The Long-Term Cognitive Development of Symbolic Algebra

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This paper discusses the long-term cognitive development of the meaning of symbols in algebra, starting with symbols in arithmetic as procedure, process and concept, on to generalised arithmetic, evaluation algebra (treating expressions as evaluation processes), manipulation algebra and then axiomatic algebra. We discuss cognitive changes required to move from one form of algebra to another, changes in meaning that occur and epistemological obstacles that arise in the development.

Cognitive development of arithmetic leading to algebra

Early whole number arithmetic has been widely researched (Fuson, 1992). It is well known that *processes* of addition and subtraction become seen as *concepts* of sum and difference, a viewpoint that continues to be valuable in algebra. These phenomena are part of a more general 'process-object' theory of cognitive development (Piaget, 1985; Dienes, 1960; Davis, 1983), expressed succinctly by Dubinsky (1986, 1991) and Sfard (1991, 1995) in terms of 'encapsulation' or 'reification' of a process as an object. (See Tall, Thomas, Davis, Gray, & Simpson, 2000 for a critical summary.) This theory grows from Piaget's idea (1985, p. 45, but originated much earlier) in which "actions and operations become thematized objects of thought or assimilation". Gray & Tall (1994) took a more pragmatic view of this theory, noting that symbols in arithmetic and algebra had a duality as both process (as in the addition 5+3) and as concept (the sum 5+3 is 8). From this viewpoint, the symbolism is pivotal, allowing the thinker to switch from operations (such as addition) to concepts (such as sum). Gray and Tall introduced the term 'procept' to refer to the use of a symbol acting as a pivot between process and concept. A process of addition such as 8+6 can be carried out by decomposing numbers into sums and recomposing sums as totals, for instance counting up to 10 to get 8+2 is 10, then decomposing 6 into 2 and 4, so 8+6 is 10+4 which is 14.

The cognitive development of arithmetic shows a steady increase in subtlety from lengthy counting procedures, to more efficient procedures, to remembering relationships between numbers that allow new computations to be performed in terms of previously developed knowledge. Gray and Tall took their cue from Thurston (1990) and described the increasingly sophisticated development in terms of 'compression'. For instance the sum 3+4 may be performed in a variety of ways including:

- **Count-all**: count 3 objects, "1, 2, 3," then 4 objects, "1, 2, 3, 4," then count all the objects, "1, 2, 3, 4, 5, 6, 7," to get 7 objects.
- **Count-both**: count 3 as "1, 2, 3," then count-on 4 as "4, 5, 6, 7."
- **Count-on**: count-on 4 after 3, "4, 5, 6, 7."
- **Count-on from larger**: turn the problem round and count-on 3 after 4 as "5, 6, 7."
- **Derived fact**: "3+4 is one less than 8, so it is 7."
- **Known fact**: "3+4 is 7."

These different ways of carrying out addition of two whole numbers involve a spectrum of different levels of compression. 'Count-all' uses three simple procedures of counting physical objects starting each count from 1. 'Count-both' uses only two counts (a simple count of the first number starting from 1, then a 'count-on' of the second number after the first). 'Count-on' has only *one* counting procedure but this is now more sophisticated, requiring a double-count which counts on whilst also keeping track of how many numbers have been counted. 'Count-on-from-larger' is a shorter count-on. 'Known fact' simply requires retrieving the remembered information from memory and 'derived fact' derives the required sum from other known facts.

Thus the child begins with lengthy counting procedures and compresses them, sometimes by taking a shorter count, but more importantly, by moving to a stage where the symbolism can be used flexibly to stand for either 'a concept to think about' or 'a process to calculate'.

Davis (1983, p. 257) made a useful distinction between a 'procedure' as a specific algorithm for implementing a 'process' in an information-processing sense. It is therefore useful to reserve the term 'procedure' as a specific step-by-step algorithm, whilst using the term 'process' to include several different procedures that have the same effect. Thus count-all, count-both, count-on, count-on-from-larger are all specific procedures to accomplish the process of addition. This proves to be a useful distinction in a range of contexts, for instance, $\frac{2}{3}$ and $\frac{4}{6}$ evoke two distinct procedures for dividing up an object, which give the same equivalent fraction considered as a process. In algebra $2(a+3)$ and $2a+6$ are distinct procedures of evaluation, but are equivalent processes.

In terms of the SOLO taxonomy (Biggs & Collis, 1982), we may categorise a single procedure as *uni-structural*, several distinct procedures having the same effect as *multi-structural* and the realisation that they are essentially the same process as *relational*. The encapsulation of a process into an object is then *extended abstract*, producing an entity (a procept) which can be used as the beginning of a higher level cycle of procedure–multi-procedure–process–procept. Multi-procedures give alternate methods of operation but, while they still need to be carried through step-by-step, they give only a way of choosing a more efficient procedure. It is only when the procedures are seen as constituting a single entity that carries out the same process that they reach the relational level and may later move on to be encapsulated as mental concepts.

The various levels of sophistication available in adding two numbers reveal a fundamental obstacle to introducing algebraic ideas too early. Foster (1994) looked at solutions of equations in the following form:

$4 + 3 = \square$ ....(A)

$4 + \square = 7$ ....(B)

$\square + 4 = 7$ ....(C).

All three look like early forms of algebra. However, for younger children, they can evoke very different procedures. Equation (A) can be performed by any counting procedure, including 'count-all' equation (B) requires at least 'count-on' and equation (C) may cause more problems for some children who see it as 'at what number do I start to count-on 4 to end at 7?' For a child who knows that order of addition does not matter, equations (B) and (C) are essentially equivalent. For a child knowing the flexible relationship between addition and subtraction, both of these have the solution 7-4.

Foster found that in a Year 1 class (aged 5/6), the lower attaining children could not do either type (B) or (C), but the average and above average children could solve type (C) by turning it around. In Year 2 the above average children reached almost 100% correct, a score attained by the average children in Year 3. The lower attainers in Year 1 could not do either type (B) or type (C). Those in Years 2 and 3 were more successful with (B) than (C), but continued to have difficulties with both types.

This research shows that the introduction of boxes in equations instead of letters is interpreted by some children as different types of counting problem, rather than the intended introduction to algebra.

Generalised arithmetic

Algebra is often referred to as 'generalised arithmetic'. This seems natural from an expert logical viewpoint. But is it necessarily appropriate as the precursor of symbolic algebra? The idea of algebra as generalised arithmetic is natural for some children. For instance, in a discussion with a child aged seven years and one month, Tall (2001) explained the idea of using *n* to stand for a number and 'two *n*' or 'two times *n*' to stand for two times the number *n*. After giving and requesting a few examples for *n*=2, 3, 4, and asking about the value of 2*n*+1 for various values of *n*, the child was asked:

"Is two *n* always even? ... Or is it sometimes odd?"

[Three seconds pause.] **"Always even."**

"Why is it always even?"

"Well, if you add an even number with an even number, you end up with an even number."

"Right."

"If you add an odd number and an odd number, you come up with an even number, but if you add an even number with an odd number, you come up with an odd number."

[chuckling:] "That's very good! Who told you that?"

"I worked it out myself."

In his very first interaction with algebraic notation, this child saw 'two *n*' as being interchangeable with '*n* plus *n*' and then related this to his ideas about adding even and odd. He did this, not only of adding two even numbers or two odd numbers to get an even number, but also discussing the case of adding an even and an odd. This child had shown a rich understanding of arithmetic and took up the notation of algebra at first acquaintance in terms of a letter standing for a specific-but changeable-number. For such a child the move from generalised arithmetic to algebra is a natural development.

However, it cannot be assumed that every child who has done arithmetic is ready for more generalised notions of arithmetic expressions. Algebraic notation violates the usual sequence of reading from left to right. For instance, 2+3×4 (which is intended to convey the sum of 2 and the product of 3 times 4) involves calculating the product 3×4 first to get 12, then adding 2+12 to give 14. Often children will read the expression from left to right as '2+3' (which is 5) 'times 4' which is 20. This variation in sequence of operation places traps in the transition from arithmetic to algebra which catch the unwary.

The shift from arithmetic in everyday situations to the synthetic symbolism of generalised arithmetic and algebra involves more complex expressions that cause a difficult transition for many. This transition is made more difficult by the change in meaning of the symbolism, In arithmetic, the expression 7+4 is an *operational* procept in the sense that it has a built-in counting procedure to give the result. In algebra, however, the symbol 7+*x* is first an expression for a process of evaluation, which cannot be performed until *x* is known. The difficulty of conceiving of an algebraic expression as the solution to a problem has been described as *lack of closure* (Collis, 1972). Davis, Jockusch & McKnight (1978) made a similar observation that 'this is one of the hardest things for some seventh-graders to cope with; they commonly say, "But how can I add 7 to *x*, when I don't know what *x* is?"' In the same vein, Matz (1980) commented that, in order to work with algebraic expressions, children must "relax arithmetic expectations about well-formed answers, namely that an answer is a number". Kieran (1981) similarly commented on some children's inability to "hold unevaluated operations in suspension". All of these can now be described as the problem of manipulating symbols that—for many students—represent potential processes (or specific procedures) that they cannot carry out, yet are expected to treat as manipulable entities. Essentially, even when children can handle general arithmetic, they may see algebra expressions as unencapsulated processes rather than manipulable procepts. Many students remain process-oriented (Thomas, 1994), thinking primarily in terms of mathematical processes and procedures, causing them to view equations in terms of the results of substitution into an expression (Kota & Thomas, 1998).

Evaluation algebra

Given that many children have difficulty with generalised arithmetic, Tall & Thomas (1991) used the computer to provide meaning for an expression 'A+3'. First the student has the experience of typing A=2 followed by PRINT A+3 to return the result 5. Likewise A=3, PRINT A+3 gives 6. Whatever value A is set to be, printing A+3 gives the numerical result of the calculation A+3. In terms of process-concept, we see the process of evaluation being performed implicitly by the computer. The student does not have to do any calculations. Not only can meaning be given to a range of expressions, but printing the values of expressions such as 2*(A+1) and 2*A+2 will always give the same numerical outputs, allowing the student to sense the equivalence of these expressions. Likewise, printing 2+3*A and 5*A will see the difference between these, while 5*A and (2+3)*A and A*(2+3) are all equivalent.

Using the computer in this way may assist students to give meaning to the various ways of writing expressions, including equivalent expressions, which involve different procedures of evaluation yet give the same input-output process.

Subsequently the same approach has been transferred to hand-calculators which allow the storage of numbers in stores labelled by letters (Graham & Thomas, 2000). The same principles apply. One of the novel teaching aspects of the module was the use of *screen snaps*, where students were given a screen view and required to reproduce it on their own calculator screen (figure 1).

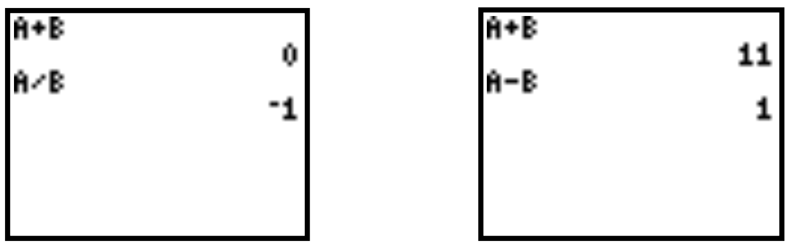

Figure 1: Screen snaps to be reproduced, requiring the student to find the values of A, B in each case

These screen snaps have the advantage of encouraging beginning algebra students to engage in reflective thinking using variables. This is beneficial since, unlike experienced mathematicians, they do not attempt to reproduce them by using algebraic procedures, but by assigning various values to the variables and predicting and testing outcomes.

The experiments with BASIC programming on a computer and variables on a graphic calculator both significantly improved the conceptual solutions of the experimental students compared with a corresponding control group. This has two important outcomes. The first is that, in today's technology, evaluation algebra has suddenly become important in a way it never was before the advent of the computer. Then it was essential to manipulate algebra to solve equations. Now spreadsheets use evaluation algebra, providing an environment to lay out relationships and make calculations and predictions without the need to perform any symbol manipulation.

The second outcome is that this approach to algebra has the potential to give meaning to algebraic expressions as processes of evaluation, and can lay the foundations to equivalent expressions with different procedures representing the same process. We suggest that equivalence is an essential ingredient to understanding the manipulation of symbols.

Manipulation algebra

What we term 'manipulation algebra' is the algebra of using letters as variables to express relationships as equations and solving those equations algebraically. It is the form of algebra that is found in traditional school syllabuses. It is often perceived by the general public as being impossibly difficult:

> For some, audits and root canals hurt less than algebra. Brian White hated it. It made Julie Beall cry. Tim Broneck got an F-minus.Tina Casale failed seven times. And Mollie Burrows just never saw the point. This is not a collection of wayward students, of unproductive losers in life. They are regular people from the Sacramento region, with jobs and families, hobbies and homes. And a common nightmare in their past.
> (Deb Kollar, *Sacramento Bee* (California), December 11, 2000.)

Given the perennial call for the improvement of algebra teaching and the regular failure to accomplish this task, it is important to ask *why* this is so. Crowley (2000) carefully interviewed students at Community College who obtained grade B in their college algebra but performed differently in the following pre-calculus course. She found that those who continued to be successful 'had readily accessible links to alternative procedures and checking mechanisms' and 'had tight links between graphic and symbolic representations'. They succeeded even though they 'made a few execution errors.' Others who succeeded in the earlier course but 'had serious difficulties with the next, [They] had links to procedures, but did not have access to alternate procedures when those broke down. They did not have routine, automatic links to checking mechanisms. They did not link graphical and symbolic representations unless instructed to do so. … They showed no evidence that they had compressed mathematical ideas into procepts.' Crowley (2000, pp. 209, 210).

On a reform calculus course using graphic calculators, McGowen (1998) found the bifurcation between students in an algebra course occurred almost from the beginning. The higher attainers were able to cope with new ideas when they met them, slightly less successful students took a week more to sustain success, but the lower attainers were sporadic in their success (McGowen & Tall, 2001). On asking students to draw maps of their developing conceptual structures, the higher attainers revealed concept maps which grew organically from previous maps whilst the low attainers tended to draw each successive concept map anew without connecting ideas together coherently (McGowen & Tall, 1999).

The learning of algebra by using a collection of procedures may help students to pass exams in algebra, but it may not prepare them for future developments. In practice, students give their own cognitive meanings to algebraic operations (MacGregor & Stacey, 1993). Many fail to give meanings that agree with standard mathematical meanings. This can be made worse by using short-term strategies that can (seem to) help at one stage, but fail later. For instance, the subject is still widely introduced by a technique that is called 'fruit salad algebra' in which letters stand for objects, such as $3a+2b$ being interpreted as 3 apples plus 2 bananas. This can give short term success, such as adding $3a+2b$ to $4a+3b$ to get $7a+5b$ by imagining apples and bananas being put together. Such an image soon outlives its usefulness when expressions such as $3ab$ are used. Is it three apples and bananas? It certainly is not 3 applies *times* bananas.

We contend that manipulation algebra can only be given flexible meaning if the algebraic expressions can be seen both as evaluation processes and as manipulable concepts. They then become procepts. However, whereas arithmetic procepts are *operational*, with a built-in algorithm of calculation, algebraic procepts have only a *potential* process of evaluation when the variables are given numerical values. This gives rise to the epistemological obstacle mentioned earlier in which algebraic expressions are not accepted as 'answers' by many children. This is the obstacle which we tried to overcome by using the computer to give meaning to evaluation and equivalence of expressions.

When powers are introduced in algebra, with a^n meaning 'n lots of a multiplied together', this can be introduced meaningfully, and the power law $a^m a^n = a^{m+n}$ also has a natural meaning. But when fractional or negative powers such as $a^{1/2}$ or a^{-1} are introduced, their meaning must be deduced from the power law. Such algebraic expressions are termed *implicit* procept because their properties are implied by the power law—a 'law' which has only been exemplified for positive integer powers. This new use of the power symbolism marks a significant change in meaning for algebra concepts. It is an early example of deduction from an assumed (and unproven) rule that arises in a more formal context much later.

Axiomatic algebra

In the introduction of more general algebraic systems such as vector spaces, the theoretic development may be performed in two contrasting ways. One is to extend previous experience, moving from linear equations in one or two variables to linear equations in several variables. Essentially this is an extension of manipulation algebra.

Another method of developing linear algebra is to start anew with a set of axioms for a vector space over a field. Harel & Tall (1991) noted the cognitive difference between these two routes. The first involves expansion of cognitive structure, the second involves reconstruction and is therefore cognitively more difficult. This therefore leads to two distinct forms of algebra, one having a *technical* structure which expands previous experience to handle more variables, the other having an *axiomatic* structure based on definitions and deductions. The axiomatic structure occurs in pure mathematics, the technical structure is used more often in applications of mathematics. The technical structure is also the type of formal algebra programmed in symbol manipulators.

Success with symbol manipulators in learning is, frankly, patchy. Research using both graphical and symbolic software often shows increased conceptual insight without affecting paper-and-pencil skills (eg. Palmiter, 1991). Sun (1993)—reported in Monaghan, Sun & Tall (1994)—found that students using Derive were more likely to get correct results for routine problems, but had little conceptual understanding, explaining concepts in terms of the succession of key-strokes for specific formulae. Roddick (1997) found Calculus & Mathematica was more helpful for (*technical*) mathematics in physics and engineering, and also more helpful in developing problem-solving strategies (which could then be used with the symbol manipulator). Meel (1995) also found students using the same approach were better problem-solvers. Considering forty PhD theses researching the use of computers in calculus (often using symbolic manipulators), Tall, Smith & Piez (in progress) found graphical software with a suitable curriculum was usually conceptually supportive, but there was a range of outcomes for the symbolic side in different contexts. Understanding the role of symbol manipulators in using and understanding symbol manipulation is still ripe for further investigation.

The meaning of 'laws' and proof change in the transition from elementary to axiomatic algebra. In elementary algebra, 'laws' are built on experiences of the operations of arithmetic. This includes such general notions as the fact that the order of addition does not matter, formulated as 'the commutative law' $a + b = b + a$. In elementary algebra this 'law' is not proved (though it may be exemplified in many ways using numbers or a visualisation of two sets put together). In axiomatic algebra the logical difficulty is removed, or avoided, by specifying such generalities as axioms to be assumed from which other properties are to be deduced. For the first time it is a genuine 'law' which acts as a foundation of the theory.

In axiomatic mathematics, operations such as addition and multiplication in rings and fields no longer have the same meanings as they do in ordinary arithmetic. Addition of two elements need no longer be based initially on counting or multiplication in terms of repeated addition or of multiplying two lengths to give a rectangular area. The operations are no longer the encapsulation of familiar processes. Every property is either an explicit axiom or must be deduced from those axioms, leading to a new deductive form of algebraic structure. The transition from manipulative algebra to axiomatic algebra therefore involves a major discontinuity in development.

The experience of students meeting formalism for the first time shows that the shift of meaning from elementary mathematics to formal theory takes a considerable time. In a foundations course at a highly-rated English university, students were given formal definitions of equivalence relation and partition and then did many exercises based on theory introduced in the lectures. Six weeks later, less than half the students gave formal responses in terms of definitions or theorems. (Chin & Tall, 2000).

Other research in the development of formal reasoning reveals successful students operating in quite different ways. One category of learners operates in a *formal* mode, learning definitions and deducing theorems, another operates in a *natural* mode reconstructing their concept imagery to build meaning for the definitions and theorems from their previous experiences (Pinto, 1998, Pinto & Tall, 1999).

Reflections

In this paper we have considered the cognitive differences that occur in algebra at various stages of development. When a child is doing simple arithmetic, the introduction of equations in a manner that seems, to the expert, to be an early form of algebra, may be seen by the learner as representing different procedures of arithmetic.

Algebra is often seen to be a natural extension of generalised arithmetic, however, it can only be used as a secure starting point for algebra if the students have a good sense of its meanings. Generalised arithmetic operations involve reading expressions in different orders from the usual left-to-right reading of (western) language, yielding a deep epistemological obstacle for many beginners.

Even when sense is being made of the symbols, they may be viewed as procedures or processes of evaluation (for given numerical values of the variables) rather than manipulable concepts. In this case, progress may be sufficient to use evaluation algebra in technological situations such as spreadsheets.

Manipulation algebra can be performed by learning procedural techniques, but we contend that it is better viewed in terms of (*potential*) procepts in which an expression is dually a process of evaluation and a concept for manipulation. Later developments in algebra require new cognitive techniques. In particular, fractional and negative powers operate as *implicit* procepts, whose properties must be deduced from the power law. Manipulation algebra may be expanded to wider technical contexts such as solving linear equations in several variables. Such extensions may continue to be classified as manipulation algebra.

Axiomatic algebra requires a new start with the operations (such as addition, multiplication) no longer seen in terms of their elementary meanings in arithmetic, but as given concepts with a list of axiomatic properties.

Long-term this reveals the *operational procepts of arithmetic* leading to *evaluation processes in generalised arithmetic,* to the manipulation of *potential* and *implicit procepts* in *elementary (manipulation) algebra* and *defined concepts* in *axiomatic algebra*. Each transition involves considerable cognitive reconstruction that may be grasped with relish by some, but act as potential barriers for many. The success of technology depends on the focus of attention of its use. It can help students to handle the arithmetic of negatives and fractions that enables weaker students to pass exams, without necessarily assuring them of further success. It can be used in evaluation algebra to give meaning for an expression as a process of evaluation. and also to build the concept of equivalent expressions that are the fundamental basis of algebraic manipulation.

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