \xi_i}$.
For any $\varphi \in S^k(V)$ and for any $F \in S^d(V^*)$ we write

$$<\varphi,F> = D_\varphi(F).$$

We call this pairing *the apolarity pairing.*
In general φ is of the form $\varphi(t_0,...,t_n)=\sum_{i_0+...+i_n=k}\alpha_{i_0,...,i_n}t_0^{i_0}...t_n^{i_n}$ and F is of the form $F(\xi_0,...,\xi_n)=\sum_{j_0+...+j_n=d}f_{i_0,...,i_n}\xi_0^{j_0}...\xi_n^{j_n}$. Then

$$D\varphi(F)=(\sum_{i_0+...+i_n=k}\alpha_{i_0,...,i_n}\partial_0{}^{i_0}...\partial_n{}^{i_n})(F).$$

We see that F is derived $i_0+...+i_n=k$ times. So we obtain a homogeneous polynomial of degree d-k on V^*.
Fixed $F\in S^d(V^*)$ we have the map

$$ap_F^k:S^k(V)\rightarrow S^{d-k}(V^*),\ \varphi\mapsto D_\varphi(F).$$

The map ap_F^k is linear and we can consider the subspace $Ker(ap_F^k)$ of $S^k(V)$.

Definition 13. *A homogeneous form $\varphi\in S^k(V)$ is called apolar to a homogeneous form $F\in S^d(V^*)$ if $D_\varphi(F)=0$,in other words if $\varphi\in Ker(ap_F^k)$. The vector subspace of $S^k(V)$ of apolar forms of degree k to F is denoted by $AP_k(F)$.*

Example 10. *We consider the case $d=2$, $n=2$, $k=1$. Let $Q\in S^2(V^*)$ be a quadratic form on V^*, we write $Q(\xi_0,\xi_1,\xi_2)=\sum_{i,j=0}^2 q_{ij}\xi_i\xi_j$, then*

$$\frac{\partial(Q)}{\partial\xi_0}=2q_{00}\xi_0+2q_{01}\xi_1+2q_{02}\xi_2$$
$$\frac{\partial(Q)}{\partial\xi_1}=2q_{01}\xi_0+2q_{11}\xi_1+2q_{12}\xi_2$$
$$\frac{\partial(Q)}{\partial\xi_2}=2q_{02}\xi_0+2q_{12}\xi_1+2q_{22}\xi_2$$

We consider $\varphi(t_0,t_1,t_2)=\alpha_0t_0+\alpha_1t_1+\alpha_2t_2$. Then $D_\varphi=\alpha_0\frac{\partial}{\partial\xi_0}+\alpha_1\frac{\partial}{\partial\xi_1}+\alpha_2\frac{\partial}{\partial\xi_2}$. The apolarity map is
$ap_Q^1(\varphi)=D_\varphi(Q)=\xi_0(2q_{00}\alpha_0+2q_{01}\alpha_1+2q_{02}\alpha_2)+\xi_1(2q_{01}\alpha_0+2q_{11}\alpha_1+2q_{12}\alpha_2)+$
$\xi_2(2q_{02}\alpha_0+2q_{12}\alpha_1+2q_{22}\alpha_2)$. In a compact form

$$ap_Q^1(\varphi)=\sum_{i=0}^2\frac{\partial(Q)}{\partial\xi_i}(\varphi)\xi_i.$$

In general if $dim(V)=n+1$ we have $ap_Q^1:V\rightarrow V^*$ defined by

$$ap_Q^1(\varphi)=\sum_{i=0}^n\frac{\partial(Q)}{\partial\xi_i}(\varphi)\xi_i.$$

2.4 Dual homogeneous Forms

We fix $d=2$ and consider the space S^2V^* of quadric forms on V. On a form $Q\in S^2V^*$ is associated a matrix $A=(a_{ij})$ and we can write

$$Q=\sum_{i,j=0}^n a_{i,j}t_it_j.$$

The apolarity map is given by

$$ap_Q^1:V\rightarrow V^*,\ v\mapsto D_v(Q)=\sum_{i=0}^n\frac{\partial Q}{\partial t_i}(v)t_i.$$

Now we define a bilinear form $B_Q:V\times V\rightarrow k$ by $B_Q(v,w)=<w,ap_Q^1(v)>$.

EXAMPLE 11. *The case d=2, n=2, k=1. Let $Q \in S^2 V^*$ be a quadratic form on V, we write $Q = \sum_{i,j=0}^{2} q_{ij} t_i t_j$, then*
$ap_Q^1(v) = (2q_{00}v_0 + 2q_{01}v_1 + 2q_{02}v_2)t_0 + (2q_{11}v_1 + 2q_{01}v_0 + 2q_{12}v_2)t_1 + (2q_{22}v_2 + 2q_{02}v_0 + 2q_{12}v_1)t_2$
$= (2q_{00}v_0 + 2q_{01}v_1 + 2q_{02}v_2,\ 2q_{11}v_1 + 2q_{01}v_0 + 2q_{12}v_2,\ 2q_{22}v_2 + 2q_{02}v_0 + 2q_{12}v_1)$.
We see that $Mat(ap_Q^1) = 2(q_{ij})$ and so Q is nondegenerate if and only if the linear map ap_Q^1 is invertible.

Also in the general case the quadric form Q is nondegenarate if and only if the linear map $ap_Q^1 : V \to V^*$ is invertible. In this case we have the inverse map $(ap_Q^1)^{-1} : V^* \to V$ that induces a bilinear map

$$B_Q^{-1} : V^* \times V^* \to k, \text{ defined by } B_Q^{-1}(f,g) = \langle g, (ap_Q^1)^{-1}(f) \rangle.$$

From the construction we deduce that the quadric forms Q^* on V^* is given by the inverse of the matrix of Q and is the unique quadric forms on V^* such that $B_{Q^*} = B_Q^{-1}$. The quadric form Q^* is called the *dual quadric form of Q*. By the definition of ap_Q^1 we see that this map sends the vector v in the tangent space of $\boldsymbol{V}(Q)$ in v, so the dual quadric Q^* is the locus in $\mathbb{P}V^*$ of tangent hyperplanes of the quadric $Q \subseteq \mathbb{P}V$.

EXAMPLE 12. *We fix n=3 and consider the quadric $Q = 2t_0^2 + 3t_1^2 + 2t_2^2 + t_3^2$. The apolar map is given by*

$$ap_Q^1(v) = \sum_{i=0}^{3} \frac{\partial Q}{\partial t_i}(v) t_i = 4v_0t_0 + 6v_1t_1 + 4v_2t_2 + 2v_3t_3.$$

The associated matrix and the inverse matrix are

$$Mat(ap_Q^1) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2Mat(Q) \qquad \textit{and} \qquad Mat({ap_Q^1}^{-1}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

We conclude that the dual quadric form is $Q^* = \frac{1}{4}\xi_0^2 + \frac{1}{6}\xi_1^2 + \frac{1}{4}\xi_2^2 + \frac{1}{2}\xi_3^2$.

2.4.1 Catalecticant matrices and dual homogeneous forms

We want to generalize the notion of dual quadric form in the case $d = 2k$ with $d > 2$. We begin constructing the *k-th catalecticant matrix* associated to a homogeneous form $F \in S^d V^*$. We consider the apolarity map

$$ap_F^k : S^k V \longrightarrow S^{d-k} V^*, \ \varphi \mapsto D_\varphi(F).$$

We write the polynomials F and φ in the form

$$F = \sum_{i_0+\ldots+i_n=d} \frac{d!}{i_0!\ldots i_n!} f_{i_0,\ldots,i_n} t_0^{i_0} \ldots t_n^{i_n},$$
$$\varphi = \sum_{j_0+\ldots+j_n=k} \frac{k!}{j_0!\ldots j_n!} \varphi_{j_0,\ldots,j_n} \xi_0^{i_0} \ldots \xi_n^{i_n}.$$

Let $\{\frac{k!}{j_0!\ldots j_n!} t_0^{j_0} \ldots t_n^{j_n}\}$ be a basis of $S^k V^*$ and $\{\frac{(d-k)!}{i_0!\ldots i_n!} t_0^{i_0} \ldots t_n^{i_n}\}$ be a basis of $S^{d-k} V^*$, both ordered lexicographically, then the matrix of the linear map ap_F^k is called the *k-th catalecticant matrix* of the form F and denoted by $Cat_F(k, d{-}k, n{+}1)$. It is a matrix of size

$$dim(S^k V) \times dim(S^{d-k} V) = \binom{n+k}{n} \times \binom{n+d-k}{n}.$$

If we consider the basis $\{t_0^{i_0}\ldots t_n^{i_n}\}_{i_0+\ldots+i_n=k}$ of $S^k V^*$, the basis $\{t_0^{j_0}\ldots t_n^{j_n}\}_{j_0+\ldots+j_n=d-k}$ of $S^{d-k} V^*$ and write

$$F = \sum\nolimits_{i_0+\ldots+i_n=d} f_{i_0,\ldots,i_n} t_0^{i_0}\ldots t_n^{i_n},$$
$$\varphi = \sum\nolimits_{j_0+\ldots+j_n=k} \varphi_{j_0,\ldots,j_n} \xi_0^{i_0}\ldots \xi_n^{i_n}.$$

The matrix of ap_F^k with respect these basis is $\frac{(d-k)!}{d!}\cdot Cat_F(k,d\text{-}k,n+1)$.

EXAMPLE 13. *The case n=2, d=3, k=1.*
We have $ap_F^1: S^1 V \to S^2 V^*$ *with* $F \in S^3 V^*$
$F = f_{3,0,0} t_0^3 + 3f_{2,1,0} t_0^2 t_1 + 3f_{2,0,1} t_0^2 t_2 + 3f_{1,2,0} t_0 t_1^2 + 3f_{1,0,2} t_0 t_2^2 + 6f_{1,1,1} t_0 t_1 t_2 + f_{0,3,0} t_1^3 + 3f_{0,2,1} t_1^2 t_2 + 3f_{0,1,2} t_1 t_2^2 + f_{0,0,3} t_2^3$.
$\varphi = \varphi_0\xi_0 + \varphi_1\xi_1 + \varphi_2\xi_2$.
$ap_F^1(\varphi) = D_\varphi(F) =$
$t_0^2(3f_{3,0,0}\varphi_0 + 3f_{2,1,0}\varphi_1 + 3f_{2,0,1}\varphi_2) + 2t_0 t_1(3f_{2,1,0}\varphi_0 + 3f_{1,2,0}\varphi_1 + 3f_{1,1,1}\varphi_2) +$
$2t_0 t_2(3f_{2,0,1}\varphi_0 + 3f_{1,1,1}\varphi_1 + 3f_{1,0,2}\varphi_2) + t_1^2(3f_{1,2,0}\varphi_0 + 3f_{0,3,0}\varphi_1 + 3f_{0,2,1}\varphi_2) +$
$2t_1 t_2(3f_{1,1,1}\varphi_0 + 3f_{0,2,1}\varphi_1 + 3f_{0,1,2}\varphi_2) + t_2^2(3f_{1,0,2}\varphi_0 + 3f_{0,1,2}\varphi_1 + 3f_{0,0,3}\varphi_2)$.
So the catalecticant matrix is

$$Cat_F(1,2,2) = \begin{pmatrix} 3f_{3,0,0} & 3f_{2,1,0} & 3f_{2,0,1} \\ 3f_{2,1,0} & 3f_{1,2,0} & 3f_{1,1,1} \\ 3f_{2,0,1} & 3f_{1,1,1} & 3f_{1,0,2} \\ 3f_{1,2,0} & 3f_{0,3,0} & 3f_{0,2,1} \\ 3f_{1,1,1} & 3f_{0,2,1} & 3f_{0,1,2} \\ 3f_{1,0,2} & 3f_{0,1,2} & 3f_{0,0,3} \end{pmatrix}$$

Now we consider the special case $d = 2k$, $F \in S^{2k} V^*$ and the apolarity map

$$ap_F^k: S^k V \longrightarrow S^k V^*.$$

We define a symmetric bilinear form

$$\Omega_F: S^k V \times S^k V \longrightarrow k,\ (\varphi_1,\varphi_2) \mapsto <\varphi_2, ap_F^k(\varphi_1)>.$$

The restriction of Ω_F to the diagonal gives a quadratic form on $S^k V$. The matrix associated to the quadric form Ω_F is the catalecticant matrix $Cat_F(k,k,n)$. It is a square matrix of size $dim(S^k V) = \binom{n+k}{k}$. For $n=1$ this matrix is known as a *Hankel matrix*. The quadratic form Ω_F is called nondegenerate if and only if $det(Cat_F(k,k,n)) \neq 0$.

DEFINITION 14. *Let* $F \in S^{2k} V^*$ *be a homogeneous form on V. Then F is called nondegenerate if* Ω_F *is a nondegenerate quadratic form on* $S^k V$.

EXAMPLE 14. *Case d=4, n=1. We have* $F = f_{4,0} t_0^4 + f_{3,1} t_0^3 t_1 + f_{2,2} t_0^2 t_1^2 + f_{1,3} t_0 t_1^2 + f_{0,4} t_1^4$.
The catalecticant matrix of F is

$$Cat_F(2,2,1) = \begin{pmatrix} f_{4,0} & f_{3,1} & f_{2,2} \\ f_{3,1} & f_{2,2} & f_{1,3} \\ f_{2,2} & f_{1,3} & f_{0,4} \end{pmatrix}$$

PROPOSITION 13. *Let $F \in S^{2k}V^*$ be a nondegenerate form. Then there exists a unique homogeneous form $\mathcal{F} \in S^{2k}V$ such that $\Omega_{\mathcal{F}} = \mho_F$.*

Proof: The quadratic form $\mho_F$ is defined by the matrix $adj(Cat_k(F)) = (c_{uv}{}^*)$ and we have

$$\mho_F = \sum c_{un}{}^* \xi^u \xi^v.$$

We consider the form $\mathcal{F} \in S^{2k}V$ defined by

$$\mathcal{F} = \sum_{|u+v|=2k} \frac{d!}{(u+v)!} c_{uv}{}^* \xi^{u+v}.$$

Then for any $t^{\boldsymbol{i}} = t_0^{i_0} \dots t_n^{i_n} \in S^k V^*$ we have

$$D_{t^{\boldsymbol{i}}}(\mathcal{F}) = \sum_{u+v \geq i} \frac{d!}{(u+v)!} c_{uv}{}^* \frac{(u+v)!}{(u+v-i)!} \xi^{u+v-i} = \sum_{|j|=k} \frac{d!}{j!} c_{ij}{}^* \xi^j.$$

So the matrix of the linear map $S^k V^* \to S^k V$ defined by $\Omega_{\mathcal{F}}$ is equal to the matrix $adj(Cat_k(F))$ and the quadratic form $\Omega_{\mathcal{F}}$ is the dual of the quadratic form Ω_F. $\square$

DEFINITION 15. *Let $l, L \in V^*$ be two linear form. We say that l and L are conjugate with respect to a nondegenerate form $F \in S^{2k}V^*$ if*

$$\Omega_{\mathcal{F}}(l^k, L^k) = 0.$$

2.5 Sums of Powers

For any finite set of points $p_1, \dots, p_h \in \mathbb{P}V$ we consider the linear space of homogeneous forms F of degree d on $\mathbb{P}V$ such that $\boldsymbol{V}(F)$ contains the points $p_1, \dots, p_h$ and we denote it by

$$L_d(\mathbb{P}V, p_1, \dots, p_h) = \{F \in S^d V \mid p_i \in \boldsymbol{V}(F)\ \forall\ 1 \leq i \leq h\}.$$

DEFINITION 16. *An unordered set of points $\{[l_1], \dots, [l_h]\}$ in $\mathbb{P}V^*$ is a polar h-polyhedron of $F \in S^d V$ if*

$$F = \lambda_1 l_1^d + \dots + \lambda_h l_h^d$$

for some nonzero scalars $\lambda_1, \dots, \lambda_h \in k$ and moreover the l_i^d are linearly independent in $S^d V^*$.

PROPOSITION 14. *Let $F \in S^{2k}V^*$ and let $\{l_1, \dots, l_h\}$ be a polar h-polyhedron for F, where the l_i^k are linearly independent in $S^k V^*$. Then each pair l_i, l_j is conjugate with respect to the polynomial F.*

Proof: We have $F = l_1^{2k} + \dots + l_h^{2k}$ and

$$\Omega_F = \sum_{i=1}^h \Omega_{l_i^{2k}} = \sum_{i=1}^h (l_i^k)^2.$$

So it is sufficient to prove the assertion for quadratic forms. We choose a coordinate system such that $l_i = t_0$, $l_j = t_1$ and $F = t_0^2+t_2^2+\ldots+t_n^2$. Then $F = \xi_0^2+\ldots+\xi_n^2$ and $\Omega_F(l_i^k,l_j^k) = \Omega_F(t_0^k,t_1^k) = 0$. □

The ground field is algebraically closed so we can write $F=L_1^d+\ldots+L_h^d$ as sums of powers of linear forms. This fact admits a geometrical interpretation. Let

$$\nu_d{:}\mathbb{P}^n\to\mathbb{P}^N,\ [x_0{:}\ldots{:}x_n]\mapsto[x_0^d{:}x_0^{d-1}x_1{:}\ldots{:}x_n^d],\ \text{with } N=\tbinom{n+d}{d}\text{-}1,$$

be the d-Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. The projective space $\mathbb{P}^N$ with $N=\binom{n+d}{d}$-1 parametrizes the homogeneous forms of degree d on $\mathbb{P}V$. The Veronese variety $V_{d^n}^n=\nu_d(\mathbb{P}^n)$ is the locus of polynomials that are powers of linear forms on $\mathbb{P}V$.
So $l_1,\ldots,l_h$ is a polar h-polyhedron of F if and only if F lies on the secant *(h-1)*-plane of $V_{d^n}^n$ passing through $l_1^d,\ldots,l_h^d$.
We know that for the Veronese variety we have

$$expdim(Sec_{h-1}(V_{d^n}^n)) = min\{h\cdot n+h\text{-}1,\ N\}.$$

It is clear that for sufficiently large values of h the variety of secant *(h-1)*-planes is the space $\mathbb{P}^N$ and each homogeneous polynomial of degree d admits a decomposition in the sums of h d-powers of linear factors. It is as much clear that for some values of h $Sec_{h-1}(V_{d^n}^n)$ is a proper subvariety of $\mathbb{P}^N$ and there is a open Zariski subset of $\mathbb{P}^N$ whose points are polynomials that don't admit a decomposition in h d-powers. Let V_4^2 be the Veronese surface in $\mathbb{P}^5$. One expects that $\dim(\mathrm{Sec}_1(\mathrm{V}_4^2))=5$ but we have seen that $\mathrm{Sec}_1(\mathrm{V}_4^2)$ is a cubic hypersurface in $\mathbb{P}^5$ and that the generic conic does not admit a decomposition in the sum of two squares of linear forms.

Lemma 1. *The set $\mathcal{P}=\{[l_1],\ldots,[l_h]\}$ is a polar h-polyhedron of F if and only if*

$$L_d(\mathbb{P}V,[l_1],\ldots,[l_h]) \subseteq AP_d(F)$$

and the inclusion is not true if we delete any $[l_i]$ from $\mathcal{P}$.

Proof: Let $\varphi \in S^dV$ be a homogeneous polynomial of degree d and *let* $l_i \in V^*$ a linear form on V.
We have $<\varphi,l_i^d>=0$ if and only if $(\sum_{i_0+\ldots+i_n=k}\varphi_{i_0,\ldots,i_n}\partial_0{}^{i_0}\ldots\partial_n{}^{i_n})(l_i^d)=0$ if and only if $(\sum_{i_0+\ldots+i_n=k}\alpha_{i_0,\ldots,i_n}l_0^{i_0}\ldots l_n^{i_n})=0$ if and only if $\varphi([l_i])=0$, where $[l_i] = [l_0{:}\ldots{:}l_n]$. Therefore

$$<l_1^d,\ldots,l_h^d>^{\perp}=\{\varphi \in S^dV \mid <\varphi,l_i^d>=0\}=\{\varphi \in S^dV \mid \varphi([l_i])=0\}=L_d(\mathbb{P}V,[l_1],\ldots,[l_h]).$$

If the conditions of the lemma are satisfied we have

$$F \in AP_d(F)^{\perp} \subseteq L_d(\mathbb{P}V,[l_1],\ldots,[l_h])^{\perp} = <l_1^d,\ldots,l_h^d>$$

and F is a linear combination of the l_i^d. If the $l_1^d,\ldots,l_h^d$ are linearly dependent there exists a proper subset $\mathcal{P}'$ of $\mathcal{P}$ such that $<\mathcal{P}'>=<\mathcal{P}>$, we can suppose $\mathcal{P}' = \{[l_1];\ldots;[l_{h-1}]\}$. Then

$$AP_d(F)^{\perp} \subseteq L_d(\mathbb{P}V,p_1,\ldots,p_h)^{\perp} = <\mathcal{P}'>.$$

We have $<\mathcal{P}'>^{\perp} = L_d(\mathbb{P}V,[l_1],...,[l_h]) \subseteq AP_d(F)$ contradicting the hypothesis. This prove that $\mathcal{P}$ is a polar polyhedron of F.
Now suppose that $\mathcal{P}$ is a polar polyhedron of F. Then $F \in <\mathcal{P}>$ and $L_d(\mathbb{P}V,[l_1],...,[l_h]) = <\mathcal{P}>^{\perp} \subseteq <F>^{\perp} = AP_d(F)$.
Suppose that $L_d(\mathbb{P}V,[l_1],...,[l_h]) \subseteq AP_d(F)$. Then $F \in AP_d(F)^{\perp} \subseteq L_d(\mathbb{P}V,[l_1],...,[l_h])^{\perp} = <l_1^d,...,l_{h-1}^d>$. So we can write

$$F=\lambda_1 l_1^d+...+\lambda_h l_h^d = \alpha_1 l_1^d+...+\alpha_{h-1} l_{h-1}^d.$$

This implies

$$(\lambda_1-\alpha_1)l_1^d+...+(\lambda_{h-1}-\alpha_{h-1})l_{h-1}^d+\lambda_h l_h^d=0$$

in contradiction whit the linear independence of the l_i^d. □

Now we consider $(\mathbb{P}V^*)^h$ whit its structure of algebraic variety. Under the action of the symmetric group we obtain another algebraic variety, the symmetric power

$$(\mathbb{P}V^*)^{(h)}=\frac{(\mathbb{P}V^*)^h}{S_h}.$$

We denote by $\boldsymbol{VSP}(F,h)^o$ the subset of $(\mathbb{P}V^*)^{(h)}$ consisting of polar h-polyhedra of F. It is natural to see $\boldsymbol{VSP}(F,h)^o$ in the symmetric power in fact we are not interested in the order of the linear forms l_i.
By lemma 1 $\boldsymbol{VSP}(F,h)^o$ is a locally closed subset of $(\mathbb{P}V^*)^{(h)}$ but it is not compact.
For example consider the family of polynomials $\lambda(X_0+X_1)^2$-X_0^2-X_1^2. For any $\lambda \notin \{0,1\}$ we have a decomposition in three factors but for $\lambda=0$ we have two factors and for $\lambda=1$ we obtain the product $2X_0X_1$. This shows that the limit of an additive decomposition in general is not additive.
Now it is natural to look for a compactification of the set $\boldsymbol{VSP}(F,h)^o$. We have different possibilities. Let F be a generic homogeneous polynomial of degree d in $n+1$ variables and let $\{L_1,...,L_h\}$ be a h-polar polyhedron of F. We write

$$F = \lambda_1 L_1^d+...+\lambda_h L_h^d.$$

The polynomials $L_1,...,L_h$ are points in $(\mathbb{P}^n)^*$ so $Z = \{L_1,...,L_h\}$ is a subscheme of dimension zero and length h in $(\mathbb{P}^n)^*$ and so Z is a point in the Hilbert scheme $Hilb_h(\mathbb{P}^n)^*$ of the subschemes of dimension zero and length h of $(\mathbb{P}^n)^*$. Via the injective morphism

$$\boldsymbol{VSP}^o(F,h) \to Hilb_h(\mathbb{P}^n)^*, \text{ defined by } \{L_1,...,L_h\} \mapsto Z$$

we can see $\boldsymbol{VSP}^o(F,h) \subseteq Hilb_h(\mathbb{P}^n)^*$ and so we have a compactification of the variety of power sums

$$\boldsymbol{VSP}_H(F,h) = \overline{\boldsymbol{VSP}^o}(F,h) \subseteq \mathrm{Hilb}_h(\mathbb{P}^n)^*.$$

From another viewpoint we can consider $L_1^d,...,L_h^d$ as points on the Veronese variety $V_{d^n}^n \subseteq \mathbb{P}^N$ with $N = \binom{n+d}{d}$ - 1. These points generate a $(h$-$1)$-plane in $\mathbb{P}^N$ and define a point in the Grassmannian $\boldsymbol{G}(h\text{-}1,N)$. For $h < N$ - n We have an injective morphism

$$\boldsymbol{VSP}^o(F,h) \to \boldsymbol{G}(h\text{-}1,N), \text{ defined by } \{L_1,...,L_h\} \mapsto <L_1^d,...,L_h^d>.$$

In this way we can see $\boldsymbol{VSP}^o_G(F,h) \subseteq \boldsymbol{G}(h\text{-}1,N)$ and we obtain another compactification

$$\boldsymbol{VSP}_G(F,h) = \overline{\boldsymbol{VSP}^o}(F,h) \subseteq \mathbf{G}(\text{h-1,N}).$$

The points in the set $\boldsymbol{VSP}(F,h) \backslash \boldsymbol{VSP}(F,h)^o$ are called *generalized polar polyhedra* and the variety $\boldsymbol{VSP}(F,h)$ is called *the variety of power sums of F.*

LEMMA 2. *The Hilbert scheme $Hilb_h(\mathbb{P}^1)$ of 0-subschemes of length h of $\mathbb{P}^1$ is a nonsingular scheme of dimension h.*

Proof: Any homogeneous polynomial $P \in S^h V$ vanishes at exactly h points in $\mathbb{P}^1$ counted with multiplicity and so determine a point in $Hilb_h(\mathbb{P}^1)$. Conversely any point in $Hilb_h(\mathbb{P}^1)$ is a collection of h points with multiplicity and so can be seen as the locus of zeros of an homogeneous polynomial $P \in S^h V$. We have a map

$$\varphi:\mathbb{P}(S^h V) \to Hilb_h(\mathbb{P}^1)\ ,\ P_Z \mapsto Z.$$

Where Z is the locus of zeros of P_Z. If $P_Z = \lambda Q_Z$ with $\lambda \in k^*$, then P_Z and Q_Z vanish at the same subscheme Z and the map φ is well defined. If $P_Z, Q_Z \in S^h V$ vanish at the same subscheme Z then they differ for a non zero constant and defines the same point in $\mathbb{P}(S^h V)$. So the map φ is injective.
Let $Z = \{p_1, ..., p_r\}$ be a point in $Hilb_h(\mathbb{P}^1)$, where the element p_i has multiplicity k_i and $k_1 + ... + k_r = h$. We write $p_i = [\alpha_i : \beta_i] \in \mathbb{P}^1$, then the polynomial

$$P_Z = (\beta_1 x_0 - \alpha_1 x_1)^{k_1} ... (\beta_r x_0 - \alpha_r x_1)^{k_r}$$

is the unique, up to scalar, homogeneous polynomial of degree h vanishing on Z. We get the morphism

$$\psi : Hilb_h(\mathbb{P}^1) \to \mathbb{P}(S^h V), Z \mapsto P_Z.$$

Clearly ψ is the inverse of φ, so φ is an isomorphism and $Hilb_h(\mathbb{P}^1) \cong \mathbb{P}(S^h V)$ is a nonsingular scheme of dimension h. □

PROPOSITION 15. *In the cases n=1,2 for a general polynomial $F \in S^d V$ the variety $\boldsymbol{VSP}(F,h)$ is either empty or a smooth variety of dimension*

$$dim(\boldsymbol{VSP}(F,h)) = h(n+1) - \binom{n+d}{d}.$$

Proof: We consider $\boldsymbol{VSP}(F,h)$ as the closure of $\boldsymbol{VSP}(F,h)^o$ in the Hilbert scheme $Hilb_h(\mathbb{P}^n)^*$. We have already seen that $\boldsymbol{VSP}(F,h)$ can be empty if h is too small. Let X be the incidence variety defined as follow

$$X = \{(Z,F) \in Hilb_h(\mathbb{P}^n)^* \times S^d V \ /\ Z \in \boldsymbol{VSP}(F,h)\}.$$

We have two projection maps

$$\varphi: X \to Hilb_h(\mathbb{P}^n)^*,\ (Z,F) \mapsto Z \text{ and } \psi: X \to S^d V,\ (Z,F) \mapsto F.$$

Let $Z \in Hilb_h(\mathbb{P}^n)^*$ be a point in the Hilbert scheme. We can see Z as a set $\{l_1,\ldots,l_h\}$ with $l_i \in \mathbb{P}V^*$. The polynomial $F=l_1^d+\ldots+l_h^d$ is such that $Z \in$ ***VSP***(F,h) and $\varphi(Z,F)=Z$, so φ is surjective.
Let $F \in S^dV$ be a polynomial. If ***VSP***(F,h) is not empty there is a decomposition of F in h factors $\{l_1,\ldots,l_h\}$ that is a point Z in the Hilbert scheme such that $\psi(Z,F)=F$. This proves that ψ is surjective.
Note that
$\varphi^{-1}(Z)=\{(Z,F) \mid F \text{ has } Z=\{l_1,\ldots,l_h\}$ as polar h-polyhedron$\}=\{(Z,F) \mid F=\lambda_1 l_1+\ldots+\lambda_h l_h\}$, so $dim(\varphi^{-1}(Z)) = h$. Moreover

$$\psi^{-1}(F)=\{(Z,F) \mid Z \text{ is a polar h-polyhedron of } F\} = \boldsymbol{VSP}(F,h).$$

Applying the theorems on the dimension of the fibres we have

$$dim(X) = dim(\varphi^{-1}(Z)) + dim(Hilb_h(\mathbb{P}^n)^*) = h+nh = (n+1)h,$$
$$dim(X) = dim(\psi^{-1}(F)) + dim(S^dV) = dim(\boldsymbol{VSP}(F,h)) + \binom{n+d}{d}.$$

Equaling the two expressions we obtain $dim(\boldsymbol{VSP}(F,h)) = h(n+1) - \binom{n+d}{d}$.
We can identify a point $Z \in Hilb_h(\mathbb{P}^n)^*$ with a unordered set $\{[l_1],\ldots,[l_h]\}$. We have $dim(L_d(\mathbb{P}V^*,Z)=dim(S^dV)$-$h$ if and only if $dim(<l_1^d,\ldots,l_h^d>,S^dV)=h$.
Recall that $L_d(\mathbb{P}V^*,Z)=<l_1^d,\ldots,l_h^d>^\perp$. The h-uples of linearly independent vectors are an open Zariski subset of $(S^dV)^h$. So we have an open Zariski subset $U \subseteq Hilb_h(\mathbb{P}^n)^*$ such that for any point $Z \in U$, $dim(L_d(\mathbb{P}V^*,Z))=dim(S^dV)$-$h$.
We fix a point $Z \in U$ and consider the fibre

$$\varphi^{-1}(Z) = \{F \in S^dV \mid Z \text{ is a polar h-polyhedron of } F\} = \{F \in S^dV \mid F=\lambda_1 l_1^d+\ldots+\lambda_h l_h^d\}$$
$$= \{F \in S^dV \mid F \in <l_1^d,\ldots,l_h^d>\} \subseteq L_d(\mathbb{P}V^*,[l_1,\ldots,[l_h])^\perp.$$

But $Z \in U$ implies that $l_1^d,\ldots,l_h^d$ are linearly independent and this is a open condition on the coefficients of the linear combinations $F=\lambda_1 l_1^d+\ldots+\lambda_h l_h^d$. So the fibre $\varphi^{-1}(Z)$ is an open Zariski subset of the linear space $L_d(\mathbb{P}V^*,Z)^\perp$, moreover the Hilbert scheme of 0-subscheme of length h of $\mathbb{P}^n$ is nonsingular in the cases $n = 1,2$. This show that $\varphi^{-1}(Z)$ is nonsingular for any $Z \in U$.
If X has a singular point it will be a singular point for some fibre $\varphi^{-1}(Z)$ then X is nonsingular.
The fibres of the second projection are the varieties ***VSP***(F,h). From Bertini theorem we deduce that for an open Zariski subset of S^dV the varieties ***VSP***(F,h) are smooth. □

2.5.1 Waring rank and Alexander-Hirschowitz's theorem

To any quadratic form $Q \in S^2V$ one can associate its rank defined as the smallest number r such that $Q = l_1^2+\ldots+l_r^2$, for some linear forms $l_1,\ldots,l_r$. We want to generalize this definition to any homogeneous polynomial $F \in S^dV$.

DEFINITION 17. *The waring rank of $F \in S^dV$ is the smallest number r such that*

$$F = l_1^d+\ldots+l_r^d$$

for some linear forms $l_1,\ldots,l_r$. We denote the Waring rank of F by wrk(F).

On the other hand *wrk(F)* is the smallest number h such that ***VSP****(F,h)* is not empty so for a generic $F \in S^d V$ one expects that

$$wrk(F) = \left\lceil \tfrac{1}{n+1}\tbinom{n+d}{d} \right\rceil.$$

This is almost always true, J. Alexander and A. Hirschowitz proved, using Terracini's lemma, that the following are the only exceptional cases:

d	**n**	**wrk(F)**
2	*arbitrary*	$n+1$
3	4	8
4	2	6
4	3	10
4	4	15

For a proof see
A.Hirschowitz, J.Alexander, *Polynomial interpolation in several variables.* J. of Algebraic Geometry, 4 (1995).
The theorem in its original form is the answer to the following interpolation problem.
Let $P_1,\ldots,P_h \in \mathbb{A}^n$ be points in general position. Consider the vector space H of polynomials $f \in k[x_1,\ldots,x_n]_{\leq d}$ of degree $\leq d$ such that $f(P_i) = a_i$ and $\frac{\partial}{\partial x_j} f(P_i) = b_{i,j}$ for any $i = 1,\ldots,s$ and $j = 1,\ldots,n$. What is the codimension of H?
It is clear that the expected codimension of H is

$$expcodim(H) = min\{(n+1)h, \tbinom{n+d}{d}\}.$$

Alexander and Hirschowitz classified the defective cases.

Theorem 8. *(<u>Alexander-Hirschowitz</u>) The vector space H has the expected codimension with the following exceptions*

- $d = 2$, $2 \leq h \leq n$;
- $n = 2$, $d = 4$, $h = 5$;
- $n = 3$, $d = 4$, $h = 9$;
- $n = 4$, $d = 3$, $h = 7$;
- $n = 4$, $d = 4$, $h = 14$.

Via Terracini's lemma it is possible to reformulate the theorem in terms of defectivity of some secant varieties to the Veronese varieties. We reformulate our problem in projective terms as follows.
Let $P_1,\ldots,P_h \in \mathbb{P}^n$ be points in general position. Consider the vector space H of hypersurfaces $X_f = \boldsymbol{V}(f) \subseteq \mathbb{P}^n$, where f is a homogeneous degree d polynomials, such that X_f passes through P_i and X_f is singular in P_i for any $i = 1,\ldots,s$. What is the codimension of H?
Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the d-Veronese embedding and let V be the corresponding Veronese variety. Then the hypersurface $X_f \subseteq \mathbb{P}^n$ corresponds to an hyperplane section $H_f \cap V$ of V. Since ν_d is an isomorphism we have that

$$X_f \text{ is singular in } P_i \text{ for any } i = 1, ..., h \Leftrightarrow H_f \supseteq T_{\nu_d(P_i)} V \text{ for any } i = 1, ..., h.$$

LEMMA 3. *(<u>Terracini</u>) If $X \subseteq \mathbb{P}^n_k$ is an irreducible variety, with char(k) = 0, then*

$$<T_{P_1}X,...,T_{P_h}X> \; = \; T_z Sec_h(X)$$

for any z in a open subset $U \subseteq Sec_h(X)$, with $P_1,...,P_h \in X$ and $z \in <P_1,...,P_h>$.

An immediate corollary is that

$$dim(Sec_h(X)) = dim(<T_{P_1}X,...,T_{P_h}X>)$$

So to know the dimension of $Sec_h(X)$ is equivalent to know the dimension of the space generated by the tangent spaces of X in h points.
For the Veronese variety V we have that $dim(Sec_h(V)) = dim(<T_{P_1}V,...,T_{P_h}V>) = min\{hn+(h-1),N\}$ if and only if the $T_{P_i}V$ are independent. From this point of view Alexander-Hirschowitz's theorem says that the only defective Veronese varieties are the following

$$V^2_{4^2},\ V^3_{4^3},\ V^4_{4^4},\ V^4_{3^4} \text{ and } V^n_{2^n} \text{ for any } n>0.$$

The next proposition compares the Waring rank of a homogeneous form $F \in S^{2k}V^*$ with the rank of the associated quadratic form Ω_F.

PROPOSITION 16. *Let $F \in S^{2k}V^*$ be a homogeneous form and let Ω_F be the associated quadric form. Then the Waring rank of F is grater or equal than the rank of Ω_F. In particular if F is nondegenerate then*

$$wrk(F) \geq \binom{k+n}{n}.$$

<u>*Proof*</u>: Let $h = wrk(F)$ be the Waring rank of $F \in S^dV^*$ with $d = 2k$. We write

$$F = L_1^d + ... + L_h^d.$$

Since Ω_F is linear with respect to F we have $\Omega_F = \sum_{i=1}^h \Omega_{L_i^{2k}}$. We can choose coordinates such that L_i is the coordinate function t_0. In this way the catalecticant matrix of L_i^{2k} is the matrix with *1* at the upper left corner and *0* elsewhere. The associated quadric form is $(t_0^k)^2$ so $\Omega_{L_i^{2k}} = (L_i^k)^2$ and we have

$$\Omega_F = \sum_{i=1}^h \Omega_{L_i^{2k}} = \sum_{i=1}^h (L_i^k)^2.$$

We have written Ω_F as sum of h squares of linear forms so we conclude that

$$rank(\Omega_F) \leq h = wrk(F).$$

If F is nondegenerate then Ω_F is a non degenerate quadratic form, its associated matrix is $Cat_F(k,k,n)$ that is a square matrix of size $\binom{k+n}{n} = rank(\Omega_F)$. □

PROPOSITION 17. *Let $F \in S^{2k}V^*$ be a general homogeneous form of degree 2k. Then*

$$wrk(F) > rank(\Omega_F)$$

except in the following cases, where the equality take place:

- $k = 1$;
- $n = 1$;
- $n = 2$, $k \leq 4$;
- $n = 3$, $k = 2$.

Proof: If $k = 1$ then F is a quadratic form and so $wrk(F) = rank(F) = rank(\Omega_F)$. If $n = 1$ then we have $wrk(F) = k+1$. The catalecticant matrix of F is a square matrix of size $k+1$ and so $rank(\Omega_F) = k+1 = wrk(F)$.
If $n = 2$ we get $wrk(F) \geq \frac{1}{3}(k+1)(2k+1)$ and $rank(\Omega_F) = \frac{1}{2}(k+1)(k+2)$. We have $wrk(F) > rank(\Omega_F)$ if and only if $k^2\text{-}3k\text{-}4 > 0$ if and only if $k > 4$. By Alexander-Hirschowitz's theorem we have
$wrk(F) = 6 = rank(\Omega_F)$ if $k = 2$;
$wrk(F) = 10 = rank(\Omega_F)$ if $k = 3$;
$wrk(F) = 15 = rank(\Omega_F)$ if $k = 4$.
If $n = 3$ *we have* $wrk(F) > \frac{1}{24}(2k+3)(2k+2)(2k+1) > \binom{k+3}{3} = \frac{1}{6}(k+3)(k+2)(k+1)$ *if and only if* $2k^2\text{-}2k\text{-}9 > 0$ if and only if $k > 2$. For $k = 2$ we get $wrk(F) = 10$. Finally for $n > 3$ the disequality $wrk(F) \geq \frac{1}{n+1}\binom{2k+n}{n} > \binom{k+n}{n}$ is verified for any $k > 1$. □

Chapter 3

Mukai's Theorem

The interest in varieties of power sums theory has been reawaken in *1992* by a work of *S.Mukai*, who gave a construction of $\mathbf{VSP}(F_d, h)^o$ in the cases

$$(n,d,h)=(2,\ 2,\ 3),\ (2,\ 4,\ 6),\ (2,\ 6,\ 10)$$

for a general polynomial F_d and also constructed a smooth compactication $\mathbf{VSP}(F_d, h)$ which turned out to be a *Fano threefold* in the first two cases and a *K3* surface in the third case. The construction of Mukai employs a generalization of the concept of the dual quadratic form to forms of arbitrary even degree $d = 2k$. The Mukai's theorem is probably the best work in varieties of power sums theory.

3.1 Mukai's skew-symmetric form

Let $\omega \in \bigwedge^2 V$ be a skew-symmetric bilinear form on V^*. We consider a basis $\{t_0,\dots,t_n\}$ of V and the dual basis $\{\xi_0,\dots,\xi_n\}$ of V^*. Then $\omega \in \bigwedge^2 V$ that is generated by the elements of type $\omega_{ij} = \xi_i \wedge \xi_j$. We define a Poisson bracket on generators in the following way

$$\{,\}_{\omega_{ij}}:\ S^{k+1}V^* \times S^{k+1}V^* \to S^{2k}V^*$$
$$\{f,g\}_{\omega_{ij}} = \{f,g\}_{\xi_i\wedge\xi_j} = D_{\xi_i}(f)D_{\xi_j}(g) + D_{\xi_j}(f)D_{\xi_i}(g).$$

Extending by linearity we obtain a skew-symmetric bilinear form

$$\{,\}_\omega:\ S^{k+1}V^* \times S^{k+1}V^* \to S^{2k}V^*.$$

Let $F \in S^{2k}V^*$ be a nondegenerate form and $\check{F} \in S^{2k}V$ be its dual form. For each $\omega \in \bigwedge^2 V$ we define $\sigma_{\omega,F} \in (\bigwedge^{2k+1} V)^*$ by

$$\sigma_{\omega,F}(f,g) = \check{F}(\{f,g\}_\omega).$$

Theorem 9. *(<u>S. Mukai</u>) Let F be a nondegenerate form in $S^{2k}V^*$ and let N be its Waring rank. Let Ω_F be the quadratic form associated to F and assume that $N = rank(\Omega_F) = \binom{n+2k}{n}$. For any $\mathcal{P} = \{[l_1],\dots,[l_N]\} \in$ **VSP**$(F,N)^o$ let $E(\mathcal{P})$ be the linear span of the powers l_i^{k+1} in $S^{k+1}V^*$*

$$E(\mathcal{P}) = <l_1^{k+1},\dots,l_N^{k+1}> \subseteq S^{k+1}V^*.$$

Then we have

i *$E(\mathcal{P})$ is isotropic with respect to each form $\sigma_{\omega,F}$;*

ii *$ap_F^{k-1}(S^{k-1}V) \subseteq E(\mathcal{P})$;*

iii *For any $\varphi \in S^{k-1}V$, $G \in S^{k+1}V^*$ and any $\omega \in \bigwedge^2 V^*$, we have $\sigma_{\omega,F}(D_\varphi(F),G) = 0$. In other words $ap_F^{k-1}(S^{k-1}V)$ is contained in the radical of each $\sigma_{\omega,F}$.*

Proof: We check that $\sigma_{\omega,F}(l_i^{k+1},l_j^{k+1}) = 0$ for any i,j. We compute $\sigma_{\omega,F}(l_i^{k+1},l_j^{k+1}) = \check{F}(\{l_i^{k+1},l_j^{k+1}\}_\omega) = \check{F}(\{l_i^k,l_j^k\})\omega(l_i,l_j) = \Omega_{\check{F}}(l_i^k,l_j^k)\omega(l_i,l_j) = 0$, since by proposition 14 of chapter 2 the pair l_i^k, l_j^k is conjugate with respect to F.
We note that for $\varphi \in S^{k-1}V$ we have $D_\varphi(l_i^{2k}) = \frac{(2k)!}{(k+1)!}\sum_{i=1}^N D_\varphi(l_i^{k-1})l_i^{k+1}$, in fact we derive the l_i^{2k} *k-1* times. Therefore the elements in $ap_F^{k-1}(S^{k-1}V)$ are in the form

$$\sum_{i=1}^N \lambda_i l_i^{k+1} \in E(\mathcal{P}).$$

To prove the last assertion we compute
$\{D_\varphi(F),G\}_{\omega_{ij}} = \{D_\varphi(F),G\}_{\xi_i\wedge\xi_j} = D_{\xi_i}(D_\varphi(F))D_{\xi_j}(G) + D_{\xi_j}(D_\varphi(F))D_{\xi_i}(G) = D_{\varphi\xi_i}(F)D_{\xi_j}(G) - D_{\varphi\xi_j}(F)D_{\xi_i}(G)$.
Now for any $A,B \in S^kV^*$ we have $\check{F}(AB) = \Omega_{\check{F}}(A,B) = <\Omega_{\check{F}}^{-1}(A),B>$. Therefore $\sigma_{\omega_{ij},F}(D_\varphi(F),G) = \check{F}(\{D_\varphi(F),G\}_{\omega_{ij}}) = \check{F}(\ D_{\xi_i}D_\varphi(F)D_{\xi_j}(G) - D_{\xi_j}D_\varphi(F)D_{\xi_i}(G)) = \check{F}(D_{\xi_i}D_\varphi(F)D_{\xi_j}(G)) - \check{F}(D_{\xi_j}D_\varphi(F)D_{\xi_i}(G)) = <\varphi\xi_i,D_{\xi_j}(G)> - <\varphi\xi_j,D_{\xi_i}(G)> = D_\varphi(D_{\xi_i\xi_j}(G) - D_{\xi_j\xi_i}(G)) = D_\varphi(0) = 0$. □

3.2 The Mukai Map

LEMMA 4. *Let V be a k-vector space and let W be a subspace of V. Then $\frac{V}{W} \cong W^\perp$.*

Proof: Let $\Pi: V \to W^\perp$ be the projection map. Then Π is a surjective k-linear morphism and we note that

$$ker(\Pi) = \{v \in V \mid \Pi(v) = 0\} = W.$$

Therefore the map $\overline{\Pi}:\frac{V}{W} \to W^\perp$ defined by $v+W \mapsto \Pi(v)$ is an isomorphism of k-vector spaces. □

LEMMA 5. *We identify S^kV with $(S^kV^*)^*$ and let $d = deg(F)$. Then*

$$ap_F^k(S^kV)^\perp = AP_{d-k}(F).$$

Proof: For any $\varphi_k \in S^kV$ and $\varphi'_{d-k} \in S^{d-k}V$ we have $<\varphi'_{d-k},ap_F^k(\varphi_k)> = <\varphi'_{d-k},<\varphi_k,F> > = <\varphi'_{d-k}\varphi_k,F> = <\varphi_k,<\varphi'_{d-k},F> > = ap_F^{d-k}(\varphi'_{d-k})(\varphi_k)$.
Thus, if $<\varphi'_{d-k},ap_F^k(\varphi_k)> = 0$ for all φ_k we have $ap_F^{d-k}(\varphi'_{d-k})(\varphi_k) = 0$ for all φ_k. By nondegenerancy of the apolarity pairing we get $ap_F^{d-k}(\varphi'_{d-k}) = 0$ *i.e.* $\varphi'_{d-k} \in AP_{d-k}(F)$.
Conversely if $\varphi'_{d-k} \in AP_{d-k}(F)$ then $ap_F^{d-k}(\varphi'_{d-k}) = 0$ and $ap_F^{d-k}(\varphi'_{d-k})(\varphi_k) = 0$ for all

φ_k. So $<\varphi'_{d-k}, ap_F^k(\varphi_k)> = 0$ for all φ_k *i.e.* $\varphi'_{d-k} \in ap_F^k(S^k V)^\perp$. □

Let $F \in S^{2k} V^*$ be a nondegenerate form and assume that (k,n) is one of the exceptional cases of proposition 17 of chapter 2, then

$$N_k = wrk(F) = rank(\Omega_F) = \binom{n+k}{n}.$$

We know that $\boldsymbol{VSP}(F,N_k)^o \neq \varnothing$ for general enough F. Let $\mathcal{P} \in \boldsymbol{VSP}(F,N_k)^o$ and

$$\overline{E}(\mathcal{P}) = \frac{<l_1^{k+1},\ldots,l_{N_k}^{k+1}>}{ap_F^{k-1}(S^{k-1}V)}.$$

The space $\overline{E}(\mathcal{P})$ is a subspace of $W = \frac{S^{k+1}V}{ap_F^{k-1}(S^{k-1}V)}$.
By lemma 4 we have $W \cong ap_F^{k-1}(S^{k-1}V)^\perp$ and so $W^* \cong ap_F^{k-1}(S^{k-1}V)^\perp$. By lemma 5 we get

$$W^* \cong ap_F^{k-1}(S^{k-1}V)^\perp \cong AP_{k+1}(F) \text{ hence } W \cong AP_{k+1}(F)^*.$$

In this way we can see $\overline{E}(\mathcal{P}) \subseteq \mathrm{AP}_{k+1}(\mathrm{F})^*$ as a subspace of $\mathrm{AP}_{k+1}(\mathrm{F})^*$.

PROPOSITION 18. *In the preceding notation we have*

i $dim(AP_{k+1}(F)^*) = \binom{n+k}{n-1} + \binom{n+k-1}{n-1}$;

ii $dim(E(\mathcal{P})) = N_k = \binom{n+k}{n}$;

iii $dim(\overline{E}(\mathcal{P})) = \binom{n+k-1}{n-1}$.

Proof:

i Since F is nondegenerate $AP_k(F) = ker(ap_F^k) = \{0\}$, hence $ker(ap_F^{k-1}) = \{0\}$. Therefore the map $ap_F^{k-1}: S^{k-1}V \rightarrow S^{k-1}V^*$ is an isomorphism of vector spaces and

$$dim(ap_F^{k-1}(S^{k-1}V)) = dim(S^{k-1}V^*) = \binom{n+k-1}{n}.$$

Now $AP_{k+1}(F)^* \cong W = \frac{S^{k+1}V*}{ap_F^{k-1}(S^{k-1}V)}$, therefore we have
$dim(AP_{k+1}(F)^*) = \binom{k+1+n}{n} - \binom{k-1+n}{n} = \frac{(n+k-1)!}{(n-1)!(k+1)!}(n+k+k+1) = \binom{n+k}{n-1} + \binom{n+k-1}{n-1}$.

ii Let $\mathcal{P} = \{[l_1],\ldots,[l_{N_k}]\}$ be a N_k-polar polyhedron of F. We have to prove that $l_1^{k+1},\ldots,l_{N_k}^{k+1}$ are linearly independent that is equivalent to prove that the space of hypersurfaces containing $[l_1],\ldots,[l_{N_k}]$ has dimension $\binom{n+k+1}{n} - N_k = \binom{n+k}{n-1}$ i.e.

$$dim(L_{k+1}(PV^*,[l_1],\ldots,[l_{N_k}])) = \binom{n+k}{n-1}.$$

Case $n = 1$) We have to prove that $dim(L_{k+1}(\mathbb{P}V^*,[l_1],\ldots,[l_{N_k}])) = \binom{1+k}{0} = 1$ where $N_k = \binom{1+k}{1} = k+1$. This is clear because given $k+1$ points in $\mathbb{P}^1$ we have only one degree $k+1$ homogeneous polynomial vanishing on the $k+1$ points.
Case $k = 1$. In this case $N_k = \binom{1+n}{n} = n+1$. We have to prove that

$$dim(L_2(\mathbb{P}V^*,[l_1],\ldots,[l_{n+1}])) = \binom{1+n}{n-1} = n(n+1).$$

The space of quadrics has dimension $\binom{n+2}{n}$ so the space of quadrics containing $[l_1],\dots,[l_{n+1}]$ has dimension $\binom{n+2}{n}$ - 1 - $(n+1) = n(n+1)$ because the l_i^2 are independent.

Case $n = 2,\ k = 2$) We have to prove that $dim(L_3(\mathbb{P}V^*,[l_1],\dots,[l_6]))=\binom{2+2}{1}=4$, in this case $N_k = \binom{4}{2} = 6$. Suppose that $dim(L_3(\mathbb{P}V^*,[l_1],\dots,[l_6])) > 4$. Since $AP_2(F) = \{0\}$ no conics passes through the 6 points, in particular no 4 points are collinear. We take a conic K through the five points $[l_1],\dots,[l_5]$ and two points x,y on K such that each component of C contains ≥ 4 points.

Since $dim(L_3(\mathbb{P}V^*,[l_1],\dots,[l_6],x,y)) > 2$ there exists three linearly independent cubics C_j such that C_i has 7 common points with K. By Bezout's theorem we see that the cubics contain K. The residual lines have to pass through $[l_6]$ and we get a 2-dimensional family of lines through a point but this is impossible.

Case $n = 2,\ k = 3$) In this case $N_k = \binom{5}{2} = 10$. We have to prove that

$$dim(L_4(\mathbb{P}V^*,[l_1],\dots,[l_{10}])) = \binom{5}{1} = 5.$$

Suppose that $dim(L_4(\mathbb{P}V^*,[l_1],\dots,[l_{10}])) > 5$, since $AP_3(F) = \{0\}$ no cubics passes through the ten points in particular no 5 points are collinear and no 8 points are on a conic. Let K be a conic through $[l_1],\dots,[l_5]$ and let x,y,z,w four points of K such that each component on K contains ≥ 5 points.

Then $dim(L_4(\mathbb{P}V^*,[l_1],\dots,[l_{10}],x,y,z,w)) > 1$ and there exist two independent quartics Q_i such that Q_i and K have 9 common points. By Bezout's theorem K is a component of Q_i. So there exists a line of conics through $[l_6],\dots,[l_{10}]$ and this forces $[l_6],\dots,[l_9]$ to be collinear. Repeating the same argument for the points $[l_6],\dots,[l_{10}]$ yields the collinearity of $[l_1],\dots,[l_4]$. Then $[l_1],\dots,[l_4],[l_6],\dots,[l_9]$ are on a conic, a contradiction.

Case $n = 2,\ k = 4$) In this case $N_k = \binom{6}{4} = 15$. We have to prove that

$$dim(L_5(\mathbb{P}V^*,[l_1],\dots,[l_{15}])) = \binom{6}{1} = 6.$$

Since $AP_4(F) = \{0\}$ no quartics passes through the 15 points in particular no 13 points are on a cubic, no 10 points are on a conic and no 6 points are collinear. Suppose that $dim(L_5(\mathbb{P}V^*,[l_1],\dots,[l_{15}])) > 6$. Let L be the line generated by $[l_1],[l_2]$, we take 4 points $x,y,z,w \in L$. Then $dim(L_5(\mathbb{P}V^*,[l_1],\dots,[l_{15}],x,y,z,w)) > 2$ and there exist 3 independent quintics C_i such that C_i and L have 6 common points. In this way we find a projective plane of quartics containing the 13 points $[l_3],\dots,[l_{15}]$ but generically the space of quartics through 13 points is a projective line. We have three possibilities.

The family of quartics is the union of a cubic for the 13 points with the lines of $\mathbb{P}^2$, but this is impossible because no 13 points are on a cubic.

The family of quartics is the union of a conic for the 10 points with the conics of $\mathbb{P}^2$ through 3 points, but this is impossible because no 10 points are on a conic.

The family of quartics is the union of a line for the 6 points with the cubics of $\mathbb{P}^2$ through 7 points, but this is impossible because no 6 points are collinear.

Case $n = 3,\ k = 2$) In this case $N_k = \binom{3+2}{3} = 10$. We have to prove that *dim*(L_3($\mathbb{P}V^*$,$[l_1]$,...,$[l_{10}]$)) = $\binom{3+2}{2}$ = *10*. Suppose *dim*(L_3($\mathbb{P}V^*$,$[l_1]$,...,$[l_{10}]$)) > *10*. Since $AP_2(F)$ = {*0*} no quadrics passes through the *10* points, in particular no *7* points are on plane. There exists a unique quadrics Q through the *9* points $[l_1]$,...,$[l_9]$. Then the *9* points impose independent conditions to the quadrics and the *10* points $[l_1]$,...,$[l_{10}]$ impose independent conditions to the cubics.

iii We have $\overline{E}(\mathcal{P}) = \frac{<l_1^{k+1},...,l_{N_k}^{k+1}>}{ap_F^{k-1}(S^{k-1}V)}$. We compute $dim(\overline{E}(\mathcal{P})) = N_k$ - $dim(ap_F^{k-1}(S^{k-1}V)) = \binom{n+k}{n}$ - $\binom{n+k-1}{n} = \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{n-1}$.

□

We have seen that for any $\mathcal{P} = \{[l_1],...,[l_{N_k}]\} \in$ ***VSP***$(F,N_k)^o$ the space

$$\overline{E}(\mathcal{P}) = \frac{<l_1^{k+1},...,l_{N_k}^{k+1}>}{ap_F^{k-1}(S^{k-1}V)}$$

is a subspace of dimension $\binom{n+k-1}{n-1}$ of the $\binom{n+k}{n-1} + \binom{n+k-1}{n-1}$ dimensional vector space $AP_{k+1}(F)^*$ i.e. a point in the Grassmannian $\boldsymbol{G}(\binom{n+k-1}{n-1},AP_{k+1}(F)^*)$. We get the regular map

$$\mathfrak{MuK}: \boldsymbol{VSP}(F,N_k)^o \longrightarrow \boldsymbol{G}(\binom{n+k-1}{n-1},AP_{k+1}(F)^*),\ \mathcal{P} \mapsto \overline{E}(\mathcal{P}).$$

We call this map the *Mukai map*.

PROPOSITION 19. *The Mukai map is injective.*

<u>*Proof*</u>: Let $\mathcal{P}_l = \{[l_1],...,[l_{N_k}]\}$, $\mathcal{P}_L = \{[L_1],...,[L_{N_k}]\} \in$ ***VSP***$(F,N_k)^o$ such that $\mathfrak{MuK}(\mathcal{P}_l) = \mathfrak{MuK}(\mathcal{P}_L)$ then

$$< l_1^{k+1},...,l_{N_k}^{k+1} >=< L_1^{k+1},...,L_{N_k}^{k+1} > mod(ap_F^{k-1}(S^{k-1}V)).$$

Since F is nondegenerate we have $AP_k(F) = \{0\}$ and so $AP_{k-1}(F) = \{0\}$. We have $< l_1^{k+1} - L_1^{k+1},...,l_{N_k}^{k+1} - L_{N_k}^{k+1} > \subseteq ap_F^{k-1}(S^{k-1}V) = S^{k-1}V^*$. This forces

$$< l_1^{k+1} - L_1^{k+1},...,l_{N_k}^{k+1} - L_{N_k}^{k+1} > = \{0\}$$

and so $l_j^{k+1} = L_j^{k+1}$ for any j. This implies

$$< l_1^{k+1},...,l_{N_k}^{k+1} >=< L_1^{k+1},...,L_{N_k}^{k+1} > \text{and}$$
$$dim(L_{k+1}(PV^*,[l_1],...,[l_{N_k}])) = dim(L_{k+1}(PV^*,[L_1],...,[L_{N_k}])).$$

Without loss of generality we can assume that $[l_1] \neq [L_j]$ for any j. Since

$$dim(L_k(PV^*,[l_2],...,[l_{N_k}])) = \binom{n+k}{n} - (N_k - 1) = \binom{n+k}{n} - \binom{n+k}{n} + 1 > 0$$

we can find a form φ of degree k vanishing on the last N_k - *1* points. If $L \in V$ is a linear form on V^* vanishing on $[l_1]$ but not containing any $[L_j]$ then $\varphi L \in L_{k+1}(PV^*,[l_1],...,[l_{N_k}])$ $= L_{k+1}(PV^*,[L_1],...,[L_{N_k}])$. The form φL vanishes on any $[L_j]$ and this force φ to vanish on any $[L_j]$. Therefore we have $\varphi \in L_k(PV^*,[L_1],...,[L_{N_k}]) \subseteq AP_k(F)$. This implies $AP_k(F) \neq \{0\}$, a contraddiction because F is nondegenerate.
Therefore we may assume $[l_1] = [L_1]$. Now if $[l_2] \neq [L_j]$ for any $k \geq 2$, we repeat the

argument replacing $[l_1]$ with $[l_2]$ and obtain another contraddiction. Proceeding in this way we show that $\mathcal{P}_l = \mathcal{P}_L$.
The same proof works for generalized polar polyhedra. Let Z and Z' be two generalized polar polyhedra such that

$$L_{k+1}(\mathbb{P}V^*,Z) = L_{k+1}(\mathbb{P}V^*,Z').$$

We suppose $Z \neq Z'$ and choose a subscheme Z_P of Z of length N_k-1 which is not a subscheme of Z'. Since

$$dim(L_k(\mathbb{P}V^*,Z_P)) \geq \binom{n+k}{n} - N_k - 1 > 0$$

there exists a nonzero $\varphi \in L_k(\mathbb{P}V^*,Z_P)$. The sheaf $\mathcal{I}_Z/\mathcal{I}_{Z_P}$ is a skyscraper sheaf concentrated in P and it is annihilated by the maximal ideal $\mathfrak{m}_P$, so $\mathfrak{m}_P\mathcal{I}_{Z_P} \subseteq \mathcal{I}_Z$. We choose a linear form L vanishing at P but not vanishing at any subscheme of Z'. Then $L\varphi \in L_{k+1}(\mathbb{P}V^*,Z) = L_{k+1}(\mathbb{P}V^*,Z')$ and hence $\varphi \in L_k(\mathbb{P}V^*,Z')$, a contradiction since F is nondegenerate. □

3.3 Mukai's Theorem

Recall that we have the linear map

$$\bigwedge^2 V \longrightarrow \bigwedge^2 VS^{k+1}V,\ \omega \mapsto \sigma_{\omega,F}.$$

We know that for any $\varphi \in S^{k-1}V$, $G \in S^{k+1}V^*$, $\omega \in \bigwedge^2 V^*$, $\sigma_{\omega,F}(D_\varphi(F),G) = 0$. Therefore the previous map defines an injective map

$$\bigwedge^2 V \longrightarrow \bigwedge^2 AP_{k+1}(F).$$

Let $\mathcal{N} \subseteq \bigwedge^2 AP_{k+1}(F)$ be the image of this map, then $\mathcal{N}$ is a subspace of the space of the 2-forms on $AP_{k+1}(F)^*$. Let

$$\boldsymbol{G}(\binom{n+k-1}{k},AP_{k+1}(F)^*)_{\mathcal{N}} \subseteq \boldsymbol{G}(\binom{n+k-1}{k},AP_{k+1}(F)^*)$$

be the subvariety of the Grassmannian consisting of the subspaces of $\bigwedge^2 AP_{k+1}(F)$ that are isotropic with respect all the 2-forms in $\mathcal{N}$. Since $\overline{E}(\mathcal{P})$ is isotropic with respect all the 2-forms in $\mathcal{N}$ we have

$$\mathfrak{MuK}(\boldsymbol{VSP}(F,N_k)) \subseteq \boldsymbol{G}(\binom{n+k-1}{k},AP_{k+1}(F)^*)_{\mathcal{N}}.$$

We know that the map $\mathfrak{MuK}$: $\boldsymbol{VSP}(F,N_k) \longrightarrow \boldsymbol{G}(\binom{n+k-1}{n-1},AP_{k+1}(F)^*)$ is injective. Therefore we have
$dim(\mathfrak{MuK}(\boldsymbol{VSP}(F,N_k))) = dim(\boldsymbol{VSP}(F,N_k)) = (n+1)N_k - \binom{n+2k}{n} = (n+1)\binom{n+k}{n} - \binom{n+2k}{n}$.
We report in the following table the cases in which we are interested

n	k	$dim(\mathfrak{MuK}(\mathbf{VSP}(F,N_k)))$
1	$arbitrary$	1
$arbitrary$	1	$\binom{n+1}{2}$
2	2	3
2	3	2
2	4	0
3	2	5

We denote by $\boldsymbol{G} = \boldsymbol{G}(h,E)$ the Grassmannian of h-subspace of a vector space E. Recall the exact sequence on the Grassmannian

$$0 \mapsto \mathcal{S}_G \longrightarrow \mathcal{E}_G \longrightarrow \mathcal{Q}_G \mapsto 0$$

where $\mathcal{S}_G$ is the universal bundle whose fibre in $x \in \boldsymbol{G}$ is the h-subspace corresponding to x. To give a section $s\colon \boldsymbol{G} \to \mathcal{S}_G$ of $\mathcal{S}_G$ is equivalent to give h-regular function $\boldsymbol{G} \to k^h$ in fact $\mathcal{S}_G$ is locally trivial of rank h. The locus of zeros of this h regular function defines a subvariety of codimension $\leq h$ of the Grassmannian and the equality holds for a Zariski open subset of sections because generically the h functions are independent. In this way we can associate to a section $s\colon \boldsymbol{G} \to \mathcal{S}_G$ of $\mathcal{S}_G$ a subvariety of $\boldsymbol{G}$ that we denote by $\boldsymbol{Z}(s)$.
In our case the universal bundle on $\boldsymbol{G}(\binom{k+1}{1}, AP_{k+1}(F)^*)$ has rank $k+1$. A 2-form on $\mathcal{E}_G$ defines by restriction a 2-form on $\bigwedge^2 \mathcal{S}_G$ whose associated subvariety has codimension $\leq \binom{k+1}{2} = rank(\bigwedge^2 \mathcal{S}_G)$ and the equality holds for a Zariski open subset of sections.
For any h-dimensional subspace of section the locus of common zeros has codimension $\leq h\binom{k+1}{2}$ and again the equality hold for a Zariski open subset of sections.
Since $dim(\mathcal{N}) = dim(\bigwedge^2 V) = \binom{n+1}{2}$ in our case the expected codimesnion and the expected dimension for $\boldsymbol{G}(\binom{n+k-1}{k}, AP_{k+1}(F)^*)_{\mathcal{N}}$ are

$$expcodim(\boldsymbol{G}(\tbinom{n+k-1}{k}, AP_{k+1}(F)^*)_{\mathcal{N}}) = \binom{\binom{n+k-1}{k}}{2}\frac{1}{2}n(n+1);$$
$$expdim(\boldsymbol{G}(\tbinom{n+k-1}{k}, AP_{k+1}(F)^*)_{\mathcal{N}}) = \binom{n+k-1}{n-1}\binom{n+k}{n-1} - \binom{\binom{n+k-1}{k}}{2}\frac{1}{2}n(n+1).$$

For $n = 1$ $expcodim(\boldsymbol{G}(1, AP_{k+1}(F)^*)_{\mathcal{N}}) = 0$, $\boldsymbol{G}(1, AP_{k+1}(F)^*)_{\mathcal{N}} = \boldsymbol{G}(1, AP_{k+1}(F)^*)$ and $dim(\boldsymbol{G}(1, AP_{k+1}(F)^*)_{\mathcal{N}}) = 1$.
For $n = 2$ we have $expcodim(\boldsymbol{G}(\binom{k+1}{k}, AP_{k+1}(F)^*)_{\mathcal{N}}) = 3\binom{k+1}{2}$ and

$$expdim(\boldsymbol{G}(\tbinom{k+1}{k}, AP_{k+1}(F)^*)_{\mathcal{N}}) = 3\binom{k+1}{2} = (k+1)(k+2) - 3\binom{k+1}{2} = \frac{1}{2}(1+k)(4-k).$$

In the cases $k = 1,2,3,4$ we have

k	$expdim(\mathbf{G}(\binom{k+1}{k}, AP_{k+1}(F)*)_{\mathcal{N}})$
1	3
2	3
3	2
4	0

We see that the expected dimension of $\boldsymbol{G}(\binom{n+k-1}{k}, AP_{k+1}(F)^*)_{\mathcal{N}}$ is equal to the dimension of $\boldsymbol{VSP}(F,N_k)$ in the cases $n = 1$ and $n = 2$, $k = 1,2,3,4$. In all other cases it is strictly less.

THEOREM 10. *(<u>S. Mukai</u>) Let $F \in S^d V^*$ be a generic polynomial of degree $d = 2k$. We assume $n = 3$ and $k \leq 4$. Then*

$$\boldsymbol{VSP}(F,N_k) = \boldsymbol{VSP}(F,\tbinom{k+2}{2}) \cong \boldsymbol{G}(\tbinom{k+1}{k}, AP_{k+1}(F)^*)_{\mathcal{N}} = \boldsymbol{G}(k+1, AP_{k+1}(F)^*)_{\mathcal{N}}.$$

M1 *If $n = 2$ and $k = 1$ then $d = 2$, $N_k = 3$ and $dim(\boldsymbol{VSP}(F_2,3)) = 3$.*
The variety $\boldsymbol{VSP}(F_2,3)$ is a smooth Fano 3-fold of genus 21 and index 2.

M2 *If $n = 2$ and $k = 2$ then $d = 4$, $N_k = 6$ and $dim(\boldsymbol{VSP}(F_2,3)) = 3$.*
The variety $\boldsymbol{VSP}(F_4,6)$ is a smooth Fano 3-fold of genus 12 and index 1.

M3 *If $n = 2$ and $k = 3$ then $d = 6$, $N_k = 10$ and dim($\boldsymbol{VSP}$(F_2,3)) = 2.*
The variety $\boldsymbol{VSP}$(F_6,10) is a smooth K3 surface.

M4 *If $n = 2$ and $k = 4$ then $d = 8$, $N_k = 15$ and dim($\boldsymbol{VSP}$(F_2,3)) = 0.*
The variety $\boldsymbol{VSP}$(F_4,6) is a set of 16 points.

<u>*Proof*</u>: We know that for $n = 2$ the varieties $\boldsymbol{VSP}(F,\binom{k+2}{2})$ are irreducible and smooth. We compute their dimensions

k	$dim(\mathbf{VSP}(F,\binom{k+2}{2}))$
1	3
2	3
3	2
4	0

Via the Mukai map we are associating to each polyhedra $\mathcal{P} \in \boldsymbol{VSP}(F,\binom{k+2}{2})$ a point in $\boldsymbol{G}(k+1,(AP_{k+1}F)^*)$. We have $dim((AP_{k+1}F)^*) = 2k + 3$ and $dim(\boldsymbol{G}(k+1,(AP_{k+1}F)^*)) = (k + 3)(k + 1)$. In our cases

k	$dim((AP_{k+1}\mathrm{F})^*)$	$\mathbf{G}(k+1,(AP_{k+1}\mathrm{F})^*)$	$dim(\mathbf{G}(k+1,(AP_{k+1}\mathrm{F})^*))$
1	5	$\mathbf{G}(2,5)$	6
2	7	$\mathbf{G}(3,7)$	12
3	9	$\mathbf{G}(4,9)$	20
4	11	$\mathbf{G}(5,11)$	30

A basis for a 3-dimension space $\mathcal{N}$ of sections of $\mathcal{E} = \bigwedge^2\mathcal{S}_G{}^*$ defines a section of the vector bundle $\mathcal{E}^{\oplus 3} = \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{E}$. The bundle $\mathcal{E}$ is generated by global section and by Bertini theorem on sections of a vector bundle we know that a generic section of $\mathcal{E}$ is smooth.
Therefore the locus of zeros $\boldsymbol{Z}(s)$ of a generic section s of $\mathcal{E}$ is a smooth subvariety of $\boldsymbol{G}(k+1,(AP_{k+1}F)^*)$ and its codimesnion in equal to $3\binom{k+1}{2} = \frac{3}{2}k(k+1)$. We compute

$$dim(\boldsymbol{Z}(s)) = \binom{n+k-1}{n-1}\binom{n+k}{n-1} - \frac{3}{2}k(k+1) = \frac{1}{2}(1+k)\text{-}(4\text{-}k).$$

We assume $k \leq 4$ and so $dim(\boldsymbol{Z}(s)) \geq 0$.
The normal bundle $\mathcal{N}_{\boldsymbol{Z}(s),G}$ is isomorphic to $\mathcal{E}^{\oplus 3}$. It is know that the determinant of the tangent bundle of $\boldsymbol{G} = \boldsymbol{G}(h,N)$ is given by

$$c_1(\boldsymbol{G}) = Nc_1(\mathcal{S}_G{}^*)$$

and that the determinant of $\bigwedge^2\mathcal{S}_G$ is given by

$$c_1(\textstyle\bigwedge^2\mathcal{S}_G) = (h\text{-}1)c_1(\mathcal{S}_G{}^*).$$

In our case we have $N = dim((AP_{k+1}F)^*) = 2k + 3$, $h = k + 1$ and

$$c_1(\mathcal{E}^{\oplus 3}) = 3c_1(\mathcal{E}) = 3c_1(\textstyle\bigwedge^2\mathcal{S}_G{}^*) = 3(k + 1 \text{ - } 1)c_1(\mathcal{S}_G{}^*) = 3kc_1(\mathcal{S}_G{}^*);$$
$$c_1(\boldsymbol{G}) = (2k + 3)c_1(\mathcal{S}_G{}^*).$$

By adjunction formula we have $K_{\boldsymbol{Z}(s)} = K_{\boldsymbol{G}} + det(\mathcal{N}_{\boldsymbol{Z}(s),G})$ i.e. on the Chern classes we have

$$c_1(\boldsymbol{Z}(s)) = c_1(\boldsymbol{G}) + c_1(\mathcal{E}^{\oplus 3}) = (2k + 3 - 3k)c_1(\mathcal{S}_G{}^*) = (3-k)c_1(\mathcal{S}_G{}^*).$$

We write $c_1(\boldsymbol{Z}(s)) = (3\text{-}k)c_1(\mathcal{O}_{\boldsymbol{Z}(s)}(1))$, where $\mathcal{O}_{\boldsymbol{Z}(s)}(1)$ is the restriction of $det(\mathcal{S}_G{}^*)$ on $\boldsymbol{Z}(s)$. We note that

$$\mathcal{O}_{\boldsymbol{Z}(s)}(1) \cong det(\mathcal{S}_G{}^*) \otimes \mathcal{O}_{\boldsymbol{Z}(s)}$$

is the sheaf associated to the *Plücker embedding* of the Grassmannian and the global sections of $\mathcal{O}_{\boldsymbol{Z}(s)}(1)$ are the hyperplane sections of the Grassmannian in its *Plücker embedding*. Therefore $\mathcal{O}_{\boldsymbol{Z}(s)}(1)$ is ample.
If $k < 3$ then $c_1(\boldsymbol{Z}(s)) = (3\text{-}k)c_1(\mathcal{O}_{\boldsymbol{Z}(s)}(1))$ with $3\text{-}k > 0$. Then the anticanonical sheaf is ample and $\boldsymbol{Z}(s)$ is a smooth Fano 3-fold.
If $k = 3$ then $c_1(\boldsymbol{Z}(s)) = 0$ and the canonical sheaf is trivial. Then $\boldsymbol{Z}(s)$ is a smooth $K3$ surface.
If $k = 4$ then $dim(\boldsymbol{Z}(s)) = 0$ and the rank of $\mathcal{E}^{\oplus 3}$ is given by

$$rank(\mathcal{E}^{\oplus 3}) = 3 \cdot rank(\mathcal{E}) = 3\binom{k+1}{2} = 30 = dim(\boldsymbol{G}(5,11)).$$

The generic section of $\mathcal{E}^{\oplus 3}$ vanishes on a finite number of points equal to the Chern number $c_{30}(\mathcal{E}^{\oplus 3}) = 16$. □

Chapter 4

A New Viewpoint on VSP

In this chapter we state some new results in varieties of power sums theory. First we prove by geometrical methods Hilbert and Sylvester theorems. Then we give an alternative proof of Dolgachev - Kanev theorem and with the same method we will find that ***VSP***$(F_2, 4)$ is a Grassmannian. Furthermore we give a method to reconstruct a 4-polar polyhedron for a plane cubic. Finally we prove a theorem on varieties of power sums rationality.

Remark 6. *(Partial Derivatives) Let $\{L_1, ..., L_h\}$ be a h-polar polyhedron for the homogeneus polynomial $F \in k[x_0, ..., x_n]_d$. We write*

$$F = \lambda_1 L_1^d + ... + \lambda_h L_h^d.$$

The partial derivatives of F are homogeneus polynomials of degree $d-1$ decomposed in h linear factors

$$F_{x_i} = \lambda_1 \alpha_{i_1} d L_1^{d-1} + ... + \lambda_h \alpha_{i_h} d L_h^{d-1}, \text{ for any } i = 0, ..., n.$$

Then $\boldsymbol{VSP}(F,h)^o \subseteq \boldsymbol{VSP}(F_{x_i},h)^o$, taking clousures we have $\boldsymbol{VSP}(F,h) \subseteq \boldsymbol{VSP}(F_{x_i},h)$. The polynomial F has $\binom{n+l}{l}$ partial derivatives of order l. Cleary these derivatives are homogeneus polynomials of degree $d-l$ decomposed in h-linear factors. Then we have $\boldsymbol{VSP}(F,h) \subseteq \boldsymbol{VSP}(F_{x_1^{l_1},...,x_n^{l_n}},h)$, where $l_1 + ... + l_n = l$.

Remark 7. *(Projections) Let $H \subseteq \mathbb{P}^N$ be a l-plane. We consider a $(N-l-1)$-plane E such that $H \cap E = \varnothing$. Then any $l+1$ - plane containing H intersects E in a point. Conversely for any point $p \in E$ is uniquely determined an $l+1$ - plane $< p, H >$ containing H. We con project $\mathbb{P}^N$ in $E \cong \mathbb{P}^{N-l-1}$ via the rational map*

$$\pi : \mathbb{P}^N \setminus H \dashrightarrow \mathbb{P}^{N-l-1}, \text{ defined by } p \mapsto < p, H > \cap E.$$

4.1 Hilbert's and Sylvester's Theorems

In this section we study two cases where the variety of power sums is a single point. We will give two proofs for the Hilbert's theorem.

Theorem 11. *(Hilbert) The variety of power sums **VSP**$(F_5, 7)$, parameterizing all decompositions in seven linear factors of a homogeneous quintic polynomial in three variables, is a point.*

Proof:

1. We consider F_5 as point in $\mathbb{P}^{20}$. We have the Veronese embedding

$$\nu_5 : \mathbb{P}^2 \longrightarrow \mathbb{P}^{20}$$

whose image is the Veronese variety $V = V^2_{25}$. By Alexander-Hirschowitz's theorem we know that he variety of secant *6*-plane of V has dimension

$$dim(Sec_6(V)) = 7 \cdot 2 + 7 - 1 = 20$$

so any quintic homogeneous polynomial in three variables admits a decomposition in seven linear factors.
Let $\{l_1,...,l_7\}$ be a polar *7*-polyhedron of F_5. We write

$$F_5 = \lambda_1 l_1^5 + ... + \lambda_7 l_7^5.$$

The partial derivatives of F_5 are homogeneous polynomials of degree four and the second partial derivatives of F_5 are homogeneous polynomials of degree three. By Schwarz theorem the second mixed derivatives are equal so we have six second partial derivatives of F_5 that we denote by F_{xx}, F_{yy}, F_{zz}, F_{xy}, F_{xz}, F_{yz}.
The second partial derivatives are decomposed in the seven linear factor $l_1,...,l_7$.
Now we look at the $\mathbb{P}^9$ parameterizing the homogeneous polynomial of degree three in x,y,z. We consider the Veronese embedding

$$\nu_3 : \mathbb{P}^2 \longrightarrow \mathbb{P}^9, \text{ with } V' = V^2_9 = \nu_3(\mathbb{P}^2).$$

In $\mathbb{P}^9$ we have the seven points $l_1^3,...,l_7^3 \in V'$, the $\mathbb{P}^5$ spanned by the second partial derivatives and denoted by $H^5_\partial = <F_{xx},...,F_{yz}>$. Moreover we have the $\mathbb{P}^6$ spanned by $l_1^3,...,l_7^3$ that contains H^5_∂, we denote it by $H^6_l = <l_1^3,...,l_7^3>$.
Now we suppose that there is a second decomposition of F_5 in seven linear factors

$$F_5 = \eta_1 L_1^5 + ... + \eta_7 L_7^5.$$

This gives rise to a second decomposition for the second partial derivatives in the factors $L_1^3,...,L_7^3$. In $\mathbb{P}^9$ we have the $\mathbb{P}^6$ spanned by $L_1^3,...,L_7^3$ that contains H^5_∂, we denote it by $H^6_L = <L_1^3,...,L_7^3>$. Since

$$dim(H^6_l) + dim(V') = dim(H^6_L) + dim(V') = 8 < 9$$

the 6 - planes H^6_l, H^6_L intersects V' exactly in the points L_i^3 and l_i^3 respectively, and since there exist i, j such that $L_i \neq l_j$, we have that $H^6_l \neq H^6_L$.
Moreover H^5_∂ does not intersect V' since $dim(V') + dim(H^5_\partial) < 9$. We project $\mathbb{P}^9$ in $\mathbb{P}^3$ via the *6*-planes containing H^5_∂. We denote the projection by

$$\pi : \mathbb{P}^9 \backslash H^5_\partial \dashrightarrow \mathbb{P}^3.$$

The variety $\overline{V} = \pi(V^{'})$ is a surface in $\mathbb{P}^3$ with $deg(\overline{V}) = 9$. The projections of H^6_l and H^6_L determine on $\overline{V}$ two points $x,y \in \overline{V}$ of multiplicity 7. We consider the line $R = <x,y>$ that intersect $\overline{V}$ with multiplicity at least 14, but $deg(\overline{V}) = 9$ implies that $R \subseteq \overline{V}$.
The line R determine a 7-plane H^7 in $\mathbb{P}^9$ whose intersection with $V^{'}$ contains a curve $\Gamma \subseteq H^7 \cap V^{'}$. The 7-plane H^7 contains H^6_l and H^6_L so we can estimate $deg(\Gamma)$ intersecting with H^6_l. We have $\Gamma \cdot H^6_l \leq 7$.
We know that any curve in $V^{'}$ has degree multiple of three, so we have only two possibilities

$$deg(\Gamma) = 3 \text{ or } deg(\Gamma) = 6.$$

We write $H^7 = H^8_1 \cap H^8_2$ as intersection of two hyperplanes. Then $H^8_1 \cap V^{'} = X_1$ and $H^8_2 \cap V^{'} = X_2$ are curves of degree 9 with Γ as a common component. The curves $X_1,\ X_2$ corresponds in $\mathbb{P}^2$ to two cubic curves $C_1,\ C_2$ with a common irreducible component $\overline{\Gamma}$. We have two cases:

- $C_1 = \overline{\Gamma} \cup K_1$ and $C_2 = \overline{\Gamma} \cup K_2$ with $K_1,\ K_2$ conics.
- $C_1 = \overline{\Gamma} \cup R_1$ and $C_2 = \overline{\Gamma} \cup R_2$ with $R_1,\ R_2$ lines.

In the first case $\Gamma = \nu_3(\overline{\Gamma})$ is a twisted cubic curve contained in H^7 and

$$H^6_l \cdot \Gamma = H^6_L \cdot \Gamma = 3,$$

say $H^6_l \cap \Gamma = \{l^3_1, l^3_2, l^3_3\}$ and $H^6_L \cap \Gamma = \{L^3_1, L^3_2, L^3_3\}$. The image of K_1 via ν_3 is a curve of degree 6, $\overline{K_1}$ that passes through $\{l^3_4, l^3_5, l^3_6, l^3_7\}$ and intersects Γ in $\overline{\Gamma}_1 = 2$ points. Similarly the image of K_2 via ν_3 is a curve of degree 6, $\overline{K_2}$ that passes through $\{L^3_4, L^3_5, L^3_6, L^3_7\}$. Now the set $\{l^3_4, l^3_5, l^3_6, l^3_7\}$ is contained in the hyperplane section $H^8_2 \cap V^{'} = \Gamma \cup \overline{K_2}$. Conversely the set $\{L^3_4, L^3_5, L^3_6, L^3_7\}$ is contained in the hyperplane section $H^8_1 \cap V^{'} = \Gamma \cup \overline{K_1}$. But $\overline{K_1}$ and $\overline{K_1}$ intersect in exactly $K_1 \cdot K_2 = 4$ points and so $\{L^3_4, L^3_5, L^3_6, L^3_7\} = \{l^3_4, l^3_5, l^3_6, l^3_7\}$. In particular there are four points on $H^6_l \cap H^6_L$ that don't lie in H^5_∂.
In the second case $\Gamma = \nu_3(\overline{\Gamma})$ is a rational normal curve of degree 6 and

$$H^6_l \cdot \Gamma = H^6_L \cdot \Gamma = 6.$$

Then the images of R_1 and R_2 are two conics $\overline{R_1}$ and $\overline{R_2}$ that passes through the remaining points say l^3_7 and L^3_7 respectively. We note that L^3_7 is in the hyperplane section $H^8_1 \cap V^{'} = \Gamma \cup \overline{R_1}$ and l^3_7 is in the hyperplane section $H^8_2 \cap V^{'} = \Gamma \cup \overline{R_2}$. Since $\overline{R_1}$ and $\overline{R_2}$ intersect in $R_1 \cdot R_2 = 1$ point, we have $l^3_7 = L^3_7$ and we find a point on $H^6_l \cap H^6_L$ that don't lie in H^5_∂.
In any case we find a point on $H^6_l \cap H^6_L$ that does not lie on H^5_∂ because it lies on $V^{'}$. So $H^6_l = H^6_L$, a contradiction. $\square$

2. The partial derivatives of F_5 are three homogeneous polynomials of degree four F_x,F_y,F_z decomposed in seven factors. We consider the Veronese embedding

$$\nu_4{:}\mathbb{P}^2 \rightarrow \mathbb{P}^{14}, \text{ with } V = V_{16}^2 = \nu_4(\mathbb{P}^2).$$

We suppose to have two different *7*-polar polyhedra $\{l_1,...,l_7\}$ and $\{L_1,...,L_7\}$. In $\mathbb{P}^{14}$ we have the two *6*-plane H_l^6 and H_L^6, the *2*-plane $H_\partial^2 = <F_x,F_y,F_z>$ and the k-plane $H^k = <H_l^6,H_L^6>$.
Let H^{13} be a generic hyperplane in $\mathbb{P}^{14}$ containing H^k. Then $H^{13} \cap V = \Gamma$ is a curve of degree *16*. Now we prove that the intersection $H^{13} \cap V$ is transversal and so the curve Γ is smooth.
By Bertini's theorem if the generic hyperplane containing H^k has non transversal intersection in x at V then $x \in H^k$. Let $\{x_1,...,x_s\} = H^k \cap V$ such that $H^k \supseteq T_{x_i}V$ for any i=*1,...,*l. Let H^{12} be a *12*-plane contained in H^{13} such that $H^{12} \supseteq H^k$ then $H^{12} \cdot V = 16$. Now the x_i have multiplicity at least two for any i=*1,...,*l and so $H^{12} \cdot V = 16 \geq 12-k+s+l$. Now s=*14* implies k=*10* and *16*$\geq$*12-10+14+*l implies l=*0*. When s decreases of one also k decreases of one and so l is constant and equal to zero.
Then Γ is smooth and corresponds to a smooth quartic curve in $\mathbb{P}^2$, so $g(\Gamma) = 3$.
Then $H^k \cdot \Gamma \leq 14$. Let Π be a hyperplane in H^{13} such that $H^k \subseteq \Pi$. We have $\Pi \cdot \Gamma = 16$. Let Δ be the linear system determined on Γ by the hyperplanes in H^{13} containing H^k, we have

$$\Delta = H^k \cdot \Gamma + \{\textit{12-k points}\} = H^k \cdot \Gamma + g_{12-k}^{12-k}.$$

In fact the family of the hyperplanes in $\mathbb{P}^{13}$ containing a fixed k-plane have dimension *13-k-1 = 12-k*. Now we have a divisor D on Γ with $deg(D) = 12\text{-}k$ and $dim(H^0(\Gamma,\mathcal{O}_\Gamma(D))) = 13\text{-}k$. By Riemann-Roch theorem on the divisor D we have

$$h^0(D) - h^0(K_\Gamma - D) = deg(D)+1-g(\Gamma) = 12\text{-}k+1\text{-}3 = 10\text{-}k.$$

Now $h^0(D) = 13\text{-}k$ implies $h^0(K_\Gamma - D) = 3$. For the canonical divisor K_Γ we have that K_Γ is the divisor associated to the sheaf

$$\mathcal{O}_{\mathbb{P}^2}(-3+4) \otimes \mathcal{O}_\Gamma = \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_\Gamma.$$

In other words K_Γ is the class of divisors determined on Γ by the lines of $\mathbb{P}^2$. We write $D = p+q$, the K_Γ - D is the class of divisor in K_Γ vanishing on D so is the divisor cut on Γ by the line $<p,q>$ and $h^0(K_\Gamma - D) = 1$, a contradiction. □

Theorem 12. *(Sylvester) Let* $F_3 = F_3(x,y,z,w)$ *be a homogeneous polynomial of degree three. The variety of power sums* ***VSP***$(F_3,5)$*, parameterizing all decompositions in five linear factors of a homogeneous cubic polynomial in four variables, is a point.*

Proof: The polynomial F_3 is a point in $\mathbb{P}^{19}$. We consider the Veronese variety $V = V_{27}^3$ parameterizing the *3*-powers of linear factor on $\mathbb{P}^3$ and its variety of secant *4*-planes. We have

$$dim(Sec_4(V)) = 5\cdot 3+5-1 = 19.$$

So the generic cubic polynomial on $\mathbb{P}^3$ admits a decomposition in five linear factors.
We suppose that there are two different 5-polar polyhedra $\{l_1,\ldots,l_5\}$ and $\{L_1,\ldots,L_5\}$ for F_3. So we have two different decomposition for the partial derivatives of F_3. The partial derivatives of F_3 are four quadric polynomials decomposed in five linear factors in two different ways.
The partial derivatives generate a $\mathbb{P}^3$ denoted by $H^3_\partial = <F_x,F_y,F_z,F_w>$ in the $\mathbb{P}^9$ parameterizing the quadric polynomials. We consider the Veronese embedding

$$\nu_2{:}\mathbb{P}^3 \rightarrow \mathbb{P}^9 \text{ with } V^{'} = V^3_8 = \nu_2(\mathbb{P}^3).$$

The two 5-polar polyhedra are two sets of five points on $V^{'}$ that generate two $\mathbb{P}^4$ denoted by

$$H^4_l = <l^2_1,\ldots,l^2_5> \text{ and } H^4_L = <L^2_1,\ldots,L^2_5>.$$

The 4-planes H^4_l and H^4_L both contain the 3-plane H^3_∂.
We project $\mathbb{P}^9$ in $\mathbb{P}^5$ via the 4-planes containing H^3_∂. We have a well defined map

$$\pi{:}\mathbb{P}^9 \backslash H^3_\partial \dashrightarrow \mathbb{P}^5.$$

In $\mathbb{P}^5$ we have the 3-fold $\bar{V}^{'} = \pi(V^{'})$ of degree 8. On $\bar{V}^{'}$ we have two 5-fold points $x = \pi(H^4_l)$ and $y = \pi(H^4_L)$. The line $R = <x,y>$ intersects $\bar{V}^{'}$ with multiplicity 10 and $deg(\bar{V}^{'}) = 8$ implies that R is contained in $\bar{V}^{'}$.
Now $\pi^{-1}(R) = H^5 \cong \mathbb{P}^5$ and we have a curve $\Gamma \subseteq H^5 \cap V^{'}$ corresponding to the line R. We note that Γ, H^4_l and H^4_L are contained in $H^5 \cong \mathbb{P}^5$. So we can estimate $deg(\Gamma)$ intersecting it with H^4_l. We have

$$\Gamma \cdot H^4_l \leq V^{'} \cdot H^4_l = 5.$$

So $deg(\Gamma) \leq 5$, but the curves in $V^{'}$ are all of even degree and we have only two possibilities

$$deg(\Gamma) = 2 \text{ or } deg(\Gamma) = 4.$$

- We suppose $deg(\Gamma) = 2$. Then $\Gamma \cdot H_l = \Gamma \cdot H_L = 2$ and we can assume

$$H^5 \cap V^{'} \supseteq \Gamma \cup l^2_1 \cup l^2_2 \cup l^2_3 \cup L^2_1 \cup L^2_2 \cup L^2_3.$$

  Now we consider the linear system $|\mathcal{I}_{H^5}(1)|$ of the hyperplanes in $\mathbb{P}^9$ containing H^5. Then $dim(|\mathcal{I}_{H^5}(1)|) = 9\text{-}5\text{-}1 = 3$. Any hyperplane of the linear system $|\mathcal{I}_{H^5}(1)|$ cuts a surface of degree 8 on $V^{'}$ that corresponds to a quadric surface of $\mathbb{P}^3$ containing the line $X = \nu_2^{-1}(\Gamma)$ and the points l_i, L_i for $i = 1,2,3$. In this way we get a linear system of quadrics $\Lambda \subseteq |\mathcal{O}_{\mathbb{P}^3}(2)|$ and $dim(\Lambda) = 3$, we write $\Lambda = <Q_1,\ldots,Q_4>$. Suppose that all the quadric in Λ are singular, the singular locus is contained in the base locus. If L_1 is a singular point then all the lines $< L_1, L_j >$, $< L_1, l_j >$ are contained in any quadric of Λ and so are in the base locus, a contradiction. If any quadric in Λ has a singular point on X then X and the singular point impose 4 conditions, if we impose to the quadrics to contain the L_i, l_i the dimension of Λ becomes smaller that 3, a contradiction. Then Λ contains a smooth quadric and so the generic quadric in Λ is smooth.

Now $X = \nu_2^{-1}(\Gamma)$ is a line and $Q_1 \cap Q_2 = X \cup R$ where R is a twisted cubic curve. On a quadric X it is a divisor of type *(1,0)* and $X \cup R$ is of type *(2,2)* so R is of type *(a,b)* where *(1+a,b) = (2,2)*. We conclude that *(a,b) = (1,2)* and $R \cdot X = 2$. Then $Q_3 \cdot R = 6$ and we have two points of R on X, so $Q_1 \cap Q_2 \cap Q_3 = X \cup$ *{4 points}*. A contradiction because we have $\Gamma \cup$*{6 points}* in the base locus.

- Now we suppose $deg(\Gamma) = 4$. Then $\Gamma \cdot H_l = \Gamma \cdot H_L = 4$ and we have $H^5 \cap V' \supseteq \Gamma \cup l_1 \cup L_1$. We consider the linear system $|\mathcal{I}_{H^5}(1)|$ of the hyperplanes in $\mathbb{P}^9$ containing H^5. Then $dim(|\mathcal{I}_{H^5}(1)|) = 9\text{-}5\text{-}1 = 3$. The linear system $|\mathcal{I}_{H^5}(1)|$ gives a linear system of quadrics $\Lambda \subseteq |\mathcal{O}_{\mathbb{P}^3}(2)|$ and $dim(\Lambda) = 3$ and we write $\Lambda = <Q_1,...,Q_4>$, as in the preceding point the generic quadric in Λ is smooth. Now $X = \nu_2^{-1}(\Gamma)$ is a conic and $Q_1 \cap Q_2 = X \cup R$ where R is conic. On a quadric X it is a divisor of type *(1,1)* and $X \cup R$ is of type *(1,1)* and so $R \cdot X = 2$. Then $Q_3 \cdot R = 4$ and we have two points of R on X, so $Q_1 \cap Q_2 \cap Q_3 = X \cup l_1 \cup L_1$. Finally we have $Q_1 \cap Q_2 \cap Q_3 \cap Q_4 = X \cup l_1 \cup L_1$, the intersection with Q_4 does not change the base locus and Q_4 is in the span of Q_1, Q_2, Q_3, a contradiction.

□

Using polar forms Sylvester's theorem can be proved in another simple and beautiful way. I thank *Giorgio Ottaviani* who suggested me the sketch of this proof.

Proof: Let $F = F_3 \in \mathbb{P}^9$ be a homogeneous form of degree three. We know that a 5-polar polyhedron of F exists. The polar form of F in a point $\xi = [\xi_0 : \xi_1 : \xi_2 : \xi_3] \in \mathbb{P}^3$ is the quadric

$$P_\xi F = \xi_0 \tfrac{\partial F}{\partial x_0} + \xi_1 \tfrac{\partial F}{\partial x_1} + \xi_2 \tfrac{\partial F}{\partial x_2} + \xi_3 \tfrac{\partial F}{\partial x_3}.$$

Let $\{L_1, ..., L_5\}$ be a 5-polar polyhedron of F, then $F = L_1^3 + ... + L_5^3$. The polar form is of the type

$$P_\xi F = \textstyle\sum_{i=1}^5 \xi_i \lambda_i L_i^2$$

and it has rank 2 on the points $\xi \in \mathbb{P}^3$ on which three of the linear form L^i vanish simultaneously. These points are $\binom{5}{3} = 10$.
Now we consider the subvariety X_2 of $\mathbb{P}^9$ parametrizing the quadrics of rank 2. A quadric Q of rank 2 is the union of two plane, the planes of $\mathbb{P}^3$ are parametrized by $\mathbb{P}^{3*}$), then $dim(X_2) = 6$. To find the degree of X_2 we have to intersect with a 3-plane, that is intersection of 6 hyperplanes. So the degree of X_2 is equal to the number of quadrics of rank 2 passing through 6 general points of $\mathbb{P}^3$. If we choose three points then the plane through these points is determined, and also the quadric is determined. Then these quadric are $\frac{1}{2}\binom{6}{3} = 10$. We have seen that $dim(X_2) = 6$ and $deg(X_2) = 10$.
Now the linear space

$$\Gamma = \{P_\xi F | \xi \in \mathbb{P}^3\} \subseteq \mathbb{P}^9$$

is clearly a 3-plane in $\mathbb{P}^9$.
Then $\Gamma \cap X_2 = \{P_\xi F | rank(P_\xi F) = 2\}$ is a set of 10 points. These points have to be the

10 points we have found in the first part of the proof. Then the decomposition of F in five linear factor is unique. $\square$

4.2 Dolgachev - Kanev's Theorem

In this section we study some cases of varieties of power sums that are not single points. We will recover some well known varieties as the projective plane $\mathbb{P}^2$ and the Grassmannian of lines ***G(1,4)***.

4.2.1 Conics

We study the variety of power sums of a homogeneous polynomial of degree two in three variables decomposed in four linear factors. We give an explicit method to find all *4*-polar polyhedra of a given quadratic polynomial.

THEOREM 13. *The variety* ***VSP***$(F_2,4)$, *parameterizing the decomposition of a homogeneous polynomial of degree 2 in 3 variables in 4 linear factors, is birational to the Grassmannian* ***G****(2,4).*

Proof: We consider the Veronese variety $V = V_4^2 \subseteq \mathbb{P}^5$ and $F_2 \in \mathbb{P}^5$ as a point. Any *4*-polar polyhedron $\{L_1, ..., L_4\}$ of F_2 determines the *4* points $L_1^2, ..., L_4^2 \in V$ which span a *3*-plane $H_L = <L_1^2, ..., L_4^2>$. In this way we get the morphism

$$\psi\colon \mathbf{VSP}(\mathrm{F}_2,4)\longrightarrow\mathbf{G}(3,5), \text{ defined by } \{L_1, ..., L_4\}\mapsto H_L.$$

Now a generic *3*-plane in $\mathbb{P}^5$ intersects V in exactly *4* points counted with multiplicity, then the morphism φ is generically injective. We note that any *3*-plane spanned by a *4*-polyhedron passes trough the point F_2. Then the image of φ is contained in the subvariety $\mathbf{G}(3,5,F_2) \subseteq \mathbf{G}(3,5)$, whose points are the *3*-planes passing through F_2. We know that $\mathbf{G}(3,5,F_2)$ is isomorphic to the Grassmannian $\mathbf{G}(2,4)$. We get a generically injective morphism

$$\psi\colon \mathbf{VSP}(\mathrm{F}_2,4)\longrightarrow\mathbf{G}(2,4), \text{ defined by } \{L_1, ..., L_4\}\mapsto H_L.$$

We know that $\mathbf{VSP}(\mathrm{F}_2,4)$ and $\mathbf{G}(2,4)$ are both smooth. Furthermore

$$dim(\mathbf{VSP}(F_2, 4)) = 12 - 6 = 6 \text{ and } dim(\mathbf{G}(2, 4)) = 6.$$

Then ψ is a generically injective between two smooth varieties of the same dimension, we conclude that it is birational map and $\mathbf{VSP}(\mathrm{F}_2,4)$ is birational to $\mathbf{G}(2,4)$. $\square$

REMARK 8. *In the preceding proposition we associate to a conic F the Grassmannian* $\boldsymbol{G}(2, H_F)$, *where H_F is the hyperplane in $\mathbb{P}^{5*}$ dual to the point F. Clearly in this construction every hyperplane H gives the varieties of power sums* $\boldsymbol{VSP}(H^*, 4)$ *of $F = H^*$, that is* $\boldsymbol{G}(2, H)$. *We want to understand when the conic associated to an hyperplane H is*

singular. The answer is the following.
If H is an hyperplane in $\mathbb{P}^{5*}$ then the Grassmannian $\boldsymbol{G}(2,H)$ is the varieties of power sums of a singular conic if and only if there exists a 3-plane $\Lambda \subseteq H$ such that

$$\Lambda = (L_1^2)^* \cap (L_2^2)^*.$$

In fact in this case Λ^* is a line passing through $H^* = F$ and $\Lambda^* =< L_1^2, L_2^2 >$, then F can be written has sum of two squares and it is singular.

We can interpret the preceding construction in another way. The *3*-planes containing F are the lines in the hyperplane F^*, then $\mathbf{VSP}(F_2, 4)$ is isomorphic to $\mathbf{G}(1,4)$, that indeed is isomorphic to $\mathbf{G}(2,4)$. Then any hyperplane H in $\mathbb{P}^5$ determines the varieties of power sums of the polynomial H^* by $\mathbf{VSP}(H^*,4) \cong \mathbf{G}(1,H)$. Fixed an hyperplane H in $\mathbb{P}^5$ it is easy to reconstruct the corresponding polynomial that is simply H^*.

EXAMPLE 15. *If we consider the hyperplane $\xi_0 + \xi_1 + \xi_2 - 2\xi_3 - \xi_4 + \xi_5 = 0$ then the corresponding polynomial is* $[1:1:1:-2:-1:1]$ *i.e.* $F = x^2 + y^2 + z^2 - 2xy - xz + yz$.

By apolarity lemma we know that if F is a homogeneous polynomials of degree *2*, $\{L_1, ..., L_4\}$ is a *4*-polar polyhedron of F if and only if

$$L_2(\mathbb{P}V^*, [L_1], ..., [L_4]) \subseteq AP_2(F)$$

and the inclusion is no more true if we delete one of the L_i.
Now $AP_2(F)$ is the kernel of the linear map

$$ap_F^2 : S^2V \to k, \varphi \mapsto D_\varphi F.$$

By dimension theorem $dim_k(AP_2(F)) = 6 - 1 = 5$ i.e. $\mathbb{P}(AP_2(F))$ is an hyperplane in $\mathbb{P}(S^2V) \cong \mathbb{P}^5$.
Let R be a line contained in $\mathbb{P}(AP_2(F))$. The line R determines a pencil of conics, by apolarity lemma we know that *4*, counted with multiplicity, base points of this pencil are a *4*-polar polyhedron of F if and only if deleting one of the be base points, say L_4, the plane of conics $L_2(\mathbb{P}V^*, [L_1], ..., [L_3])$ is contained in $\mathbb{P}(AP_2(F))$. But the lines in $\mathbb{P}(AP_2(F))$ are parametrized by $G(1, \mathbb{P}(AP_2(F)))$ that has dimension *6*, and also $\mathbf{VSP}(F,4)$ has dimension *6*, so any line in $\mathbb{P}(AP_2(F))$ determines a *4*-polar polyhedron of F.

REMARK 9. *By the preceding argumentation we can give another proof of theorem 13. Let $\{L_1, ..., L_4\}$ be a 4-polar polyhedron of F, we can consider the pencil of conics*

$$L_2(\mathbb{P}V^*, [L_1], ..., [L_4]) \subseteq AP_2(F).$$

We get an injective morphism

$$\varphi : \boldsymbol{VSP}(F,4) \longrightarrow \boldsymbol{G}(1, \mathbb{P}(AP_2(F))), \{L_1, ..., L_4\} \mapsto L_2(\mathbb{P}V^*, [L_1], ..., [L_4]).$$

Since $dim(\boldsymbol{VSP}(F,4)) = dim(\boldsymbol{G}(1, \mathbb{P}(AP_2(F)))) = 6$, and since both the varieties are smooth φ has to be an isomorphism.
This interpretation allow us to write explicitly an inverse morphism. Let H be an hyperplane in $(\mathbb{P}^5)^*$ then $\boldsymbol{G}(1,H)$ is the variety of power sums of $F = H^*$. Take a line

$R \in \boldsymbol{G}(1,H)$, choose two conics K_1, K_2 in the pencil determined by R, compute the intersection $K_1 \cdot K_2$. The 0-subscheme $K_1 \cdot K_2$ of length 4 is a 4-polar polyhedron of F. In this notation the morphism

$$\psi : \boldsymbol{G}(1,H) \longrightarrow \boldsymbol{VSP}(F,4), R \mapsto K_1 \cdot K_2$$

is the inverse of φ.
In this way we get a direct method to construct all 4-polar polyhedra of a given homogeneous polynomial of degree 2.

We give an explicit example

EXAMPLE 16. *Consider the polynomial $F = x^2+2y^2-z^2+4xy-xz+yz$. The differential operator associated to a homogeneous polynomial φ of degree 2 is*

$$D_\varphi = \alpha_0 \tfrac{\partial^2}{\partial x^2} + \alpha_1 \tfrac{\partial^2}{\partial y^2} + \alpha_2 \tfrac{\partial^2}{\partial z^2} + \alpha_3 \tfrac{\partial^2}{\partial x \partial y} + \alpha_4 \tfrac{\partial^2}{\partial x \partial z} + \alpha_5 \tfrac{\partial^2}{\partial y \partial z}.$$

Applying D_φ to F we get the hyperplane in $\mathbb{P}^5$

$$\mathbb{P}(AP^2_F) = \boldsymbol{V}(2\alpha_0 + 4\alpha_1 - 2\alpha_2 + 4\alpha_3 - \alpha_4 + \alpha_5).$$

Note that F can be recovered by $\mathbb{P}(AP^2_F)$ simply dividing by 2 the coefficients of the pure derivatives. In this way we get the point $[1:2:-1:4:-1:1]$ that corresponds to F. We choose the line

$$R = \boldsymbol{V}(\alpha_0, \alpha_1, \alpha_2, 2\alpha_0 + 4\alpha_1 - 2\alpha_2 + 4\alpha_3 - \alpha_4 + \alpha_5)$$

contained in $\mathbb{P}(AP^2_F)$. On R we consider the points [0:0:0:1:2:-2] and [0:0:0:0:1:1], i.e. the conics

$$K_1 = \boldsymbol{V}(xy + 2xz - 2yx) \text{ and } K_2 = \boldsymbol{V}(xz + yz)$$

in the pencil determined by R. An easy computation show that

$$K_1 \cdot K_2 = \{[1:0:0], [0:1:0], [0:0:1], [4:-4:1]\}.$$

Then the linear forms

$$L_1 = x, L_2 = y, L_3 = z, L_4 = 4x - 4y + z$$

determine a 4-polar polyhedron of F. Indeed we have

$$F = 3x^2 + 4y^2 - \tfrac{7}{8}z^2 - \tfrac{1}{8}(4x-4y+z)^2 = 3L_1^2 + 4L_2^2 - \tfrac{7}{8}L_3^2 - \tfrac{1}{8}L_4^2.$$

REMARK 10. *We have proved, in theorem 13, that if F is a generic polynomial of degree two in three variables (n=2) then $\boldsymbol{VSP}(F,4) \cong \boldsymbol{G}(1,4)$. Ranestad and Schreier proved that if G is a generic polynomial of degree two in four variables (n=3) then we have $\boldsymbol{VSP}(G,4) \cong \boldsymbol{G}(1,4)$. We conclude that*

$$\boldsymbol{VSP}(F,4) \cong \boldsymbol{VSP}(G,4).$$

It can be interesting to write explicitly an isomorphism.

4.2.2 Plane Cubics

Now we consider the special case *(d,n,h)=(3,2,4)* of the plane cubic curves in four factors. Let F_3 be a homogeneous polynomial in three variables $F_3 = F_3(x,y,z)$. In this case we have $wrk(F_3) = 4$. We know that the variety $\boldsymbol{VSP}(F_3,4)$, parameterizing all decomposition of F_3 in powers of four linear factors, is an irreducible and nonsingular variety of dimension $dim(\boldsymbol{VSP}(F_3,4)) = 2$.

Theorem 14. *(Dolgachev - Kanev) The variety of power sums $\boldsymbol{VSP}(F_3,4)$, parameterizing all decompositions in four linear factors of a homogeneous cubic polynomial in three variables, is isomorphic to the projective plane $\mathbb{P}^2$.*

Proof: Let F_3 be a generic cubic polynomial. We are in $\mathbb{P}^9$ and we consider the Veronese variety $V = V_9^2$. For the variety of its secant *3*-planes we have

$$dim(Sec_3(V)) = min\{4\cdot 2 + 3,9\} = 9.$$

So the generic cubic polynomial admits a decomposition as sums of four linear factors. The partial derivatives of F_3 are three quadric polynomials $\frac{\partial F_3}{\partial x}$, $\frac{\partial F_3}{\partial y}$, $\frac{\partial F_3}{\partial z}$ that generate a projective plane Π in the $\mathbb{P}^5$ parameterizing the plane conics.
Let $\{[l_1],\dots,[l_4]\}$ be a polar *4*-polyhedron of F_3. We have

$$F_3 = \lambda_1 l_1^3 + \lambda_2 l_2^3 + \lambda_3 l_3^3 + \lambda_4 l_4^3.$$

The partial derivatives of F_3 are

$$\begin{aligned}\frac{\partial F_3}{\partial x} &= 3\lambda_1\alpha_1 l_1^2 + 3\lambda_2\alpha_2 l_2^2 + 3\lambda_3\alpha_3 l_3^2 + 3\lambda_4\alpha_4 l_4^2\\ \frac{\partial F_3}{\partial y} &= 3\lambda_1\beta_1 l_1^2 + 3\lambda_2\beta_2 l_2^2 + 3\lambda_3\beta_3 l_3^2 + 3\lambda_4\beta_4 l_4^2\\ \frac{\partial F_3}{\partial z} &= 3\lambda_1\gamma_1 l_1^2 + 3\lambda_2\gamma_2 l_2^2 + 3\lambda_3\gamma_3 l_3^2 + 3\lambda_4\gamma_4 l_4^2.\end{aligned}$$

The polynomials l_1^2, l_2^2, l_3^2, l_4^2 are four points on the Veronese surfaces $V_4^2 \subseteq \mathbb{P}^5$. This points generate a $\mathbb{P}^3$ that contains Π. Let $\boldsymbol{G}(5,3)$ be the Grassmanian of the projective spaces of $\mathbb{P}^5$ and let $\boldsymbol{G}(5,3,\Pi)$ the subvariety of $\boldsymbol{G}(5,3)$ parameterizing the projective spaces of $\mathbb{P}^5$ that contains Π. We have the morphism

$$\varphi\colon \boldsymbol{VSP}(F_3,4)^o \to \boldsymbol{G}(5,3,\Pi) \text{ defined by } \{[l_1],[l_2],[l_3],[l_4]\} \mapsto <l_1^2,l_2^2,l_3^2,l_4^2>.$$

We denote by $\boldsymbol{VSP}(\partial,F_3,4)^o$ the sums of power variety of the partial derivatives of F_3. We see that $\boldsymbol{VSP}(F_3,4)^o \subseteq \boldsymbol{VSP}(\partial,F_3,4)^o$ and taking the closure we have $\boldsymbol{VSP}(F_3,4) \subseteq \boldsymbol{VSP}(\partial,F_3,4)$. We have a morphism

$$\varphi\colon \boldsymbol{VSP}(F_3,4) \to \boldsymbol{G}(5,3,\Pi),\ \{[l_1],[l_2],[l_3],[l_4]\} \mapsto <l_1^2,l_2^2,l_3^2,l_4^2>.$$

Now let Λ be a projective space that contains Π. Let V' be the Veronese surface in $\mathbb{P}^5$, we know that $deg(V') = 4$ so $\Lambda \cap V'$ consists of four points counted with multiplicity l_1^2,l_2^2,l_3^2,l_4^2, and the morphism φ is injective. By duality the variety $\mathbf{G}(5,3,\Pi)$ is isomorphic to $\mathbf{G}(1,2) \cong \mathbb{P}^2$, and we have a injective morphism

$$\begin{aligned}\varphi\colon \boldsymbol{VSP}(F_3,4) &\longrightarrow \mathbb{P}^2\\ \{[l_1],[l_2],[l_3],[l_4]\} &\longmapsto <l_1^2,l_2^2,l_3^2,l_4^2>\end{aligned}$$

We know that ***VSP***$(F_3,4)$ is a smooth variety of dimension two. The map φ is a bijective morphism between smooth varieties of the same dimension then it is an isomorphism and ***VSP***$(F_3,4) \cong \mathbb{P}^2$. $\square$

4.2.3 Reconstructing polar Polyhedra

We have associated to any homogeneous polynomial F of degree *3* a plane in $\mathbb{P}^5$. Now we give a method to reconstruct all *4*-polar polyhedra of F. We begin this section with another proof of *Dolghachev-Kanev* theorem, involving apolar forms.

THEOREM 15. *(Dolgachev - Kanev) The variety of power sums* ***VSP***$(F_3,4)$*, parameterizing all decompositions in four linear factors of a homogeneous cubic polynomial in three variables, is isomorphic to the projective plane* $\mathbb{P}^2$.

Proof: Let $\{L_1, ..., L_4\}$ be a *4*-polar polyhedron of F, then it is also a *4*-polar polyhedron for the partial derivatives F_x, F_y, F_z of F.
By apolarity lemma we have that the linear space $L_2(\mathbb{P}V^*, [L_1], ..., [L_4])$ is contained in the hyperplanes $\mathbb{P}(AP_2(F_x))$, $\mathbb{P}(AP_2(F_y))$, $\mathbb{P}(AP_2(F_z))$. Since F is general these three hyperplanes intersect in a plane $H = \mathbb{P}(AP_2(F_x)) \cap \mathbb{P}(AP_2(F_y)) \cap \mathbb{P}(AP_2(F_z))$.
We get a morphism

$$\varphi : \mathbf{VSP}(F_3, 4) \longrightarrow H^*, \{L_1, ..., L_4\} \mapsto L_2(\mathbb{P}V^*, [L_1], ..., [L_4]).$$

If two pencil of conics are equal clearly they have the same base points i.e. the morphism φ is injective. Since $dim(\mathbf{VSP}(F_3, 4)) = 2 = dim(H^*)$ it is an isomorphism. $\square$

Fix a plane H in $\mathbb{P}^5$ then it represents the varieties of power sums of a polynomial F_H. Let R be a line in H then R represents a pencil of conics and by apolarity lemma the base locus of this pencil is a *4*-polar polyhedron of F_H. To find the linear forms we can take two conics K_1, K_2 and compute their intersection. In this notation the morphism

$$\psi : H^* \longrightarrow \mathbf{VSP}(F_3, 4), R \mapsto K_1 \cdot K_2,$$

is the inverse of φ.

EXAMPLE 17. *We consider the cubic polynomial*

$$F = x^3 + y^2z + xz^2.$$

Its partial derivatives are

$$F_x = 3x^2 + z^2, F_y = 2yz, F_z = y^2 + 2xz.$$

Applying the differential operator

$$D_\varphi = \alpha_0 \tfrac{\partial^2}{\partial x^2} + \alpha_1 \tfrac{\partial^2}{\partial y^2} + \alpha_2 \tfrac{\partial^2}{\partial z^2} + \alpha_3 \tfrac{\partial^2}{\partial x \partial y} + \alpha_4 \tfrac{\partial^2}{\partial x \partial z} + \alpha_5 \tfrac{\partial^2}{\partial y \partial z}.$$

to the partial derivatives we obtain

$$\mathbb{P}(AP_2(F_x)) = \boldsymbol{V}(3\alpha_0+\alpha_2), \mathbb{P}(AP_2(F_y)) = \boldsymbol{V}(\alpha_5), \mathbb{P}(AP_2(F_z)) = \boldsymbol{V}(\alpha_1+\alpha_4).$$

So the plane H is given by

$$H = (3\alpha_0+\alpha_2 = \alpha_5 = \alpha_1+\alpha_4 = 0).$$

We consider the line R contained in H given by

$$R = (3\alpha_0+\alpha_2 = \alpha_5 = \alpha_1+\alpha_4 = \alpha_3 = 0)$$

and on R we choose the points $[0:1:0:0:-1:0]$ *and* $[1:0:-3:0:0:0]$ *corresponding to the conics*

$$K_1 = (y^2-xz=0) \text{ and } K_2 = (x^2-3z^2=0).$$

These conics intersects in the four points

$$[\sqrt{3}:\sqrt[4]{3}:1], [\sqrt{3}:-\sqrt[4]{3}:1], [-\sqrt{3}:i\sqrt[4]{3}:1], [-\sqrt{3}:-i\sqrt[4]{3}:1]$$

and so

$$L_1 = \sqrt{3}x+\sqrt[4]{3}y+z, L_2 = \sqrt{3}x-\sqrt[4]{3}y+z, L_3 = -\sqrt{3}x+i\sqrt[4]{3}y+z, L_4 = -\sqrt{3}x-i\sqrt[4]{3}y+z.$$

is a 4-polar polyhedron of F.

4.3 The Grassmannian *G(1,4)*

In this section we prove that in the case $n = 3$, $d = 2$, $h = 4$ the variety ***VSP***$(F_2,4)$ is birational to the Grassmannian ***G****(1,4)* giving explicitly a birational morphism of ***VSP***$(F_2,4)$ in ***G****(1,4)*. For our proof we need to see the Veronese variety V_{16}^4 as a subvariety of the Grassmannian ***G****(1,4)*.
Ranestad and Schreier proved by more complicated methods that ***VSP***$(F_2,4)$ and ***G****(1,4)* are isomorphic.

PROPOSITION 20. *The projective space $\mathbb{P}^n$ can be embbedded in the Grassmannian **G**(1,n+1) as the 2-Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ with $N = \binom{n+2}{2}$-1. In other words the Veronese variety $V_{2^n}^n$ is a subvariety of the Grassmannian of lines **G**(1,n+1).*

Proof: Let $[x_0,\dots,x_n]$ be a point in $\mathbb{P}^n$. We consider $[x_0,\dots,x_n,0]$ and $[0{:}x_0,\dots,x_n]$ as two points in $\mathbb{P}^{n+1}$ that generate the line

$$L_{[x_0,\dots,x_n]} = <[x_0,\dots,x_n,0],[0{:}x_0,\dots,x_n]> \subseteq \mathbb{P}^{n+1}.$$

In this way we have a morphism

$$\varphi{:}\mathbb{P}^n \to \boldsymbol{G}(1,n+1), \text{ defined by } [x_0,\dots,x_n] \mapsto L_{[x_0,\dots,x_n]}$$

Now we consider a line $R = <[u_0,\dots,u_{n+1}],[v_0,\dots,v_{n+1}]>$ in ***G****(1,n+1)* and the *Plücker embedding*

$$pk{:}\boldsymbol{G}(1,n+1) \to \mathbb{P}^N,\ R \mapsto [\Delta_{0,1}{:}\dots{:}\Delta_{i,j}{:}\dots{:}\Delta_{n,n+1}].$$

Where $\Delta_{i,j}$ is the 2×2 minor given by the columns i,j of the matrix

$$\Delta = \begin{pmatrix} u_0 & u_1 & \cdots & u_n & u_{n+1} \\ v_0 & v_1 & \cdots & v_n & v_{n+1} \end{pmatrix}$$

On $\mathbb{P}^n$ we have the composition

$$\mathbb{P}^n \xrightarrow{\varphi} \boldsymbol{G}(1,n+1) \xrightarrow{pk} \mathbb{P}^N.$$

We note that

$$(pk \circ \varphi)(x_0,\dots,x_n) = pk \begin{pmatrix} x_0 & x_1 & \cdots & x_n & 0 \\ 0 & x_0 & \cdots & x_{n-1} & x_n \end{pmatrix} = [\Delta_{0,1}:\dots:\Delta_{i,j}:\dots:\Delta_{n,n+1}] =$$

$$= [x_0^2 : x_0x_1 : \dots : x_{n-1}x_n : x_n^2] = \nu_2(\mathbb{P}^n).$$

So $pk \circ \varphi$ gives an embedding of $\mathbb{P}^n$ in $\boldsymbol{G}(1,n+1) \subseteq \mathbb{P}^N$ whose image is the Veronese variety $V_{2^n}^n$. □

THEOREM 16. *Let $F = F_2(x,y,z,w)$ be a homogeneous polynomial of degree two and let $\boldsymbol{VSP}(F_2,4)$ be the variety of power sums, parameterizing all decompositions in four linear factors of a homogeneous quadric polynomial in four variables. The variety $\boldsymbol{VSP}(F_2,4)$ is birational to the Grassmannian of lines $\boldsymbol{G}(1,4)$.*

Proof: We consider F as a point in $\mathbb{P}^9$. We have the Veronese embedding

$$\nu_2 : \mathbb{P}^3 \to \mathbb{P}^9, \text{ with } V = V_8^3 = \nu_2(\mathbb{P}^2)$$

Then we consider the *Plücker embedding*

$$pk : \boldsymbol{G}(1,4) \to \mathbb{P}(\textstyle\bigwedge^2(k^5)) = \mathbb{P}^9.$$

We know that $dim(\boldsymbol{G}(1,4)) = (1+1)(4\text{-}1) = 6$ and

$$deg(\boldsymbol{G}(1,4)) = deg(\boldsymbol{G}(2,5)) = (2(5\text{-}2))! \prod_{j=1}^{2} \frac{(j-1)!}{(5-2+j-1)!} = 5.$$

Now for any 4-polar polyhedra $\{l_1,\dots,l_4\}$ of F we consider the 3-plane $\Lambda_l = <l_1^2,\dots,l_4^2>$. We have $dim(\boldsymbol{G}(1,4) \cap \Lambda_l) = 6+3\text{-}9=0$ and $deg(\boldsymbol{G}(1,4)) = 5$ implies that the intersection consists of exactly 5 points $\{p_1,\dots,p_5\}$ counted with multiplicity.
By the proposition 20 $\mathbb{P}^3$ can be embedded in $\boldsymbol{G}(1,4)$ as the Veronese variety V. So any $\mathbb{P}^3$ generated by a 4-polar polyhedra $\{l_1^2,\dots,l_4^2\}$ intersects $\boldsymbol{G}(1,4)$ in the four points $l_1^2,\dots,l_4^2$ and in a additional point $\widetilde{P}$. In this way we have a map

$$\psi : \boldsymbol{VSP}(F_2,4) \to \boldsymbol{G}(1,4), \text{ defined by } \{l_1,\dots,l_4\} \mapsto \widetilde{P}.$$

Let $\{l_1,\dots,l_4\}$ and $\{L_1,\dots,L_4\}$ two 4-polar polihedra of F and let H_l and H_L the two associated 3-spaces. If $\widetilde{P}_l = \widetilde{P}_L$ then $H_l \cap H_L$ contains the line $R = <\widetilde{P}_l,F>$.
We can assume $\widetilde{P}_l \notin V$. If $l_i \neq L_j$ for any $i,j=1,\dots,4$, then H_l and H_L generate a 5-plane Λ that intersects V in 8 points. Let $Q \in V$ be a point different from l_i and L_i. Then $<\Lambda,Q>$ is a 6-plane that intersects V in 9 points but $deg(V) = 8$, a contraddiction.
If $l_1 = L_1$ and $l_i \neq L_i$ for any $i>1$ then $H_l \cap H_L$ contains the plane $<F,l_1,P_l>$. Then $\Lambda = <H_l, H_L>$ is a 4-plane that intersects V in 7 points.

We choose two points $Q_1,Q_2 \in V$ different form l_i, L_i. Then $<\Lambda,Q_1,Q_2>$ is a *6*-plane that intersects V in *9* points, a contraddiction.
If $l_1 = L_1$ and $l_2 = L_2$ we note that the variety V is defective and so $dim(Sec_1(V)) = 6$ and we can assume that the lines $<l_1,l_2>$ and $<F,\widetilde{P}_l>$ are skew. Then $H_l \cap H_L$ contains the *3*-plane $<F,l_1,l_2,P_l>$ and $<l_1^2,...,l_4^2> = <L_1^2,...,L_4^2>$.
In this way we have proved that the map ψ is generically injective, furthermore $dim(\boldsymbol{VSP}(F_2,4)) = dim(\boldsymbol{G}(1,4))$ implies that ψ is birational. □

4.4 Polynomials on $\mathbb{P}^1$

In this section we prove some results, probably well known to the experts, about polynomials in two variables.
We fix $n = 1$. We consider the variety of power sums $\boldsymbol{VSP}(F_d,h)$ for a fixed h. If $d = 2h - 1$ then the waring rank of F_{2h-1} is

$$wrk(F) = \frac{1}{2}\binom{2h}{2h-1} = h.$$

Sylvester proved that $\boldsymbol{VSP}(F_{2h-1},h)$ is a point. In this section we prove that $\boldsymbol{VSP}(F_h,h)$ is isomorphic to $\mathbb{P}^{h-1}$.
Then we determine the variety $\boldsymbol{VSP}(F_d,h)$ for any $h \leq d \leq 2h-1$. Note that for $h = 1$ we have $h = 2h-1$, for $h = 2$ we have $2h-1 = 3$, so the first interesting case is for $h = 3$.
Let d be an integer $h \leq d \leq 2h-1$ and let ν_d:$\mathbb{P}^1 \to \mathbb{P}^d$ be the d-uple embedding then $X = \nu_d(\mathbb{P}^1)$ is the rational normal curve of degree d in $\mathbb{P}^d$. A h-polar polyhedron of F_d determines an *(h-1)*-plane. The dimension of the variety of secant *(h-1)*-planes of X is

$$dim(Sec_{h-1}(X)) = min\{h + h - 1, d\} = min\{2h - 1, d\}.$$

Since $d \leq 2h-1$ we see that $Sec_{h-1}(X)$ covers $\mathbb{P}^d$. This observation shows that for any $h \leq d \leq 2h-1$ the generic homogeneous polynomial F_d of degree d admits a decomposition in h linear factors.

THEOREM 17. *(Sylvester) Let F_{2h-1} be a homogeneous polynomial of degree 2h-1 in two variables. The variety of power sums $\boldsymbol{VSP}(F_{2h-1},h)$ parameterizing all decomposition of F_{2h-1} in h linear factors is a single point.*

Proof: We consider F_{2h-1} as a point in $\mathbb{P}^{2h-1}$ and let X be the rational normal curve of degree *2h-1* in $\mathbb{P}^{2h-1}$.
We suppose that $\{l_1,...,l_h\}$ and $\{L_1,...,L_h\}$ are two distinct h-polar polyhedra of F_{2h-1}.
Let Λ_l and Λ_L the two *(h-1)*-planes generated by the decompositions. The point F_{2h-1} belongs to $\Lambda_l \cap \Lambda_L$ so the linear space $\Gamma = <\Lambda_l,\Lambda_L>$ has dimension

$$dim(\Gamma) \leq (h\text{-}1)+(h\text{-}1) = 2h\text{-}2.$$

If Λ_l and Λ_L have only F_{2h-1} as common point then $dim(\Gamma) = (h\text{-}1)+(h\text{-}1) = 2h\text{-}2$. So Γ is an hyperplane in $\mathbb{P}^{2h-1}$ and $\Gamma \cdot X \geq 2h$. A contraddiction because $deg(X) = 2h\text{-}1$.
If Λ_l and Λ_L have k common points then Λ_l and Λ_L intersect in k+*1* points $Q_1,...,Q_k,F_{2h-1}$,

$\Lambda_l \cap \Lambda_L$ is a $\mathbb{P}^k$ and $dim(\Gamma) = 2h\text{-}2\text{-}k$. We choose k points $P_1,\ldots,P_k$ on X in general position so $H = <\Gamma,P_1,\ldots,P_k>$ is a hyperplane such that $H \cdot X \geq 2h\text{-}k\text{+}k = 2h$, a contraddiction. We conclude that the decomposition of F_{2h-1} in h linear factors is unique. $\square$

Now we consider some specific cases.

- Case $d = 3$ and $h = 3$. Let F be a cubic polynomial and let X be the twisted cubic curve in $\mathbb{P}^3$. A 3-polar polyhedron of F determines a plane containing F. Conversely any plane containing F intersects X in three points counted with multiplicity. The plains of $\mathbb{P}^3$ containing a fixed point are parametrized by $\mathbb{P}^2$. So we have a well defined injective morphism

$$\varphi\text{:}\boldsymbol{VSP}(F_3,3) \rightarrow \mathbb{P}^2, \text{ defined by } \{l_1,\ldots,l_3\} \mapsto <l_1^3,\ldots,l_2^3>.$$

  Since $dim(\boldsymbol{VSP}(F_3,3)) = 2$ we conclude that φ is an isomorphism and $\boldsymbol{VSP}(F_3,3)$ is isomorphic to $\mathbb{P}^2$.

- Case $d = 4$ and $h = 4$. In this case F is a quartic polynomial and X is the rational normal curve of degree 4 in $\mathbb{P}^4$. By analogy with the preceding case we have a bijective correspondence between the 4-polar polyhedra of F and the 3-planes in $\mathbb{P}^4$ containing the point F, that are parametrized by a $\mathbb{P}^3$. So we have a well defined injective morphism

$$\psi\text{:}\boldsymbol{VSP}(F_4,4) \rightarrow \mathbb{P}^3, \text{ defined by } \{l_1,\ldots,l_4\} \mapsto <l_1^3,\ldots,l_4^3>.$$

  As in the preceding case we conclude that $\boldsymbol{VSP}(F_3,3)$ is isomorphic to $\mathbb{P}^3$.

This two observations suggest us that $\boldsymbol{VSP}(F_h,h)$ will be isomorphic to $\mathbb{P}^{h-1}$.

PROPOSITION 21. *The variety of power sums $\boldsymbol{VSP}(F_h,h)$ is isomorphic to $\mathbb{P}^{h-1}$.*

Proof: Let F be a homogeneous polynomial of degree h. We consider the rational normal curve X of degree h in $\mathbb{P}^h$.
Any h-polar polyhedron $\{l_1,\ldots,l_h\}$ of F determines h points $l_1^h,\ldots,l_h^h \in X$. This h points span the hyperplane H_l containing F. Let $\boldsymbol{G}(h\text{-}1,h,F)$ be the variety of the hyperplanes containing F. We have a well defined morphism

$$\varphi\text{:}\boldsymbol{VSP}(F_h,h) \rightarrow \boldsymbol{G}(h\text{-}1,h,F), \text{ defined by } \{l_1,\ldots,l_h\} \mapsto <l_1^h,\ldots,l_h^h>.$$

Any hyperplane containing F intersects X in h points counted with multiplicity so φ is injective. Moreover the variety $\boldsymbol{G}(h\text{-}1,h,F)$ is isomorphic to $\mathbb{P}^{h-1}$ and $dim(\boldsymbol{VSP}(F_h,h)) = h\text{-}1$. So φ is an injective morphism between smooth varieties of the same dimension then it is an isomorphism and $\boldsymbol{VSP}(F_h,h) \cong \mathbb{P}^{h-1}$. $\square$

By Sylvester theorem we know that $\boldsymbol{VSP}(F_{2h-1},h)$ is a single point and by the preceding proposition $\boldsymbol{VSP}(F_h,h)$ is $\mathbb{P}^{h-1}$. Now it is natural to ask what happens for a generic integer d such that $h \leq d \leq 2h\text{-}1$. We begin with some particular observations.

- We fix $h = 3$ so $2h\text{-}1 = 5$. We have to controll the case $d = 4$. Let F be a homogeneous polynomial of degree 4, we consider the decompositions of F in three linear factors. The partial derivatives of F are two homogeneous polynomials F_x,F_y of degree 3 in 3 linear factors. In $\mathbb{P}^3$ we consider the twisted cubic curve X. Any decomposition $\{l_1^4,...,l_3^4\}$ of F in 3 linear factors determine a decomposition $\{l_1^3,...,l_3^3\}$ for F_x and F_y. The plane spanned by the points $l_1^3,...,l_3^3 \in X$ contains the line $R = <F_x,F_y>$. Conversely any plane containing R intersects X in three points counted with multiplicity. The planes containing a line in $\mathbb{P}^3$ are parametrized by a $\mathbb{P}^1$. So we have a well defined injective morphism

$$\varphi\text{:}\boldsymbol{VSP}(F_4,3) \to \mathbb{P}^1, \text{ defined by } \{l_1,...,l_3\} \mapsto <l_1^3,...,l_3^3>.$$

  Since $dim(\boldsymbol{VSP}(F_4,3)) = 1$ this morphism is an isomorphism and $\boldsymbol{VSP}(F_4,3)$ is isomorphic to $\mathbb{P}^1$.

- Now we fix $h = 4$ so $2h\text{-}1 = 7$. We have to controll the cases $d = 5,6$.
  For $d = 5$ we consider the partial derivatives of F that are two polynomials F_x,F_y of degree 4 in 4 linear factors. In $\mathbb{P}^4$ we are considering the rational normal curve X of degree 4 and the 3-planes containing the line $R = <F_x,F_y>$. By analogy with the preceding case we have a bijective correspondence between the 4-polar polyhedra of F and the 3-planes of $\mathbb{P}^4$ containing the line R that are parametrized by a $\mathbb{P}^2$ so we have $\boldsymbol{VSP}(F_5,4) \cong \mathbb{P}^2$.
  The case $d = 6$ is a bit more difficult in fact it is clear that it is not sufficient to consider the first partial derivatives of F to have a good correspondence. So we consider the second partial derivatives F_{xx},F_{yy},F_{xy} that are three homogeneous polynomials of degree 4 in 4 linear factors. Let X be the rational normal curve of degree 4 in $\mathbb{P}^4$. The second partial derivatives span a plane $H = <F_{xx},F_{yy},F_{xy}>$. Any decomposition $\{l_1^6,...,l_4^6\}$ determine a decomposition $\{l_1^4,...,l_4^4\}$ of the second partial derivatives and a 3-plane Γ_l spanned by $l_1^4,...,l_4^4 \in X$ containing H. Conversely any 3-plane containing H intersects X in three points counted with multiplicity. The 3-planes in $\mathbb{P}^4$ containing a fixed plane are parametrized by a $\mathbb{P}^1$. So we have an injective morphism

$$\varphi\text{:}\boldsymbol{VSP}(F_4,3)\to \mathbb{P}^1, \text{ defined by } \{l_1,...,l_4\} \mapsto <l_1^4,...,l_4^4>.$$

  Now $dim(\boldsymbol{VSP}(F_4,3)) = 1$ and so φ is an isomorphism.

The preceding observations suggest as that for any integer d such that $h \leq d \leq 2h\text{-}1$ the variety $\boldsymbol{VSP}(F_d,h)$ will be a linear space and that in order to prove this we have only to consider the right order of the partial derivatives of F.

THEOREM 18. *Let $h > 1$ be a fixed integer. For any integer d such that $h \leq d \leq 2h\text{-}1$ the variety of power sums $\boldsymbol{VSP}(F_d,h)$, parameterizing all decompositions of a homogeneous polynomial of degree d in h linear factors, is isomorphic to $\mathbb{P}^{2h-d-1}$.*

Proof: Let F be a homogeneous polynomial of degree d and let $\{L_1,...,L_h\}$ be a h-polar polyhedron of F. We write

$$F = \lambda_1 L_1^d + \ldots + \lambda_h L_h^d.$$

We consider the partial derivatives of order *d-h* > 0 of F. This partial derivatives are

$$\binom{d-h+1}{d-h} = d\text{-}h\text{+}1$$

polynomials of degree h.
Let X be the rational normal curve of degree h in $\mathbb{P}^h$.
The partial derivatives span a *(d-h)*-plane H and $L_1^d,\ldots,L_h^d \in X$ span a hyperplane in $\mathbb{P}^h$ containing H. We note that $d < 2h\text{-}1$ implies $d\text{-}h < h\text{-}1$. Let $\boldsymbol{G}(h\text{-}1,h,H)$ be the variety of the hyperplanes of $\mathbb{P}^h$ containing H. We have a well defined morphism

$$\varphi{:}\boldsymbol{VSP}(F_d,h) \to \boldsymbol{G}(h\text{-}1,h,H) \text{ , defined by } \{L_1,\ldots,L_h\} \mapsto <L_1^h,\ldots,L_h^h>.$$

Now any hyperplane containing H intersects X in h points counted with multiplicity so φ is injective. We note that $\boldsymbol{G}(h\text{-}1,h,H)$ is isomorphic to $\mathbb{P}^{2h-d-1}$. Moreover

$$dim(\boldsymbol{VSP}(F_d,h)) = 2h - \binom{d+1}{d} = 2h - d - 1.$$

So φ is an injective morphism between smooth varieties of the same dimension and then it is an isomorphism. We conclude that $\boldsymbol{VSP}(F_d,h) \cong \mathbb{P}^{2h-d-1}$. $\square$

4.5 Morphisms into Grassmannians of lines

In this section we prove that some varieties of power sums admits a finite morphism to $\boldsymbol{G}(1,r)$ for a particular r. For example we see in theorem 15 that $\mathbf{VSP}(F_3, 4)$ admits a finite morphism to $\boldsymbol{G}(1,2)$ that indeed is injective.
Let $F = F_d \in k[x,y,z]_d$ be a homogeneous polynomial of odd degree $d = 2k+1$ in three variables. For any $k \in \mathbb{N}$ we consider the partial derivatives of order k of F. These derivatives are

$$\binom{d-k+2}{2} = \tfrac{1}{2}(d-k+2)(d-k+1)$$

homogeneous polynomials of degree $d-k$. We set

$$N_k = \binom{d-k+2}{2} - 1 \text{ and } h_k = N_k - 1 = \tfrac{1}{2}(d-k+2)(d-k+1) - 2.$$

Then we consider the Grassmannian of lines $\boldsymbol{G}(1,k+1)$ and the variety of power sums $\mathbf{VSP}(F, h_k)$.

Remark 11. *We note that h_k is exactly the Waring rank of F for k = 0,1,2,3,4,5 but for $k \geq 6$ the variety $\boldsymbol{VSP}(F,h_k)$ is empty. Moreover since $n = 2$ is fixed the varieties $\boldsymbol{VSP}(F,h_k)$ are smooth.*

Now we are ready to prove the following

Proposition 22. *For any $0 \leq k \leq 5$ there exists a generically $\binom{(k+1)^2}{h_k}$ to one morphism of the variety of power sums $\boldsymbol{VSP}(F,h_k)$ to the Grassmannian of lines $\boldsymbol{G}(1,k+1)$. Where $deg(F) = d = 2k+1$.*

Proof: We consider the partial derivatives of order k of F. As we have observed before these are $\binom{d-k+2}{2} = \frac{1}{2}(d-k+2)(d-k+1)$ points in the projective space $\mathbb{P}^{N_k}$ and span an $\frac{1}{2}(d-k+2)(d-k+1)-1$ - plane H.
If $\{L_1, ..., L_{h_k}\}$ is an h_k polar polyhedron of F then each partial derivative is decomposed on the factor $L_1^{d-k}, ..., L_{h_k}^{d-k}$. Then any h_k polar polyhedron $\{L_1, ..., L_{h_k}\}$ of F determine an (h_k-1) - plane $\Pi_L =< L_1^{d-k}, ..., L_{h_k}^{d-k} >$ that contains H.
By dualization the (h_k-1) - planes of $\mathbb{P}^{N_k}$ containing a fixed $(\frac{1}{2}(d-k+2)(d-k+1)-1)$ - plane are the $(N_k-(h_k-1)-1)$ - planes contained in a $(N_k-(\frac{1}{2}(d-k+2)(d-k+1)-1)-1)$ - plane. We compute

$$N_k-(h_k-1)-1 = \binom{d-k+2}{2} - 1 - \frac{1}{2}(d-k+2)(d-k+1)+2 = 1$$
$$N_k-(\frac{1}{2}(d-k+2)(d-k+1)-1)-1 = \frac{1}{2}d^2+\frac{3}{2}d-dk-3k-1 = k+2-1 = k+1.$$

We get the morphism

$$\varphi_k : \mathbf{VSP}(\mathrm{F}, \mathrm{h}_k) \longrightarrow \mathbf{G}(1, k+1), \quad \{\mathrm{L}_1, ..., L_{h_k}\} \mapsto < L_1^{d-k}, ..., L_{h_k}^{d-k} >.$$

Let $\nu_{d-k} : \mathbb{P}^2 \longrightarrow \mathbb{P}^{N_k}$ be the $(d-k)$ - Veronese embedding and let $V = \nu_{d-k}(\mathbb{P}^2)$ be the Veronese surface. Since the L_i^{d-k} are points on the Veronese surface V and

$$dim(V)+(h_k-1) = 2+\frac{1}{2}(d-k+2)(d-k+1)-3 = N_k,$$

we see that the morphism φ is generically finite. Moreover

$$deg(V) = (d-k)^2 = (k+1)^2,$$

so any (h_k-1) - plane determines $(k+1)^2$ points counted with multiplicity on V. With this $(k+1)^2$ points we can construct $\binom{(k+1)^2}{h_k}$ polar polyhedra of F. Then the morphism φ_k is $\binom{(k+1)^2}{h_k}$ to one. $\square$

We report in the following table the cases of preceding proposition

k	d	h	$dim(\mathbf{VSP}(F_d,h))$	$dim(\mathbf{G}(1,k+1))$
0	1	1	0	0
1	3	4	2	2
2	5	8	3	4
3	7	13	3	6
4	8	19	2	8
5	11	26	0	10

Remark 12. *In particular the morphism*

$$\varphi_2 : \boldsymbol{VSP}(F_5, 8) \longrightarrow \boldsymbol{G}(1, 3)$$

maps $\boldsymbol{VSP}(F_5, 8)$ *in a divisor of the Klein quadric* $\boldsymbol{G}(1,3)$. *Furthermore we note the the morphism*

$$\varphi_1 : \boldsymbol{VSP}(F_3, 4) \longrightarrow \boldsymbol{G}(1, 2) \cong \mathbb{P}^2$$

is $\binom{(k+1)^2}{h_k} = \binom{(1+1)^2}{4} = 1$ *to one. Then* $dim(\boldsymbol{VSP}(F_3,4)) = 2 = dim(\boldsymbol{G}(1,2))$ *and we recover the isomorphism of theorem 15. Unfortunately this is the only case in which this observation works.*

Let $F = F_d \in k[x,y,z]_d$ be a homogeneous polynomial of degree d, let $C = \mathbf{V}(F) \subseteq \mathbb{P}^2$ be the plane curve of degree d defined by F. Let $\{L_1,...,L_h\}$ be an h-polar polyhedron of F. We consider $L_1,...,L_h \in (\mathbb{P}^2)^*$ as points in the dual projective plane, then we have the lines $R_1^L = L_1^*,...,R_h^L = L_h^* \subseteq \mathbb{P}^2$. The curve

$$X_{L_1,...,L_h} = R_1^L \cup ... \cup R_h^L \subseteq \mathbb{P}^2$$

is a plane curve of degree h. Then $C \cap X_{L_1,...,L_h} = Z_{L_1,...,L_h}$ is a zero subscheme of length hd of the curve C, i.e. a point in the Hilbert scheme $Hilb_{hd}(C)$. We get a morphism

$$\varphi_{d,h} : \mathbf{VSP}(F,h) \longrightarrow Hilb_{hd}(C), \text{ defined by } \{L_1,...,L_h\} \mapsto Z_{L_1,...,L_h}.$$

It can be interesting to understand when this morphism fails to be injective.

4.6 Birational geometry of VSP

In this section we state some original results about varieties of power sums rationality. In the first part we give some examples that show how to construct a cone of given degree and dimension on a Veronese variety.

4.6.1 Cones on some Veronese varieties

We construct cones of given degree and dimension on some Veronese varieties. This can be useful to write a rational map from a variety of power sums to a rational variety.

- Case *d=2, n=2, h=3*. Let $F_2 \in \mathbb{P}^5$ a homogeneous polynomial of degree two and let $V = V_4^2$ the Veronese surface in $\mathbb{P}^5$.
  Let O be a point of $\mathbb{P}^5$ that does not lie on V and let Y be the cone of the lines over V with vertex O. Then Y contains V and $dim(Y) = dim(V)+1 = 3$, $deg(Y) = deg(V) = 4$.
  Any 3-polar polyhedron $\{L_1,...,L_3\}$ of F_2 generate a plane $H_L = <L_1^2,...,L_3^2>$ with $L_i^2 \in V_4^2$, that intersects Y in 4 points counted with multiplicity, the 3 points L_1^2, L_2^2, L_3^2 and an additional point $\widetilde{P}$. So we have a map

$$\varphi\text{:}\boldsymbol{VSP}(F_2,3) \dashrightarrow Y, \text{ defined by } \{L_1,...,L_3\} \mapsto \widetilde{P}.$$

- Case *d=2, n=3, h=4*. Let $F_2 \in \mathbb{P}^9$ be a homogeneous polynomial of degree two and let $V = V_8^3$ the Veronese variety in $\mathbb{P}^9$.
  Let $P_1,P_2,P_3 \in V$ three points in general position, the P_i generate a $\mathbb{P}^2$ denoted by H. We project $\mathbb{P}^9$ in $\mathbb{P}^6$ via the $\mathbb{P}^3$ containing H. Let Π:$\mathbb{P}^9 \backslash H \dashrightarrow \mathbb{P}^6$ the projection. Then $V' = \Pi(V)$ is a variety in $\mathbb{P}^6$ with $dim(V') = dim(V) = 3$ and $deg(V') = deg(V)-3 = 5$.
  Let $X = \overline{\Pi^{-1}(V')}$ be the cone over V'. Then $X \subseteq \mathbb{P}^9$ is a variety of dimension $dim(X) = dim(V')+3 = 6$ and degree $deg(X) = deg(V') = 5$.
  Now we have $X \subseteq \mathbb{P}^9$ of dimension 6 and degree 5 containing V.
  Any 4-polar polyhedron $\{L_1,...,L_4\}$ of F_2 generate a plane $H_L = <L_1^2,...,L_4^2>$ with $L_i^2 \in V$, that intersects X in 5 points counted with multiplicity, the 4 points $L_1^2, L_2^2, L_3^2, L_4^2$ and an additional point $\widetilde{P}$. So we have a map

$$\psi\colon \boldsymbol{VSP}(F_2,4) \dashrightarrow X, \text{ defined by } \{L_1,\dots,L_4\} \mapsto \widetilde{P}.$$

Let $\{l_1,\dots,l_4\}$ and $\{L_1,\dots,L_4\}$ two 4-polar polyhedra of F and let H_l and H_L the two associated 3-spaces. If $\widetilde{P}_l = \widetilde{P}_L$ then $H_l \cap H_L$ contains the line $R = <\widetilde{P}_l,F>$.
We can assume $\widetilde{P}_l \notin V$. If $l_i \neq L_j$ for any $i,j=1,\dots,4$, then H_l and H_L generate a 5-plane Λ that intersects V in 8 points. Let $Q \in V$ be a point different from l_i and L_i. Then $<\Lambda,Q>$ is a 6-plane that intersects V in 9 points but $deg(V) = 8$, a contradiction.
If $l_1 = L_1$ and $l_i \neq L_i$ for any $i>1$ then $H_l \cap H_L$ contains the plane $<F,l_1,P_l>$. So $\Lambda = <H_l,H_L>$ is a 4-plane that intersects V in 7 points. We choose two points $Q_1,Q_2 \in V$ different form l_i, L_i. Then $<\Lambda,Q_1,Q_2>$ is a 6-plane that intersects V in 9 points, a contraddiction.
If $l_1 = L_1$ and $l_2 = L_2$ we note that the variety $Sec_1(V)$ is defective and so $dim(Sec_1(V)) = 6$, we can assume that the lines $<l_1,l_2>$ and $<F,\widetilde{P}_l>$ are skew. Then $H_l \cap H_L$ contains the 3-plane $<F,l_1,l_2,P_l>$ so $<l_1^2,\dots,l_4^2> = <L_1^2,\dots,L_4^2>$.
In this way we have proved that the map ψ is generically injective, furthermore $dim(\boldsymbol{VSP}(F_2,4)) = dim(X)$ implies that ψ is birational. So $\boldsymbol{VSP}(F_2,3)$ is birational to X.

- Case $d=2$, $n=4$, $h=5$. Let $V = V_{16}^4$ the Veronese variety in $\mathbb{P}^{14}$ and let $L,R \subset \mathbb{P}^4$ two skew lines. We consider the linear system $|\mathcal{I}_{L\cup R}(2)|$ of the quadric hypersurfaces of $\mathbb{P}^4$ containing $L \cup R$.
The linear system $|\mathcal{I}_{L\cup R}(2)|$ is a subsystem of the complete linear system $|\mathcal{O}_{\mathbb{P}^4}(2)|$ whose sections are the quadric hypersurfaces of $\mathbb{P}^4$, moreover $|\mathcal{I}_{L\cup R}(2)|$ does not have unassigned base points.
To prove the last assertion we must show that for any point $P \notin L \cup R$ there exist a quadric in $|\mathcal{I}_{L\cup R}(2)|$ that does not contain P.
Modulo an automorphism of $\mathbb{P}^4$ we can suppose

$$L = \{X_0 = X_1 = X_2 = 0\},\ R = \{X_0 = X_3 = X_4 = 0\},\ P = [1{:}0{:}0{:}0{:}0].$$

The quadric hypersurfaces $Q = \boldsymbol{V}(X_0^2 + X_1X_3 + X_2X_4)$ contains L and R but P does not lie on Q. Two quadrics $Q_1,Q_2 \in |\mathcal{I}_{L\cup R}(2)|$ intersect in a surface $Y = Q_1 \cap Q_2$ of degree 4 such that e $Y = Q_1 \cap Q_2$ of degree 4 such that

$$\omega_Y = \mathcal{O}_Y(2+2-4-1) = \mathcal{O}_Y(-1).$$

So Y is a Del Pezzo surface of degree 4 in $\mathbb{P}^4$ and we can see it as the blow up of the linear system of the plane cubics with 5 assigned base points $P_1,\dots,P_5$ not three collinear and no five on a conic.
Let $Q_3 \in |\mathcal{I}_{L\cup R}(2)|$ another quadric, Q_3 intersects Y in a curve of degree 8 that is union of the two line L,R and a curve Γ with $deg(\Gamma) = 6$. The curve Γ is obtained by cutting Y with a quadric so it corresponds to a curve of degree 6 in $\mathbb{P}_2$ with P_1,P_2 as 3-fold points and the other P_i as 2-fold points. We can suppose that L,R are the exceptional divisors of the blow up corresponding to P_1 and P_2. So on Y we have

$$\Gamma{\cdot}L = \Gamma{\cdot}R = 3.$$

Intersecting with a new quadric $Q_4 \in |\mathcal{I}_{L\cup R}(2)|$ we obtain *12* points on Γ but *3* points are on L an *3* are on R so if $\widetilde{Y} = Bl_{L\cup R}(\mathbb{P}^4)$ is the blow up of $\mathbb{P}^4$ in $L \cup R$ on $\widetilde{Y}$ we miss *3+3 = 6* points.
We note that the complete linear system $|\mathcal{O}_{\mathbb{P}^4}(2)|$ has dimension

$$dim(H^0(|\mathcal{O}_{\mathbb{P}^4}(2)|)) \text{ - } 1 = \binom{4+2}{2} \text{ - } 1 = 14.$$

Imposing to a quadric to contain two skew lines is equivalent to impose to the system $|\mathcal{O}_{\mathbb{P}^4}(2)|$ six independents conditions, so we have $dim(|\mathcal{I}_{L\cup R}(2)|) = 14 \text{ - } 3 \text{ - } 3 = 8$. The blow up linear system $Bl_{L\cup R}(|\mathcal{I}_{L\cup R}(2)|)$ has degree *12 - 3 - 3 = 6* and it has dimension $dim(Bl_{L\cup R}(|\mathcal{I}_{L\cup R}(2)|)) = dim(|\mathcal{I}_{L\cup R}(2)|) = 8$. Moreover the new linear system on $\widetilde{Y}$ is without base points and induces a morphism of $\widetilde{Y}$ in $\mathbb{P}^8$ as a *4*-fold of degree *6*.

$$\begin{tikzcd}
\widetilde{Y} \arrow[rr, "\varphi"] \arrow[dr, dashed, "Bl_{L\cup R}"'] & & \mathbb{P}^8 \\
& \mathbb{P}^4 \arrow[ur] &
\end{tikzcd}$$

So via $\varphi{:}\widetilde{Y} \to \mathbb{P}^8$ we obtain a variety $\widetilde{Y}$ of dimension *4* and degree *6*. In $\mathbb{P}^{14}$ we consider the cone of the $\mathbb{P}^6$ containing a fixed $\mathbb{P}^3$ over $\widetilde{Y}$. This cone is a variety X of dimension $dim(X) = 4{+}6 = 10$ and $deg(X) = 6$. Moreover $\widetilde{Y}$ is obtained by the blow up of the linear system $|\mathcal{I}_{L\cup R}(2)|$ that is a subsystem of the complete linear system $|\mathcal{O}_{\mathbb{P}^4}(2)|$ giving the *2*-uple embedding of $\mathbb{P}^4$ in $\mathbb{P}^{14}$, so the cone X contains the Veronese variety V. Now any *5*-polar polyhedra $\{L_1,\ldots,L_5\}$ of F determines a *4*-plane $H_L = <L_1^2,\ldots,L_5^2>$ whose intersection with X consists of *6* points counted with multiplicity. Five points are the L_i^2 and we have an additional point $\widetilde{P}$. Since $dim(\boldsymbol{VSP}(F_2,5)) = 10$ we have a rational map

$$\psi{:}\,\boldsymbol{VSP}(F_2,5) \dashrightarrow X, \text{ defined by } \{L_1,\ldots,L_5\} \mapsto \widetilde{P}.$$

4.6.2 Quadrics and Cubics

In this section we state the rationality of some varieties of power sums. In particular we consider homogeneous polynomials of degree two $F_2 \in k[x_0,\ldots,x_n]_2$ decomposed in $h = n + 1 = wrk(F_2)$ linear factors and we prove that the varieties $\boldsymbol{VSP}(F_2,n{+}1)$ are rational.

PROPOSITION 23. *The variety* $\boldsymbol{VSP}(F_2,3)$ *is birational to a smooth quartic Del Pezzo threefold.*

Proof: Let $F_2 \in \mathbb{P}^5$ a homogeneous polynomial of degree two and let $V = V_4^2$ the Veronese surface in $\mathbb{P}^5$.
The homogeneous ideal $\mathbb{I}(V)$ is generated by quadric forms. Let Q_1 and Q_2 two quadric forms in the ideal $\mathbb{I}(V)$ then $X = Q_1 \cap Q_2$ has dimension *3*, degree *4* and contains V. Moreover the canonical sheaf of X is

$$\omega_X \cong \mathcal{O}_X(2+2\text{-}5\text{-}1) = \mathcal{O}_X(\text{-}2).$$

So the anticanonical divisor is $-K_X = 2H = (dim(X)\text{-}1)H$, where H is the hyperplane section. So X is a smooth quartic Del Pezzo threefold in $\mathbb{P}^5$.
Any 3-polar polyhedron $\{L_1,...,L_3\}$ of F_2 generate a plane $H_L = <L_1^2,...,L_3^2>$ with $L_i^2 \in V_4^2$. The plane H_L intersects X in 4 points counted with multiplicity, the 3 points L_1^2, L_2^2, L_3^2 and an additional point $\widetilde{P}$. So we have a rational map

$$\varphi : \boldsymbol{VSP}(F_2,3) \dashrightarrow X, \text{ defined by } \{L_1,...,L_3\} \mapsto \widetilde{P}_L.$$

If $\widetilde{P}_L = \widetilde{P}_l$ we have two planes Π_L and Π_l containing the line $<F,\widetilde{P}_L>$. The point F is very general we can assume $F \notin V$.
If $L_1 = l_1$ and $L_i \neq l_i$ for any $i>1$ then Π_L and Π_l generate a $\mathbb{P}^2$ since F and $\widetilde{P}_L \in X$ are genaral and we can assume $l_i \notin <F,\widetilde{P}_L>$.
If $L_i \neq l_i$ for any i then Π_L and Π_l generate a 3-plane Λ and the intersection $\Lambda \cap V$ can have dimension 0 or 1. If $dim(\Lambda \cap V) = 0$ then Λ intersects V in 6 points, a contraddiction because $deg(V) = 4$. If $\Lambda \cap V$ is a curve C. We write $\Lambda = H_1 \cap H_2$ as intersection of two hyperplanes, then C corresponds to a plane curve $\widehat{C}$ that is a common component of two conics, so $deg(\widehat{C}) = 1$ and $deg(C) = 2$. But we have $H_L \cdot C = 3$, a contradiction. Then $\Pi_L = \Pi_l$ and the map φ is generically injective. Now $dim(\boldsymbol{VSP}(F_2,3)) = 3 = dim(X)$ implies that φ is birational. □

We have seen in chapter 3 The Mukai's theorem which states that $\boldsymbol{VSP}(F_2,3)$ is a smooth Fano threefold that indeed is birational to a smooth quartic Del Pezzo threefold. Now we come to an important theorem. Let $F \in k[x_0 : ... : x_n]_2$ be a homogeneous polynomial, by Alexander-Hirschowitzs theorem we know that $wrk(F) = n + 1$, and we have the following

THEOREM 19. *Let F be a homogeneous polynomial of degree two in the $n+1$ variables $x_0,\ldots,x_n$. Then for any $n>0$ the variety $\boldsymbol{VSP}(F,n+1)$ is rational.*

<u>*Proof*</u>: We have $d = deg(F) = 2, h = n+1$ and $N = \frac{1}{2}(n+2)(n+1)\text{-}1$.
Modulo an automorphism of $\mathbb{P}^n$ we can write F in the form

$$F = x_0^2 + \ldots + x_n^2.$$

Let Λ^{N-n} be a generic $(N\text{-}n)$-plane in $\mathbb{P}^N$ that does not contain F. We consider the generic quadric $G \in \Lambda^{N-n}$ and the pencil of quadrics $\lambda F\text{-}G$ generated by F and G.
Let $Q \in M(n+1)$ be the symmetric matrix representing the generic quadric on $\mathbb{P}^n$ then the hypersurface $X = \boldsymbol{V}(det(Q))$ is a hypersurface of degree $n+1$ in $\mathbb{P}^N$ parameterizing the singular quadrics.
Since F and G are generic quadratic forms the line $<F,G>$ will intersect X in $n+1$ points that represent the cones $C_0,\ldots,C_n$ in the pencil $\lambda F\text{-}G$. If we denote by $v_i \in \mathbb{P}^n$ the vertex of the cone C_i for $i=0,\ldots,n$, then via the Veronese embedding $\nu_2 : \mathbb{P}^n \to \mathbb{P}^N$ we find $n+1$ points $\nu_2(v_i)$ on the Veronese variety $V_{2^n}^n$.
If A is the matrix of G then the cones in the pencil $\lambda F\text{-}G$ are determined by the values of λ such that $det(\lambda I\text{-}A) = 0$, in other words the cones are determined by the eigenvalues

$\lambda_0,\dots,\lambda_n$ of A that we can suppose distinct because G is general. Then for any $i=0,\dots,n$ we have

$$\lambda_i I\text{-}A = Mat(C_i) \text{ and } v_i = Sing(C_i) = Ker(\lambda_i I\text{-}A).$$

We see that the vertex v_i of the cone C_i is the eigenvector of A corresponding to the eigenvalue λ_i. In the basis $\{v_0,\dots,v_n\}$ the matrix A is in the form

$$A = \begin{pmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

and G is in the form $G = \lambda_0 v_0^2 + \dots + \lambda_n v_n^2$. We note that

$$\lambda_i I - A = \begin{pmatrix} \lambda_i - \lambda_0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda_i - \lambda_n \end{pmatrix}$$

Since $\lambda_i \neq \lambda_j$ for $i \neq j$ we have

$$v_i = Ker(\lambda_i I - A) = (0,\dots, \underbrace{1}_{i-thplace} ,\dots,0).$$

Note that the basis $\{v_0,\dots,v_n\}$ is orthonormal, so the matrix of F remains the identity after the change of basis. For the Veronese embedding we have

$$\nu_2(\alpha_0 x_0 + \dots + \alpha_n x_n) = \alpha_0^2 x_0^2 + \dots + \alpha_n^2 x_n^2 + \{mixed\ terms\}$$
$$\nu_2(v_0) = \nu_2([1:0:\dots:0]) = x_0^2,\dots,\nu_2(v_n) = \nu_2([0:0:\dots:1]) = x_n^2.$$

In this way we see that $F,G \in <x_0^2,\dots,x_n^2>$ and we can define a map

$$\psi : \Lambda^{N-n} \dashrightarrow \boldsymbol{VSP}(F,n+1), \text{ defined by } G \mapsto \{v_0,\dots,v_n\}.$$

Now we define another map

$$\varphi : \boldsymbol{VSP}(F,n+1) \dashrightarrow \Lambda^{N-n}, \text{ defined by } \{L_0,\dots,L_n\} \mapsto G_L = <L_0^2,\dots,L_n^2> \cap \Lambda^{N-n}.$$

We want to prove that ψ is the inverse of φ.
If $G_L = <L_0^2,\dots,L_n^2> \cap \Lambda^{N-n}$ with $\{L_0,\dots,L_n\} \in \boldsymbol{VSP}(F,n+1)$ then we can write

$$G = \lambda_0 L_0^2 + \dots + \lambda_n L_n^2$$

and since the diagonalizing bases is orthonormal we can assume

$$F = L_0^2 + \dots + L_n^2.$$

We consider the pencil $\lambda F\text{-}G_L$ whose associated matrix in the basis $\{L_0,\dots,L_n\}$ is

$$B = \begin{pmatrix} \lambda - \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda - \lambda_n \end{pmatrix}$$

Then for $\lambda = \lambda_i$, i=0,...,n we have the cones in the pencil λF-G_L.
For $\lambda = \lambda_j$ we get

$$Ker(\lambda_j I\text{-}B) = (0,\ldots,\underbrace{1}_{j-thplace},\ldots,0)$$

that represents the form L_j. So we have $\psi(G_L) = \{L_0,\ldots,L_n\}$ and this prove that

$$\psi \circ \varphi = Id_{\boldsymbol{VSP}(F,n+1)^o}.$$

Now we fix $G \in \Lambda^{N-n}$ and we have $\psi(G) = \{L_0,\ldots,L_n\}$ with $G \in <L_0^2,\ldots,L_n^2>$. On the other hand $\varphi(\{L_0,\ldots,L_n\}) = G_L = <L_0^2,\ldots,L_n^2> \cap \Lambda^{N-n}$, but the points G, G_L are containes in $<\mathrm{L}_0^2,\ldots,\mathrm{L}_n^2> \cap \Lambda^{N-n}$ implies $G = G_L$ and this prove that

$$\varphi \circ \psi = Id_{\Lambda^{N-n}}.$$

We conclude that the maps ψ and φ defines a birational isomorphism between ***VSP***$(F,n+1)$ and Λ^{N-n}. $\square$

Remark 13. *We consider two particular cases when $d = 3$.*

- *Let $F_3 \in \mathbb{P}^9$ be a homogeneous polynomial and let $V = V_9^2 \subseteq \mathbb{P}^9$ be the Veronese variety. Let $P_1,P_2,P_3 \in \mathbb{P}^2$ be three points in general position.*
  Let $|\mathcal{I}_{P_i}(3)| \subseteq |\mathcal{O}_{P^2}(3)|$ be the linear system of the plane cubics containing P_1,P_2,P_3. Then we have

  $$deg(|\mathcal{I}_{P_i}(3)|) = 9 \text{ and } h^0(|\mathcal{I}_{P_i}(3)|) = 9\text{-}3 = 7.$$

  The linear system $|\mathcal{I}_{P_i}(3)|$ is without unassigned base points and so blowing up $\mathbb{P}^2$ in P_1,P_2,P_3 we obtain a very ample linear system $Bl_{P_i}(|\mathcal{I}_{P_i}(3)|)$ such that

  $$deg(Bl_{P_i}(|\mathcal{I}_{P_i}(3)|)) = 9\text{-}3 = 6 \text{ and } h^0(Bl_{P_i}(|\mathcal{I}_{P_i}(3)|)) = 7.$$

  The linear system $Bl_{P_i}(|\mathcal{I}_{P_i}(3)|)$ gives an embedding of $\widehat{\mathbb{P}^2} = Bl_{P_i}(\mathbb{P}^2)$ in $\mathbb{P}^6$ as a Del Pezzo surface of degree 6 that we denote by Y.

  $$\begin{tikzcd} \widehat{\mathbb{P}^2} \arrow[rr] \arrow[dr, dashed, "Bl_{P_i}"'] & & \mathbb{P}^6 \\ & \mathbb{P}^2 \arrow[ur] & \end{tikzcd}$$

  Now let X be the cone over Y constructed by the 3-planes containing the plane $<\nu_3(P_1),\nu_3(P_2),\nu_3(P_3)>$, then $deg(X) = 6$ and $dim(X) = 2+3 = 5$. Let $\{L_1,\ldots,L_5\}$ be a 5-polar polyhedron of F_3. We consider the 4-plane $H_L =< L_1^3,\ldots,L_5^3 >$ with $L_i^3 \in V_3^2$. The 4-plane H_L intersects X in 6 points counted with multiplicity, the 5 points and an additional point $P_L \in X$. In this way we get a rational map

  $$\varphi\colon\ \boldsymbol{VSP}(F_3,5) \dashrightarrow X,\ \{L_1,\ldots,L_5\} \mapsto P_L.$$

- *Let $F_3 \in \mathbb{P}^{19}$ be a homogeneous polynomial of degree 3 in four variables. The partial derivatives of F_3 are homogeneous polynomials of degree two F_x,F_y,F_z,$F_w \in \mathbb{P}^9$. We denote by $H^3_\partial = <F_x,F_y,F_z,F_w>$ the 2-plane spanned by the derivatives. We consider the Veronese variety $V = V^3_8 \subseteq \mathbb{P}^9$, then any 6-polar polyhedron $\{L_1,...,L_6\}$ determines a 5-plane $H^5_L = <L_1,...,L_6>$ that contains H^3_∂ and intersects V in exactly L_1,...,L_6 since 5+3<9.*
  Now we consider a 4-plane Λ^4 such that $\Lambda^4 \cap H^3_\partial = \varnothing$ so $H^5_L \cap \Lambda^4 = \{P_L\}$. We define the rational map

  $$\varphi\colon \textbf{\textit{VSP}}(F_3,6) \dashrightarrow \Lambda^4, \text{ defined by } \{L_1,...,L_6\} \mapsto P_L.$$

4.7 Maps between VSP

Let F be a homogeneous polynomial of degree d in $n+1$ variables and let $\{L_1, ..., L_h\}$ be an h-polar polyhedron of F, we write

$$F = L_1^d + ... + L_h^d.$$

Let H be an hyperplane in $\mathbb{P}^n$, we denote by $\widehat{F}$ the restriction to H of F. Then $\widehat{F}$ is a homogeneous polynomial of degree d in n variables. Since $\widehat{L_i^d} = \widehat{L_i}^d$ we have

$$\widehat{F} = \widehat{L_1}^d + ... + \widehat{L_h}^d$$

where the $\widehat{L_i} = L_{i|H}$ are linear forms on $\mathbb{P}^{n-1}$. In this way we get a rational map

$$\varphi_H : \mathbf{VSP}(F, h) \dashrightarrow \mathbf{VSP}(\widehat{F}, h),\ \{L_1, ..., L_h\} \mapsto \{\widehat{L_1}, ..., \widehat{L_h}\}.$$

We want to give a geometrical description of this map. We can assume $H = \{x_n = 0\}$. The polynomial F is of the form

$$F = \textstyle\sum_{i_0+...+i_n=d} f_{i_0,...,i_n} x_0^{i_0} ... x_n^{i_n}.$$

To restrict F on H means to kill the monomials in which x_n compares. This monomials form a space of dimension $\binom{n+d-1}{d-1}$.
So we are projecting $\mathbb{P}^N$ in $\mathbb{P}^{\widehat{N}}$ from a $\binom{n+d-1}{d-1} - 1 - plane$ Π, where $N = \binom{n+d}{d} - 1$ and $\widehat{N} = \binom{n-1+d}{d}$. The projection maps F in $\widehat{F}$. The $h - polyhedron$ $\{L_1, ..., L_h\}$ determine the zero subscheme of length h, $\{L_1^d, ..., L_h^d\}$ on the Veronese variety $V^n_{d^n}$ that spans a $(h-1) - plane$ H_L passing through F. This $h - plane$ is projected in a $(h-1) - plane$ passing through $\widehat{F}$ and $h - secant$ to the Veronese variety $V^{n-1}_{d^{(n-1)}}$ if and only if H_L does not intersect the center of projection Π. This is the reason why a priori we can not say that φ_H is a morphism.

Example 18. *We fix $d = 2$. We have*

$$F = \alpha_0x_0^2 + \alpha_1x_1^2 + \alpha_2x_2^2 + \alpha_3x_3^2 + \alpha_4x_0x_1 + \alpha_5x_0x_2 + \alpha_6x_0x_3 + \alpha_7x_1x_2 + \alpha_8x_1x_3 + \alpha_9x_2x_3.$$
$$\widehat{F} = \alpha_0x_0^2 + \alpha_1x_1^2 + \alpha_2x_2^2 + \alpha_4x_0x_1 + \alpha_5x_0x_2 + \alpha_7x_1x_2.$$

We are projecting $\mathbb{P}^9$ in the 5-plane given by the equation

$$\{X_3 = X_6 = X_8 = X_9 = 0\}$$

from a 3-plane.

Since the dimension of $\mathbf{VSP}(F,h)$ is $h(n+1)-\binom{n+d}{d}$ the expected dimension of the variety $X \subseteq Sec_{h-1}V_{d^n}^n$ of the $(h-1)-planes$ passing though F is $h(n+1)-\binom{n+d}{d}+(h-1)$. If $h(n+1)-\binom{n+d}{d}+(h-1)+dim(\Pi) < N$ the X does not intersect Π, but this inequality is equivalent to $h(n+2) < 1-\binom{n+d-1}{d-1}$ that is never verified. So we expect that the maps of the form φ_H are never morphisms.
However it can be interesting to understand if for some n, h, d one can find a plane H such that φ_H is generically injective. In this way we can study the birational geometry of varieties of power sums from another viewpoint.

Conclusions

As we have seen few varieties of sums of powers have been classified. To resume the main results of this work we report a table updated with our contributions.

d	**n**	**h**	**VSP($\mathbf{F_d}$, h)**	**Reference**
2h-1	1	h	1 point	Sylvester
h$\leq$ d $\leq$ 2h-1	**1**	**h**	$\mathbf{P^{2h-d-1}}$	**Massarenti and Mella**
2	2	3	quintic Fano threefold	Mukai [Muk92]
3	2	4	P^2	Dolgachev and Kanev [DK93]
3	**2**	**4**	**New proof of D-K Th.**	**Massarenti and Mella**
2	**2**	**4**	**birational to G(1,4)**	**Massarenti and Mella**
2,3	**2**	**4**	**Reconstruction of Decompositions**	**Massarenti and Mella**
4	2	6	Fano threefold of genus twelve	Mukai [Muk92]
5	2	7	1 point	Hilbert, Richmond, Palatini
5	**2**	**7**	**New proof of Hilbert Th.**	**Massarenti and Mella**
6	2	10	K3 surface of genus 20	Mukai [Muk92]
7	2	12	5 points	Dixon and Stuart
8	2	15	16 points	Mukai [Muk92]
2	3	4	**G**(1,4)	Ranestad and Schreier [RS00]
3	3	5	1 point	Sylvester's Pentahedral Theorem
3	**3**	**5**	**New proof of Sylvester Th.**	**Massarenti and Mella**
3	4	8	W	Ranestad and Schreier [RS00]
3	5	10	S	Iliev and Ranestad [IR01b]
2	**n**	**n+1**	**VSP(F,n+1) rationality**	**Massarenti and Mella**

Our next object is the study the birational geometry and the rational connection of varieties of power sums in the case $d \geq 3$.

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