THINKING THROUGH THREE WORLDS OF MATHEMATICS

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The major idea in this paper is the formulation of a theory of three distinct but interrelated worlds of mathematical thinking each with its own sequence of development of sophistication, and its own sequence of developing warrants for truth, that in total spans the range of growth from the mathematics of new-born babies to the mathematics of research mathematicians. The title of this paper is a play on words, contrasting the act of 'thinking through' several existing theories of cognitive development, and 'thinking through' the newly formulated theory of three worlds to see how different individuals may develop substantially different paths on their own cognitive journey of personal mathematical growth.

INTRODUCTION

The International Group for the Psychology of Mathematics Education is a broad organisation with many 'voices' expressing a wide range of issues in mathematical teaching, learning and thinking. So broad are the views of its members that it moves over the years in ways that may not seem to present an overall universal picture. Yet there are themes that occur which focus on the psychology of cognitive growth of different individuals in mathematics education that begin to recur and link together to build a global framework. In this paper such themes are drawn out to formulate a long-term theory of cognitive development from conception to mature adult that encompasses a wide range of different paths taken by individuals, from discalculic children who make little progress, to research mathematicians who move forward the boundaries of the subject.

THINKING THROUGH A RANGE OF THEORIES

It is probably difficult for those of us looking at the huge range of current mathematics education research to imagine the state of the theory when the International Group for the Psychology of Mathematics Education was first conceived. Psychology was still in the grips of behaviorism, with the teaching of mathematics largely in the hands of mathematical practitioners and general educators, and just a few major theorists, such as Piaget (1965), Dienes (1960) and Bruner (1966), having something to say that had particular relevance in mathematics. At the time, Piagetian theories held sway, with an emphasis on successive stages of development and a particular focus on the transitions between stages. Underlying Piagetian theory was a tripartite theory of abstraction: *empirical abstraction* focusing on how the child constructs meaning for the properties of objects, *pseudo-empirical abstraction*, focusing on construction of meaning for the properties of actions on objects, and *reflective abstraction* focused on the idea of how 'actions and operations become thematized objects of thought or assimilation' (Piaget, 1985, p. 49). Meanwhile, in a somewhat different direction, Bruner focused on three distinct ways in which ‘the individual translates experience into a model of the world’, namely, enactive, iconic and symbolic (Bruner 1966, p.10). The foundational symbolic system is language, with two important symbolic systems especially relevant to mathematics: number and logic (ibid. pp. 18, 19).

Our founding President, Efraim Fischbein, with his wide experience of psychology and mathematics, was from the very beginning interested in three distinct aspects of mathematical thinking: fundamental *intuitions* that he saw as being widely shared, the *algorithms* that give us power in computation and symbolic manipulation, and the *formal* aspect of axioms, definitions and formal proof (Fischbein, 1987).

Our second President, Richard Skemp, balanced his professional knowledge of mathematics and psychology with both theory and practice, not only producing his own textbook series for both primary and secondary schools, but also developing a general theory of increasingly sophisticated human learning (Skemp, 1971, 1979). He saw the individual having receptors to receive information from the environment and effectors to act on the environment forming a system he referred to as ‘delta-one’; a higher level system of mental receptors and effectors (delta-two) reflected on the operations of delta-one. This two level system incorporate three distinct types of activity: *perception* (input), *action* (output) and *reflection*, which itself involves higher levels of perception and action.

The emphases in these three-way interpretations of cognitive growth are very different, but there are underlying resonances that appear throughout. First there is a concern about how human beings come to construct and make sense of mathematical ideas. Then there are different ways in which this construction develops, from real-world perception and action, real-world enactive and iconic representations, fundamental intuitions that seem to be shared, via the developing sophistication of language to support more abstract concepts including the symbolism of number (and later developments), the increasing sophistication of description, definition and deduction that culminates in formal axiomatic theories.

In geometry, van Hiele (1959, 1986) has traced cognitive development through increasingly sophisticated succession of levels. His theory begins with young children perceiving objects as whole gestalts, noticing various properties that can be described and subsequently used in verbal definitions to give hierarchies of figures, with verbal deductions that designate how, if certain properties hold, then others follow, culminating in more rigorous, formal axiomatic mathematics. In a recent article, van Hiele (2002) asserts differences between his theory and those of others, for instance, denying a change of ‘level’ between arithmetic and algebra, but asserting a change in level from the symbolism of algebra and arithmetic and an axiomatic approach to mathematics. This suggests a significant difference between his theory of development applied in geometry and the cognitive development of arithmetic and algebra, while reasserting a distinction between elementary mathematics (by which I mean school geometry, arithmetic and algebra) and advanced mathematical thinking with its formal presentation of axiomatic theories.

Meanwhile, process-object theories such as Dubinsky's APOS theory (Czarnocha *et al.*, 1999) and the operational-structural theory of Sfard (1991) gave new impetus in the construction of mathematical objects from thematized processes in the manner of Piaget's reflective abstraction. Gray & Tall (1994) brought a new emphasis on the role of symbols, particularly in arithmetic and algebra, that act as a pivot between a do-able process and a think-able concept that is manipulable as a mental object (a *procept*). This, in its turn, amplifies and extends Fischbein's algorithmic mode of thinking, to include not only procedures, but their meanings in an integrated theory.

At the same time, the advanced mathematical thinking group of PME, organised by Gontran Ervynck in the late eighties, surveyed the transition to formal thinking and began to extend cognitive theories to the construction of axiomatic systems.

Two further strands were also emerging, one encouraged by the American Congress declaring 1990-2000 as 'the decade of the brain' in which resources were offered to expand research into brain activity. The other related to a focus on embodiment in cognitive science where the linguist Lakoff worked with colleagues to declare that all thinking processes are embodied in biological activity.

In the first of these, brain imaging techniques were used to determine low grain maps of where brain activities are occurring. Such studies focused mainly on elementary arithmetic activities (eg Dehaene 1997, Butterworth 1999), but others revealed how logical thinking, particularly when the negation of logical statements is involved, causes a shift in brain activity from the visual sensory areas at the back of the brain to the more generalised frontal cortex (Houdé *et al*, 2000). This reveals a distinct change in brain activity, consistent with a significant shift from sensory information to formal thinking. At the other end of the scale, studies of young babies (Wynn, 1992) revealed a built-in sense of numerosity for distinguishing small configurations of 'twoness' and 'threeness', long before the child had any language. The human brain has visual areas that perceive different colours, shades, changes in shade, edges, outlines and objects, which can be followed dynamically as they move. Implicit in this structure is the ability to recognize small groups of objects (one, two or three), providing the young child with a fundamental intuition for small numbers.

In the second development, Lakoff and colleagues theorized that human embodiment suffused all human thinking, culminating in an analysis of *Where Mathematics Comes From* (Lakoff & Nunez, 2000). Suddenly *all* mathematics is claimed to be embodied. This is a powerful idea on the one hand, but a classification with only one category is not helpful in making distinctions.

If one takes 'embodiment' in its everyday meaning, then it relates more to the use of physical senses and actions and to visuo-spatial ideas in Bruner's two categories of enactive and iconic representations. Following through van Hiele's development, the visual embodiment of physical objects becomes more sophisticated and concepts such as 'straight line' take on a conceptual meaning of being perfectly straight, and having no thickness, in a way that cannot occur in the real world. This development, from physical embodiment to increasingly sophisticated conceptual embodiments is quite different from the symbolic development encountered in arithmetic where *actions* on objects (such as counting and sharing) are symbolised and the symbols themselves take on a character that allows them to be mentally manipulated at a higher level. The latter may be *functionally* embodied (in that we use our hands to write symbols and think metaphorically about 'moving symbols around') but the encapsulation of processes into mental objects is fundamentally different from the reflective sensory focus on objects themselves, sufficient to place it in a different category, analogous to Sfard's (1991) distinction between operational and structural.

The category focusing on the increasing sophistication of representations of objects includes two of Bruner's three forms of representation: the enactive and iconic. Meanwhile, symbolic representations include the technical forms of number and logic that resonate with Fischbein's algorithmic and formal categories.

These re-alignments of categories are usefully seen in relation to the SOLO (Structure of Observed Learning Outcomes) theory of another president of PME, Kevin Collis (Biggs & Collis, 1982). This incorporates a revised stage theory that builds on aspects from both Piaget and Bruner, with successive stages named sensori-motor, ikonic, concrete-symbolic, formal, and post-formal. An essential aspect of this theory is that, once a stage has been constructed, it becomes available *together with previous stages*. Seeing cognitive development in a cumulative light can combine sensori-motor interactions and ikonic visuo-spatial ideas to give an embodied basis for mathematics. This goes in one direction towards geometry through the focus on properties of objects underpinned with language, in another direction, actions on embodied objects build a distinct development operating with symbols in arithmetic and algebra. All these activities grow in sophistication and the study of their properties lead on later to more formal, abstract, logical aspects. Language continues to underpin all of this activity. A visual picture is nothing without meaning being given to what it represents. While embodiment is fundamental to human development, language is essential to give the subtle shades of meaning that arise in human thought.

Taking the lead from Collis in seeing successive developments as cumulative, rather than as the replacement of earlier ways of thinking, we may now see mathematical development beginning before language with an implicit sense of numerosity. By the time the child arrives at school, sensori-motor and ikonic aspects are already working together with language making more subtle conceptions possible. This is the beginning of a van Hiele development in visuo-spatial ideas of figures in particular and other graphical concepts in general. The introduction of arithmetic (concrete-symbolic) brings a distinct mode of operation focusing on the symbolisation of counting processes as number concepts; the properties encountered in the elementary mathematics of arithmetic, algebra, geometry and calculus lead on to a new property-based focus using axiomatic definitions and proof.

THREE WORLDS OF MATHEMATICS

The foregoing discussion leads to a possible categorisation of cognitive growth into three distinct but interacting developments.

The first grows out of our *perceptions* of the world and consists of our thinking about things that we perceive and sense, not only in the physical world, but in our own mental world of meaning. By reflection and by the use of increasingly sophisticated language, we can focus on aspects of our sensory experience that enable us to envisage conceptions that no longer exist in the world outside, such as a 'line' that is 'perfectly straight'. I now term this world the 'conceptual-embodied world' or '*embodied* world' for short. This includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuospatial imagery. It applies not only the conceptual development of Euclidean geometry but also other geometries that can be conceptually embodied such as non-Euclidean geometries that can be imagined visuo-spatially on surfaces other than flat Euclidean planes and any other mathematical concept that is conceived in visuo-spatial and other sensory ways.

The second world is the world of symbols we use for calculation and manipulation in arithmetic, algebra, calculus and so on. These begin with *actions* (such as pointing and counting) that are encapsulated as concepts by using symbol that allow us to switch effortlessly from processes to *do* mathematics to concepts to *think* about. This second world I call the 'proceptual-symbolic world' or simply the '*proceptual* world'. It does not develop in the same way as the van Hiele development of geometry, but by expanding the context of counting to new contexts, sharing, using fractions, allowing debts using negative numbers, decimal representations, repeating and non-repeating decimals, real numbers, complex numbers, vectors in two and three, then n dimensions, and so on.

The third world is based on *properties*, expressed in terms of formal definitions that are used as axioms to specify mathematical structures (such as 'group', 'field', 'vector space', 'topological space' and so on). This is termed the 'formal-axiomatic world' or '*formal* world', for short. It turns previous experiences on their heads, working not with familiar objects of experience, but with axioms that are carefully formulated to *define* mathematical structures in terms of specified properties. Other properties are then deduced by formal proof to build a sequence of theorems. Within the axiomatic system, new concepts can be defined and their properties deduced to build a coherent, logically deduced theory.

JOURNEYS THROUGH THE THREE WORLDS

Different individuals take very different journeys through the three worlds. A few children have such difficulties with numbers that the phenomenon has been given the name 'discalculia'. Most children cope with the action-schema of counting leading to the development of the number concept. However, there are growing differences in the ways children cope with arithmetic. Some remain focused much longer on the procedures of counting, while others are developing more flexible number concepts.

Failure to compress counting procedures into thinkable concepts can lead to the learning of facts by rote. For many (perhaps most) individuals rote-learning can become a way of life. This may give success in a variety of routine contexts, but the longer-term Piagetian vision in which 'operations become thematized objects of thought' requires compression of knowledge into thinkable mental entities.

As an individual travels through each world, various obstacles occur on the way that require earlier ideas to be reconsidered and reconstructed, so that the journey is not the same for each traveller. On the contrary, different individuals handle the various obstacles in different ways that lead to a variety of personal developments, some of which allow the individual to progress through increasing sophistication in a meaningful way while others lead to alternative conceptions, or even failure.

For instance, the transition from whole numbers to fractions is highly complex; the embodied representation of a number as a physical collection of counters must be replaced by a sharing of an object or a collection of objects into equal parts and selecting a number of them. In new contexts, old experiences can cause serious conflicts. I call such old experiences 'met-befores'. In experiencing whole numbers, the child will encounter the idea that each number has a next number and there are none in between. This met-before can cause confusion with fractions wherein there is no 'next' fraction, and two fractions always have many others in between. Likewise, in moving from arithmetic to algebra, a typical met-before is the idea that every sum has an answer, for instance, 2+3 is 5. But an expression such as $2+3x$ has no 'answer' unless x is known. So if x is unknown, the child who regards a sum as an operation to carry out is faced with something that cannot be done. Other met-befores include ideas such as letters stand for codes (for example, $a=1$, $b=2$, etc) so $30-x$ is 6 (because x is 24), or the idea of place value that interprets 23 as two tens and a 3, so if $x=3$, then $2x$ is 23. It is my belief that such met-befores are a major source of cognitive obstacles in learning mathematics and, when conflict occurs, the safe thing is to stick to routines and learn, at best, in a procedural fashion.

Watson (2001) considers the notion of vector, which is met in school in various guises such as journeys, or forces. These met-befores can give insight in some ways, but can cause serious problems in others. For instance, if a vector is a journey, then one can follow a journey from A to B then B to C to give the journey from A to C, so that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$. But what is $\overrightarrow{BC}+\overrightarrow{AB}$? As journeys, one can go from B to C to start, but if C is not the same as A, then physically, one needs to jump from C to A before finishing the journey. At the physical embodied level of a journey, addition of journeys is not commutative, it may not even be defined. In another instance, it is a common mistake for a student to say that the sum of two vectors with the same endpoint as in figure 1 is zero. If the picture evokes a sense of two fingers pushing together, then they cancel out.

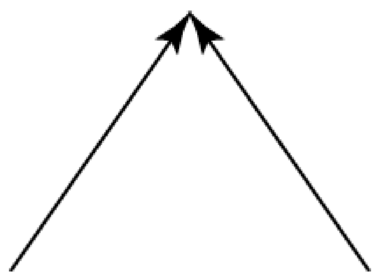

Figure 1

The concept of vector has many meanings. It has several distinct physical manifestations that lay down different met-befores that need to be resolved at more sophisticated levels. The problems just enunciated do not occur once the individual has focused on the concept of *free* vector, which has only magnitude and direction. At this level commutativity occurs for different reasons in each world of mathematics. In the embodied world, the truth of $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ follows from the properties of a parallelogram (figure 2) and meaning is supported by tracing the finger along two sides to realise that the effect is the same, whichever way one goes to the opposite corner of the figure. In the symbolic world of vectors as matrices, addition is commutative because the sum of the components is commutative. At the formal level of defining a vector space, commutativity holds because it is assumed as an axiom.

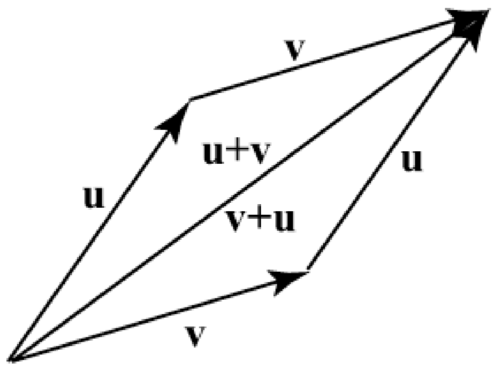

Figure 2: **u+v** is the same as **v+u**

More generally, each world develops its own 'warrants for truth' (in the sense of Rodd, 2000) in different ways. Initially something is 'true' in the embodied world because it is *seen* to be true. This is truth in the intuitive sense of Fischbein. Increasing sophistication in geometry leads to Euclidean proof, which is supported by a visual instance and proved by agreed conventions, often based on the idea of 'congruent triangles'. In arithmetic, something is 'true' because it can be calculated; in algebra, because one can carry out an appropriate symbolic manipulation such as

$$(a-b)(a+b)=(a-b)a+(a-b)b=a^2-ba+ab-b^2=a^2-b^2.$$

In the formal world, something is 'true' because it is either assumed as an axiom or definition, or because it can be *proved* from them by formal proof.

By becoming aware of the different developments in the different worlds and of the way in which experiences may work at one stage, yet create met-befores that interfere with later development, a broader, more coherent view of cognitive development becomes possible. The theory proposed builds on the fundamental human activities of perception, action and reflection and, by tracing these through the worlds of embodiment and proceptual symbolism to the formal world of mathematical proof, a global vision of mathematical growth emerges. Some make only a small journey before encountering obstacles. Some remain in the initial embodied world of perception and action and cling to procedural thinking, some reflect on embodiments and become fluent in algorithms to encapsulate them into thinkable entities. These achievements may be entirely appropriate for the use of mathematics in a wide range of situations. A few may take matters further into the world of formal mathematical thinking. The purpose of developing such a theory is to gain an overview of the full range of mathematical cognitive development. It is a goal that I suggest is appropriate for the overall study of the Psychology of Mathematics Education.

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