European Research in Mathematics Education I

Inge Schwank (Editor)

Forschungsinstitut für Mathematikdidaktik, Osnabrück 1999
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SOCIAL INTERACTIONS IN
MATHEMATICAL LEARNING SITUATIONS

PREFACE

Inge Schwank
Institute for Cognitive Mathematics, Department of Mathematics & Computer Science
University of Osnabrueck, D-49069 Osnabruck, Germany
schwank@mathematik.uni-osnabrueck.de

First of all, we would like to thank all the authors - in particular those colleagues responsible for the organization of the resulting electronic documents of their group - for their great work, that facilitated the creation of the present book in many ways.

We hope that the use of this ebook will be a source not only of ideas and methodologies in mathematics education but of pleasure.

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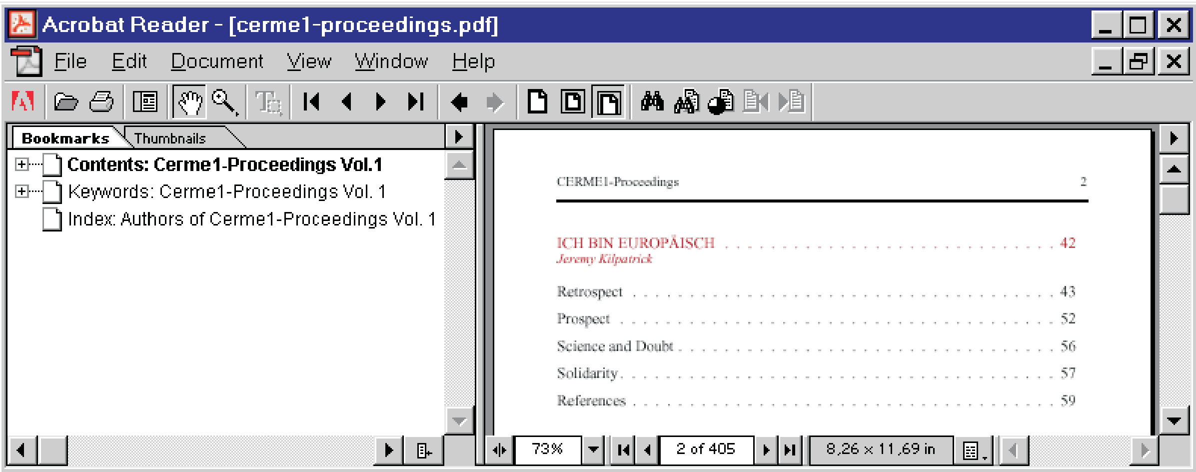

Fig. 1: Navigation features:
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- or -

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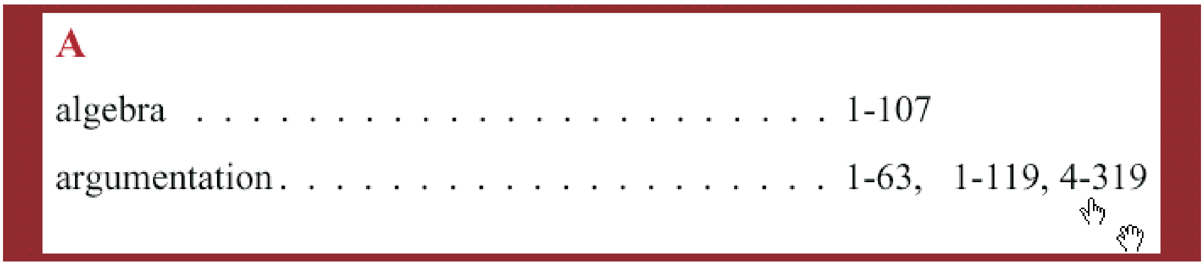

Fig. 2: Keywords: page numbers are linked areas

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A NEW EUROPEAN SOCIETY AND ITS FIRST CONFERENCE

CERME1 was the first conference of the new Society ERME, i.e., the European Society for Research in Mathematics Education. At the time of publication of these proceedings (August 1999), this Society is going through a two-year constitutive process.

It is a very exciting time for mathematics education in Europe. To launch the new society, in May 1997, mathematics educators from 16 countries met to discuss what a European society in mathematics education research might look like. The meeting took place in Haus Ohrbeck, near Osnabrueck in Germany. Representatives from these countries formed the initial Constitutive Committee of ERME. After an energetic two days, with considerable argument and voluble exchange, it was agreed that the founding philosophy of the Society should be one involving *Communication*, *Co-operation* and *Collaboration* throughout Europe.

Fundamentally we need to know more about the research which has been done and is ongoing, and the research groups and research interests in different European countries. We need to provide opportunities for collaboration in research areas and for inter-European co-operation between researchers in joint research projects.

Thus we should endeavour to be *informed* about research in different countries, to acknowledge, respect and be in a position to build on research which has already been done and is ongoing. We should create an environment for *sharing* ideas, research outcomes, research methodologies, and developing knowledge across national boundaries. We should encourage collaborative projects where researchers from a number of countries *work together* overtly to promote knowledge through research. An outline manifesto was produced which has since been developed to form a part of the developing constitution.

The first conference of the new society was planned for August 1998 with its theme *Communication*, *Co-operation* and *Collaboration* in Mathematics Education Research in Europe. It seemed of fundamental importance that the *style* of this conference should fit with its general theme. Thus, the style of the conference deliberately and distinctively moved away from research presentations by individuals towards collaborative group work. Haus Ohrbeck was chosen as venue, since it was felt its family atmosphere would support the desired style of conference.

The main feature of the conference was to be the *Thematic Group* whose members would work together in a common research area and through which they would share their individual work, think and talk together and develop a common ongoing programme of work. It was the intention that each group would engage in scientific debate with the purpose of deepening mutual knowledge about thematics, problematics and methods of research in the field. The Scientific Programme was developed relative to this basic group structure.

1. The development of the scientific programme of CERME1

1.1 Setting up the groups

The elected Programme Committee (PC) had first to consider the themes for the groups. Suggested themes were put to members of the Constitutive Committee for their comments in an email discussion with the PC. Eventually seven themes were agreed, and group leaders were sought for the seven groups. The PC strove to invite as group leaders recognised experts, each having research interest and expertise in the theme of the group. Each group had three or four leaders from different countries. One of these leaders was asked to co-ordinate the group and to be responsible for decisions and actions. In addition a balance of nations was sought in the group leadership. Of course not every person invited was able to accept, so some compromises on this balance had to be made.

The chosen groups, and the agreed group leaders who finally participated, are as follows:

1. *The Nature and Content of Mathematics and its Relationship to Teaching and Learning*
   Ferdinando Arzarello (I);
   Jean Luc Dorier (F); Lisa Hefendehl Hebeker (G); S. Turnau (Pol.).

2. *Tools and Technologies in Mathematical Didactics*
   Colette Laborde (F);
   Richard Noss (UK); Sergei Rakov (Ukr); Angel Gutierrez (S).
3. *From a Study of Teaching Practices to issues in Teacher Education*
   Konrad Krainer (A);
   Fred Goffree (NL).
4. *Social Interactions in Mathematical Learning Situations*
   Goetz Krummheuer (G);
   Joao Matos (P); Alain Mercier (F).
5. *Mathematical Thinking and Learning as Cognitive Processes*
   Inge Schwank (G);
   Emanuila Gelfman (R); Elena Nardi (Gr).
6. *School Algebra: Epistemological and Educational Issues*
   Paolo Boero (I);
   Christer Bergsten (Sw); Josep Gascon (S).
7. *Research Paradigms and Methodologies and their relation to questions in Mathematics Education*
   Milan Hejny (CR);
   Christine Shiu (UK); Juan Diaz Godino (S); Herman Maier (G).

When group leaders had agreed to act, a process for receiving and reviewing papers had to be put in place. It was agreed at the original May meeting that papers should be submitted to a group, and that, as far as possible, reviewers would be prospective members of a group. Each prospective participant could be asked to review up to three papers. In addition to participants, group leaders could identify other suitable reviewers within the topic of the group. The review process was to be based on scientific criteria and designed to be sympathetic and supportive. Thus a peer review process was set in place, with reviewers writing reviews directly to authors, and copies with recommendations for acceptance, modification or rejection to the group leaders. Group leaders would make the final decisions on accepted papers as a result of reading reviewers' comments and recommendations. Group Leaders were encouraged to recommend to authors ways of modifying their paper: additions needed, references to related papers etc. Papers not relevant to the area of the group would be rejected, or might be redirected to another group.

Papers would be read by three reviewers who would be asked to comment on the following: Theoretical framework and related literature, methodology (if appropriate), statement and discussion of results, clarity, relevance to CERME-1 audience. Specific guidelines for review were provided by the Programme Committee, based on PME (The International Group for the Psychology of Mathematics Education) reviewing criteria which were adapted slightly to the specific aims of CERME1.

Thus the process of paper selection was as follows:

1. Author submits paper to a group.
2. Paper goes through the Group's review process.
3. Comments are sent to the author, as is the review recommendation.
4. If the recommendation is *accept*, the paper is sent for reading in advance to all members of the group. This is its 'presentation'.
5. If the recommendation is *modify*, the paper is returned to the author for resubmission to Group Leaders by a given date (possibly allowing 3 weeks for further work). Providing that it is received and is satisfactory by this date, it is then sent for reading to all members of the group. If it is not possible to meet this date, participants may bring papers for distribution, but the papers will then not be read before the Group meets.
6. If the recommendation is *reject*, the paper is not presented to the group, although the author may, nevertheless, take part in the work of the group.

It was asked that the review process be as friendly and helpful as possible. Once accepted papers were known, the group leaders could then plan the work of the group.

1.2 The organisation of group working

The Thematic Groups at CERME1 were designed to be *working* groups with time given to discussion of ideas and issues in a genuinely collaborative atmosphere. Group

Leaders were asked to aim at strengthening the process of dialogue between people, aiming to support communication, co-operation and collaboration, the main themes of the conference. For this reason, 'presentation' of papers took on a rather different meaning to that usually understood. Rather than authors presenting their papers orally at the conference, papers for presentation were to be 'presented' to group members in written form for reading *before* the conference. Participants would be asked to read presented papers thoroughly in anticipation of a discussion of issues. It was emphasised that *there should be no 'oral delivery' of a paper within a group at the conference.*

The idea was that Group Leaders would draw on the accepted papers to decide on areas of interest, theories, methodologies and special questions on which the group would work together. In other words, the leaders would identify themes and subthemes related to the accepted papers, but not limited to these papers. These themes were designed to be the basis of discussion and work within the group. In working on these themes, reference could be made to the presented papers which all group members would have read in advance.

Thus, Group leaders were asked to organise the work of their group in the following ways:

1. Receive papers from prospective members of the group.
2. Read the papers and organise a peer review process.
3. Receive reviewers' comments, decide on papers for inclusion in the programme and possibly on subgroupings within the main group. Make a synthesis from accepted papers to act as a starting position for the Group. Decide on the broad themes and subthemes on which the group would work, of which the synthesis from papers is a part.
4. Organise a programme of work which would draw on papers, enlarge on themes in a progressive way, and gradually introduce new ideas and issues as appropriate to the Group's work. Modes of working might include large and small group discussions, working sessions for developing common ideas and research programmes etc. The spirit of this work should be as democratic and inclusive as is possible. Papers which are not given presentation time, should be available for reading by group members.

5. Facilitate ongoing work, after the conference, by the group or subgroups formed during the working sessions.
6. Organise informal or formal collection(s) of papers for distribution and/or publication.

In the conference, groups were given 12 hours over three days in which to meet and progress their work. Participants were strongly recommended to stay with one group for the entire conference.

Plenary Presentations from Groups

On the last morning of the conference each group was given about 12 minutes to present their work to the conference. It was suggested that this should be a synthesis of the main themes and ideas arising from their work, as well as proposed ongoing work. They were asked to communicate the style in which the group's work would be published.

1.3 Other sessions at the conference

Keynote Addresses

Two plenary keynote addresses were invited and presented by Professor Guy Brousseau of France, and Professor Jeremy Kilpatrick of the United States. We had wished also to invite Professor Vasilii Davidov of Russia, and were saddened to hear of his recent death.

Posters

To encourage new researchers, posters were invited for submission to the PC which reviewed all submissions. There was a one-hour session on in which posters were displayed and conference participants were able to talk to poster presenters. Where a poster was of direct relevance to a group, it was encouraged to be a part of that group's work.

2. Constituting ERME

Three sessions at the conference were devoted to considering the new Society ERME, its ongoing work and constitution. The ERME Board, elected at the conference, will communicate the results of these sessions.

3. Participation in CERME1

120 participants from 24 countries attended the conference. A generous grant of the "Deutsche Forschungsgemeinschaft" (German National Science Foundation) enabled 22 participants from Middle and East European Countries to attend the conference.

4. Publications from the conference

It was agreed to publish proceedings from the CERME1 electronically. They would include an account of the conference, its programme and philosophy, and the activities of groups, together with papers from the plenary speakers.

These conference proceedings have been designed to consist of one electronic book (launched on the Internet) for which the group leaders submit the individual "outcomes" of their group in a layout common to the whole book. We are extremely grateful to the group leaders for their fruitful collaboration and Inge Schwank for her outstanding editorial work.

Group leaders are responsible for publications from their group. Groups may also, individually, produce more/other publications of their own.

5. Evaluation of CERME1

During the conference, evaluation sheets were given to participants, asking for their views on and perceptions of aspects of CERME1. 46 evaluation sheets were handed in out of a conference of 120 people. The sheet's wording was as follows:

The main organisational device for the academic/scientific work of CERME1 has been the Thematic Group which was designed to promote the main aims of ERME, *Communication, Cooperation and Collaboration* in Mathematics Education Research in Europe.

1. Was your group effective in promoting these aims? Please give some specific details of what you valued or otherwise.
2. What would you recommend which might have improved the work of your group?
3. Please comment on other aspects of the conference. Are there other features which you would recommend for future meetings?

Clarification was requested on the difference between *Cooperation* and *Collaboration*. One response to this is that Collaboration means actively working together - e.g. setting up joint research projects. Cooperation is about being supportive, sharing, respecting, listening – e.g. if research projects in different countries are addressing similar questions, researchers can gain from taking seriously each other's methods and findings, sharing results, discussing outcomes etc. but without actively collaborating in the research.

What follows is a summary/overview of what was said in the sheets. It is not a statistical analysis! Numbers have largely been omitted. The sheets are available if anyone is interested in more explicit numbers.

The overwhelming response to the first question was YES. Communication was thought to be well promoted. In some cases, there were suggestions that Cooperation and Collaboration were less well promoted (more about this below). The work of the group leaders in coordinating, structuring and providing activities for the group was overwhelmingly praised. Many respondents indicated their appreciation for a firm structure which made sure that all group members were involved. Discussion rather than oral presentation was appreciated, but some felt that more space could have been given to authors of papers to elaborate on their work.

Some people said explicitly that they had valued this style of conference, and that future conferences should keep the same philosophy and style. There were far more people who said this than who said the reverse.

It was suggested that the size of the conference was important to its success, and that this style of conference depends on a relatively small number of participants (e.g. <200). Many said that the size of a group should be at most 15 people.

However, 4 people were not happy with the conference. Two of these were unhappy with their groups, suggesting that ideas were imposed by the group leaders and that there was no democracy in the group. The other two favoured a much different style of conference with a variety of types of sessions etc.

While valuing this style of conference there were some comments which suggested that a broader format might have been appreciated. These included suggestions for the following:

a) Possibility to attend two groups.
b) More posters or other types of presentation (although two people commented that having posters was not in the spirit of this conference!)
c) More opportunity for collective activity.

There was a strong feeling that the conference was too strict in its very tight programme and that more opportunity for meeting with and sharing with people from other groups would have been appreciated. Some people asked for longer and clearer breaks. Others wanted sessions built into the programme e.g. discussions of plenary talks; subplenaries; workshops. There was a suggestion however that 2 plenaries was “just right”.

The breadth of group interest was in some cases seen as a problem in that it sacrificed the depth which can be achieved by focusing more narrowly. Some people said it was hard to see into which group their paper fitted. It was indicated that for a first conference this breadth was probably necessary, but that for future conference, more focused groups would be better.

While the efforts of group leaders to encourage all members of a group to take an active part in its work was praised, there was recognition of the language difficulties involved. It is clear that, however well this conference did in trying to overcome language barriers, more thought needs to be given to what is required to allow all to participate fully despite the language problems.

There was considerable feeling that the conference could have provided more opportunity to learn about the research in other countries.

Regarding Cooperation and Collaboration, some people said that there were tentative beginnings where people in a group were suggesting ongoing work. However, it was felt by others that this conference had really only just started the process, and that much more effort would be required before true C&C could be seen to be happening.

One suggestion was that rather than starting with individual papers, groups might start with ideas for discussion from which collaborative papers might be written. A product from the work of groups was seen to be as important as the processes involved.

There was a suggestion that we might include work which bridged the divide with other disciplines.

The work of participants in reading papers in advance to prepare for work in groups was praised. This was felt to have contributed largely to the success of group work. One suggestion was for papers to include “meta-level user’s guide” to the paper. Some suggested that a written synthesis could be made in advance from several papers. One suggestion was for a survey of related literature to be provided. It was not clear who was supposed to make such provisions.

Two participants queried the quality of papers presented, suggesting that it was not totally at a level suitable for such a conference.

There were comments about the domestic and social nature of the conference. There was much appreciation for the accommodation at Haus Ohrbeck and the facility for everyone being close together on the same site. A number of people felt unhappy with the hotel being so far from the house. A few people felt the accommodation too restrictive, and would have appreciated greater facilities. The lack of email provision was remarked by a few. Some groups would have liked a bigger room. On the social side, some felt the evenings could have been livelier, perhaps with music and dancing. Some people felt that the venue should have been easier to reach. Some felt that it was expensive for the facilities it offered. These last two comments were from very small numbers of people. However, there was much appreciation for the work of the secretariat, both in setting up the conference and in running it and being so helpful to participants. The web page was also praised.

Although this summary has necessarily mentioned all comments both positive and negative which were made, it would be a mistake to feel that the negative comments overwhelmed the positive. This was not the case. An overall appreciation was indicated for the style of conference and the work which had gone into organising it.

6. Conference Programme Committee

CERME1 was organised by the following Programme Committee elected the 1997 May meeting:
Elmar Cohors-Fresenborg – *Coordinator* (Germany), Milan Hejny (Czech Rep.), Barbara Jaworski (UK), Joao Pedro da Ponte (Portugal), André Rouchier (France).

Barbara Jaworski
On behalf of the CERME1 Programme Committee
September, 1998

RESEARCH IN MATHEMATICS EDUCATION: OBSERVATION AND … MATHEMATICS

Guy Brousseau
Université Bordeaux 1/LADIST, 40 Rue Lamartine, 33405 Talence, France
Guy.Brousseau@cribx1.u-bordeaux.fr

***Abstract**: Mathematics education is a subject whose complexity justifies, in its examination, various break downs into constituent parts. But it is difficult to verify the reintegration of partial studies within the knowledge of global process. This legitimization requires specific concepts and observations, among others clinical observation, both participatory and controlled. The field of research in mathematics education is open to numerous disciplines. Part of research in the didactics of mathematics is mathematical activity aimed at mathematicians, as well as teachers. It can only succeed through interaction with the community of mathematicians.*

1. Introduction

I feel very honored that the organizing committee has provided this chance for me to address you all, and I thank its members.

I hope you will forgive me for not providing, as the century comes to an end, a panoramic overview of actions in the domains that concern us.

Since 1960 I have participated in a number of them, and would have evoked with pleasure the projects, the hopes, the actions of some of the principal European architects of research in the field of mathematics education.

These actions are alas too numerous and too complex, their originators still too close to us to allow a short yet satisfactory synthesis, one that would be acceptable to all our diverse points of view. Such an effort calls for more time, resources and talent than I can muster today.

I hope you will also forgive me for not presenting some sort of «Osnabrück program[1] » or an overview of subjects and tasks about to be tackled by the community of young researchers. I am unlikely to do better than Hilbert[2], and one sees how, even in mathematics, the march of science must submit to contingency and admit of what we might call a historical character, in other words one that escapes our own conception of necessity.

Finally, I will request your indulgence. What I have to say may seem banal to most of you, but this is neither the time nor the place for making scientific revelations. Let me only try and highlight a few comments, a few ideas around which we can come to an understanding. Such understanding might well be of use to the extended fraternity of researchers and participants with which we are dealing.

Thus I will not expound upon the points that seem to you the most urgent and most important.

Given the language and cultural differences, and differences in educative systems, we find constituted within Europe a paradigm that includes almost the whole range of educative propositions in the world at large. All the difficulties and all the obstacles that prevent a common culture and science are present. Problems about research orientation, about joining researchers to institutions, about the organization and diffusion of knowledge produced, all occur at the same level of complexity as those met on a worldwide scale. The members of our association will no doubt prove their usefulness by the way they deal with both the richness and weakness of this situation. For 40 years I have regularly witnessed small teams of researchers in mathematics education move from one institutional connection to another, as dictated by fashion and social conditions, without solving a single problem concerning epistemological foundations and the social legitimization of their work, driven along as ever by immediate necessity.

I'll limit myself to raising 3 questions.

The first concerns the relationship between a researcher in mathematics education with the subject of research, i.e. the students and their teachers at work.

The second question concerns the role of what in many countries is called didactics[3], in the diffusion and representation of knowledge in mathematics education.

This particularly involves certain forms of observation: clinical "participatory" observation, and controlled experiment.

The third concerns the relationship between the researcher in the didactics of mathematics and mathematics itself, and with mathematicians and their institutions.

When back in the 1960s I first became interested in mathematics teaching, the influence of Jean Piaget made one thing evident: the primary need for greater familiarity with spontaneous cognitive behavior of a mathematical nature, when a child is confronted with a disposition - an environment - unencumbered by didactic intentions. But then the influence of Célestin Freinet, Makarenko, and Vygotsky led me to consider the environment as at once material, social, and cultural. The specific mathematical interactions between the learning subject, and the elements pertaining to this environment, had to be isolated. From these we had to create a model situation, where constraints and evolutions could be studied. This point of departure explains the importance of psychological studies, and the study of cognitive epistemology in the study of mathematics education[4]. It also explains their eventual separation from teaching research or mathematics education proper.

2. Subject of research

The subject to be studied by researchers brought together here is « mathematics teaching. »

It is a subject at once vast, complex, and very important[5]. It would therefore seem imperative to break the subject down into its component parts, insofar as one is not discouraged by the resulting complexity. This breaking down can be done in a number of ways:

- into hypo-systems[6]: teachers, students, educational environment, knowledge, teaching and learning, etc.

- into infra-systems[7] that can be apprehended within the classic disciplines made available by culture: problems psychological, sociological, linguistic, mathematical, meta-mathematical, epistemological among others, the whole considered in terms of teaching mathematics.

Intensive development in these domains contributes still more to the complexity of approaches, for this multiplies the possibilities of independent studies and indefinitely prolongs the time needed to form researchers in our domain.

This complexity explains the variety of preoccupations of researchers in the realm of mathematics education, their difficulties over delimiting their field and communicating the results of their work, the multiplicity of their institutional ties, their isolation, and their need to come together in society. It also explains both the necessity of ESRME and justifies the diversity of its composition.

But in what measure can knowledge concerning the reductions of the system under study, or established within disciplines that only partially take into account the system, remain valid for the system as a whole? And how can they be integrated and used in didactic actions?

The disappointment caused by the ill-considered importation of knowledge from concepts or theories otherwise firmly established is well known.

For example, is the organization of knowledge adapted to a certain group mathematicians a good « model » for organizing knowledge in the case of a child, or an adult beginner? On the other hand, with what ethical, scientific, experimental right do we « simplify » and transpose cultural knowledge in order to educate, to cause students to be educated according to it?

The again, doesn't the uncontrolled intrusion and accumulation of dispositions, and the introduction of psychological reasoning into teaching, sometimes have negative effects? To what extent can we use results derived from the observation of a *subject,* isolated in a one-to-one relationship with a psychologist? Can we predict from them anything about the behavior of the *student* this subject becomes, once implicated in a process that includes both a teacher and an entire class?[8]

Neither is the simple rationalization of teaching practices and praxeology beyond reproach. Evidence indicates that ill-considered use of evaluations of objectives tends not to reduce, but increase failure.

Established knowledge imported from what we call disciplines « of reference », is necessary and important. Its integration, however, presents unique problems that are part of our field [9]. Perhaps his or her study lies at the very heart of our work, for no one will do it in our stead.

We cannot consider this integration to be transparent, nor a mere question of surface arrangement. Neither can we, a priori and without examination, allows the verification of the validity of our knowledge, relative to mathematics teaching, to rest on the simple juxtaposition of this knowledge of reference. We cannot in particular allow the indispensable empirical link of our subject to depend exclusively on concepts, institutions and methodologies adapted to other ends.

It is recognized - I for one am persuaded - that specific concepts and methods are indispensable, by which I mean ones sufficiently removed from known concepts. Should this indeed be the case, we must provide a forum for debate, a means of accepting or rejecting concepts according to appropriate criteria.

The subject of these concepts would be " the study of conditions for the diffusion of mathematical knowledge useful to men and their institutions ", containing within it a more limited project, " the study of social projects allowing institutions to appropriate, with or without their cooperation, knowledge either already constituted, or on its way to being so. "

The scientific character of research that has not entirely been " taken on " by disciplines of reference, depends on two factors. Certainly it depends on the new concepts being theoretically consistent and compatible with established knowledge in other disciplines. But it depends above all on their valid confrontation with contingency, which implies an interaction between the community with the subject under study, and its meticulous observation.

How is information gathered, how is its validity examined, and how is it then circulated, how does it manifest itself within the community? How is it discussed and transformed, and inversely how does the community act upon education, to satisfy its own need for information?[10]

It is therefore proper to ask ourselves about the interactions between our academic institutions and the mathematics education “ milieu ” - in other words the entirety of systems in interaction - which we think resides at the center of phenomena under study[11.]

The essential lies in the nature of these interactions. Which of them can insure pertinence, validity and sufficiency when the knowledge we work on is put to new uses, whatever its origin?

This point merits particular attention.

3. Forms of observation[12]

3.1 The subject of observation

The form of mathematics education most currently envisaged is one where a teacher is held responsible for the acquisition, by a group of students - a class - of designated mathematical knowledge. This acquisition occurs within a given lapse of time, and in an environment or milieu, i.e. where learning conditions are determined.

The work of a mathematics class is a good paradigm for all other teaching models[13]. It therefore adequately represents what we intend to study. Likewise the community can be represented by a single researcher (or by a small group) who takes this class as his subject. The researcher attempts to construct true and transmissible knowledge about conditions that either favor, or inhibit, the teaching process. « New » phenomena are identified. Which interactions between the class and the observer might bring this about, what is the language, the vocabulary and syntax, he can use? The relationship between researcher and subject must be taken into consideration. In any case, what would be of an epistemological (or metascientific) nature in the « hard » sciences, here becomes, if only provisionally, the subject of the field of study. The researcher is one of the elements of the system, his interactions must be analysed with the same means as those he himself uses to study the interaction between other systems.

3.2 Passive observation

A researcher can remain in strict observation, without personal intervention of any kind on the education process. However, the phenomena he intends to observe do not occur in concentrated form for all teaching processes. Thus observations must be noted in a way that allows the researcher to return to them according to the evolution of his reflections and analyses.

And because his research is habitually the result of a process, or even of a dialectic, it will be in his interest to record as much information as possible. This information will constitute the matter of his study - its « contingency ». His work then becomes that of a paleontologist: to distinguish and extract from this matter what is necessary to separate it from what is, provisionally or residually, of the order of contingency or history. This work already poses interesting theoretical and practical problems. What relation, if any, exists between the observations of a researcher recording from time to time students' or teachers' responses to questions specific to his research, and the ongoing process of the class?

Moreover, it is indispensable to create conditions that probably won't occur naturally, in order to study them. A system delivers more information when it reacts to well chosen stimulation.

Yet to intervene for purely scientific purposes in the educational system, purposes foreign to that system's mandate, raises important ethical, and consequently methodological and scientific problems.

3.3 The educative action, research - action

Another solution is to integrate one's research totally into the system. For the researcher this means sharing the different positions of those participating within the system. It also assumes the fact that he intends to transform it, not only in order to understand it, but also to reveal its social function and improve it. The researcher at once studies the entirety of interventions, their actions, and those of other participants and their results.

In the above instance researcher and teacher form a team, each taking on the role of teacher or researcher indifferently. In other instances this fusion will work with the coordinator, the group leader, the inspector, or the minister. This configuration occurs very often. It allows the acquisition of invaluable knowledge. Yet it has frequently led to considerable confusion, and sometimes contributed to very negative interventions when pushed to the extreme.

The value of knowledge acquired in these conditions is more often than not to be accepted with prudence. An assertion is not valid just because the actions associated with it succeed. Nor is it enough for the teacher to have created a few favorable conditions and excelled, etc.

We can see here and there a succession of reforms, founded on ideologies, and on the optimistic appreciation of certain haphazard results obtained by a few « researchers ». But in fact no reform corrects the errors of preceding ones. On the contrary each adds its own pitfalls to the previous ones. Teachers are in no position to integrate and regulate the flood of commands they receive from all sides, but neither can they sustain against them a coherent set of practices. It is very wrong to suppose that at the bottom of everything lie certain " traditional methods ", an indestructible adversary and indispensable support for all innovators.

No one wishes to deny teachers the right to seek out improved teaching conditions and better their students' results. We mean only to adjust the relationship between this effort and the ethics of their profession, its present technical possibilities, and the scientific knowledge of the moment.

The freedom of the intervener, whether researcher or other, to depart from established practice, should be limited by his ability to correct errors eventually resulting from his original action. Such corrections should be effected by established practice.

This freedom implies assuming partial responsibility for results, insofar as researchers become involved in action on the educative system. They must therefore regulate the actions that correspond to their initiatives according to a system of regulation prepared in advance[14].

3.4 A form of research in didactics: « participatory observation »

Of course we may reject the theoretical model « research - action », while still accepting the education process as at once the subject of observation and the disposition of experience. No doubt it is not possible to completely separate:

- conceptions of interventions that aim directly at producing a decision or a procedure for accomplishing or improving teaching,
- conceptions that aim at the possibility of establishing a valid statement about teaching conditions.

Yet if we consider the tripartite « students - teacher - researcher » as a system, a minimum number of rules can be established. These will allow the system to work in terms of its obligations and regulation.

An example exists of such a system. Its construction calls for considerable effort, but its numerous results are both original and interesting. Here are a few rules governing this system of controlled, participatory observation.

The researcher conceives and chooses

- the new conditions in terms of the learning experience he would like brought about,

- the exercises that allow verification of the intended acquisitions and of the minimum success to be obtained.
- a length of time for the intervention that includes time for the proposed lesson, and supplementary time for remedial actions should results prove insufficient.

It is best to compare teaching conditions (efforts, time necessary, etc) that give identical results.

The researcher proposes these conditions to a teacher (or a system of education). When he accepts, the teacher retains full authority to interrupt the experiment at any time, should he judge the contract is not being respected. He then takes over the work in his own way. Priority is given to insuring the best results for students, according to the criteria taken together. These results are followed up and verified within the establishment for the duration of a student's schooling, together with the research organization and the teachers, whether or not there are experiments still in progress.

The researcher observes the group of processes and determines the realization of the conditions he proposed, as well as events that seem to result from it. Experimental phases on the whole effect rather short periods of time, if only because of the amount of work that goes into organizing the data. They represent at most 10 percent of teaching time.

In order to insure a certain stability of practices, the teachers (3 for 2 classes) do « lessons » prepared and written in common. They are able to present or continue either lesson in turn. Modifications are made in cooperation with the research organization. No a priori « common pedagogical doctrine » is postulated, none is rejected, but all realizations are discussed.

These dispositions preserve the functioning and regulation of teaching under the responsibility of the « education expert », i.e. the teacher. They are balanced by others that encourage cooperation with the researchers. For example, the teacher is recruited by the research organization with administrative agreement, according to a contract of limited duration, but renewable.

The cost and cumbersomeness of this system appears excessive today. Considering the diversity of ways opened to researchers, and the exploitation and diffusion of information gathered, this disposition hardly offers an encouraging perspective. However, it seems to me essential that in some instances the community should keep open the possibility of benefitting from this sort of relationship with mathematics education.

4. Relationship between the didactics of mathematics and mathematicians

80 percent of mathematics research consists in reorganizing, reformulating, and “problematizing” mathematics that have already been « done », by the researcher himself or by others. In the majority of cases the researcher also teaches mathematics, rarely the same ones, which again calls for reorganizing, reformulating, and “reproblematizing” the area of mathematics he intents to teach, in order to adapt the mathematics to his students and their needs. Of course, to create good mathematics, he pursues other activities and exhibits other qualities, distinct from those necessary to « teach well ». But to reorganize, reformulate and “reproblematize” an area of mathematics more or less vast, constitutes the essence of didactic activity as opposed to pedagogical activity, where the specificity of transmitted knowledge does not intervene. Thus all mathematicians are practitioners and consequently connoisseurs of didactics as applied to mathematics.

Inversely, all mathematics teachers are mathematicians, or at least in the broad sense suggested by Thurston: « I call mathematician anyone who develops (improves and extends) the human comprehension of mathematics »[15.]

4.1 Difficulties and failures of didactic activities

In spite of extensive development of the means of communication and of mathematics teaching, spontaneous didactic activity has its limits. Within the community of mathematicians is regularly heard the recrimination that good syntheses are too rare to insure the verification of a consistent whole, threatened by the huge influx of disparate theorems. Without, others complain about the difficulties of diffusing mathematics in institutions where they are necessary, and about the « failure », on the part of a profession that should see to this need, to teach them. This holds true even in France, where the ratio of mathematics teachers is somewhat better than elsewhere. To know and practice didactics as an activity is no longer sufficient for mathematicians when it comes to answering both the needs of society and its appetite for control. In the same way the knowledge of economics put to use by store owners and housewives, hardly suffices when it comes to today's economic questions. Moreover, just as there exist phenomena in economics, there exist didactic phenomena, irresistible yet unsuspected by those who act, phenomena whose correction demands analysis and comprehension. The domain that defines and takes charge of these phenomena is the didactics of mathematics, or the science of conditions specific to the diffusion of mathematical knowledge useful both to institutions and to men.

Mathematicians need a science in order to carry out their dual task of research, and education. They also need a science in order to assume their social responsibility, including an epistemological vigilance vis-à-vis the diffusion of mathematical knowledge. For when it comes to judging the quality of teaching in their discipline, society turns to them for the last word. Work accomplished in the domain of mathematics didactics is destined not only for teachers and researchers, but for the entire mathematical community. Part of its own efforts must be consecrated to these questions.

The ambition to understand and facilitate the didactic task of mathematicians and teachers calls for the use of diverse disciplines, as much for practical as epistemological reasons. This explains the diversity of researchers interested in the didactics of mathematics, and a certain confusion in its given representation. Yet as in economics, didactics has its unique object , to be approached in its own, independent way as compared to other approaches - psychological, sociological, linguistic, historical and so on. There is no reason to subordinate the didactics of mathematics in some non-exclusive way to any one of these disciplines, even if their usefulness in this domain is not contested.

Phenomena linked with didactics diffusion are not, by definition, independent of the knowledge diffused. Moreover, it is recognized both within and without the community that the diffusion of mathematics, and in particular the teaching of it, is the business of those who create, and not of those who repeat. In the study of processes through which mathematics is diffused, it is a question of describing or producing mathematics, and here mathematics is doubly implicated: at once as the subject of study, and as the means of these studies, although the subject and the means do not necessarily belong to the same area of knowledge. In these conditions, and a fortiori, the core of didactic of mathematics is necessarily a work for mathematicians, namely for some researcher they discuss with mathematicians inside of their community.

To link posts in the didactics of mathematics to mathematics itself is therefore advisable. I would add that it is in the interest of mathematicians to integrate the didactics corresponding to their specific area. For on the one hand many students of mathematics will not find employment in mathematics in the strict sense, ending up teachers of mathematics, and on the other because 80 percent of their activity with be none the less of a didactic nature.

To conclude, the decision on the part of important scientific universities to attribute, at a professorial level in mathematics, a post to the didactics of mathematics, seems to me an excellent and altogether judicious initiative.

The didactics of mathematics has developed in France mainly through the efforts of mathematicians provided with an IREM (Institut de Recherche pour l'Enseignement des Mathématiques)[16], in order to examine the teaching of mathematics, and to act upon this teaching in ways far more rewarding than through mere administrative decision. Scientific didactics, both theoretical and experimental, has progressively detached itself from immediate action on the practices and reforms of education, but profound affective and technical attachments subsist. These ties seem to me not merely a legacy but a necessity. In any case we set up our IREMs according to this conception.

For nearly 30 years now the didactics of mathematics has progressed considerably and stands ready for truly important advances. On the other hand diffusion has remained rather confined, whether vis-à-vis mathematicians, more interested in other tasks, or vis-à-vis teachers and professors, caught up in storms of reform diverse in origin.

Other communities (specially in psychology or pedagogy) has a very false idea of our area of investigation.

5. Conclusion

I cannot conclude without reminding you that our work over the years has relied on an astounding number of people and researchers, both European and other. They have through very diverse means attempted to understand and improve mathematics education. I would have liked to name a few, but my choice could only be unfair. Today you might call them « precursors », but more often than not we haven't as yet appreciated to the full extent their works, nor drawn the consequences of their successes and failures. The researchers of my generation sometimes feel they've left their young colleagues in a discouraging shambles of work, vast and very rich, but very disordered. Moreover, the perspectives for action are much narrower than those we found. I cannot encourage them enough, to continue examining a domain for which I have had an abiding passion, and which has given me so much joy.

Notes

1. reference to the Erlangen Program from Klein
2. reference to his conference of 1900 in Paris
3. or what we might have called didactology
4. An importance reflected in the name retained for a research institution we proposed at the ICME congress in Karlsruhe (together with Vergnaud and Fishbein, among others) viz., « Psychology of Mathematics Education ».
5. Too much so for P. Greco, a genetic psychologist who supported enthusiastically the first steps taken towards a didactics of mathematics (see articles in the Encyclopedia Universalis among others), only he did not believe in a *scientific* study!
6. Hypo-system: all relations relative to a few sub-systems

7. Infra-system: relations concerning the entirety of sub-systems
8. Because of these problems I found myself obliged to study the articulation between « didactic » phases, where the teaching intention is announced, and « adidactic » phases, where the teacher's didactic intentions are provisionally unknown to the student, whether purposefully or not.
9. H. Steiner proposed the study of this problem by creating the TME group.
10. It is not proper here to take into consideration actions motivated by other reasons.
11. The study of the relationship between theory and practice has been undertaken by a group created by Bent Christiaensen.
12. We cannot help but evoke the work of J.M. de Ketele in this domain.
13. It engenders them, either by generalization - for example, an institution wishes to modify the mathematical means and procedures for decision making of another institution - , or by specifications or reductions of present elements - for example, the teacher has only one student, or else the teacher acts through the intermediary of diverse media, or else he becomes a fiction, as in autodidactism, etc.
14. Once again I call attention to the methodological error consisting in gross comparison of student success percentages and exercises proposed in the course of learning. These percentages are regulated variables of the system, they measure marginal differences.
15. William P. Thurston: « On proofs and progress in mathematics » (in answer to Jaffe and Quinn), in the Bulletin of the American Mathematical Society. Vol 30 (1994) 161 - 177.
16. and also the efforts of all those who pushed for the great reforms of the 60's and 70's in Europe: J. Dieudonné, G. Choquet, C. Gattegno, G. Papy, W. Servais, Anna Krigowska, Hans Freudenthal, preceded and followed by many others.

(Translated from french by William Fergusson)

ICH BIN EUROPÄISCH

Jeremy Kilpatrick
University of Georgia, 105 Aderhold, 30602-7124 Athens, GA, USA
kilpat@coe.uga.edu

All of us are familiar with Felix Klein's *Erlanger Programm*, which he published in 1872 when he became professor of mathematics at Erlangen. As with much of our history, however, we often get the details wrong. A common impression is that it was in Klein's inaugural lecture on 7 December 1872 that he laid out the program that was to change how mathematicians looked at their discipline. In fact, Klein's *Erlanger Programm* was a written paper distributed at the time of his lecture but quite separate from it. His "*Erlanger Antrittsrede*" was directed to a general audience, and its theme was mathematics education (Rowe, 1983, 1985). In the lecture, Klein deplored the growing division between humanistic and scientific education. He saw the content of mathematics as outside both, even though, if a partition is made, the vital role mathematics plays in many scientific disciplines rightly puts it with the sciences. In the lecture, Klein made no reference to research in mathematics education. Later, however, after becoming a professor at Göttingen in 1886, he was active in helping get such research underway. He eventually taught courses in mathematics education, supervising the first European doctoral degree (*Habilitation*) in mathematics education in 1911. He also pushed for the reform of the school mathematics curriculum. During Klein's lifetime, he published over 30 books and articles on mathematics education (Rowe, 1983). His activities can be seen as helping to launch mathematics education as a field of academic study.

Klein was the first president of the International Commission on the Teaching of Mathematics, which was formed in Rome in 1908 and which invited countries to undertake studies of the state of mathematics teaching. The journal *L'Enseignement Mathématique*, founded in 1899, became the commission's official journal and recorded the growing interest around the world in how mathematics is taught and learned (see Kilpatrick, 1992, for further discussion of the commission and its activities). Other European mathematicians of the time also contributed to our thinking about mathematics education, including Samuel Dickstein, Jacques Hadamard, Henri Poincaré, Guiseppe Peano, and John Perry. Influential educators such as Maria

Montessori and Charles Godfrey, among others, were helping to develop the field by turning their attention to children's learning. The two patriarchs of mathematics education in the United States at the turn of the century, David Eugene Smith and Jacob William Albert Young, were both influenced very strongly by the work being done in Europe, and specifically in Germany.

Thus we can see that mathematics education has deep roots in Europe, a continent that has changed dramatically over the past century. It is useful to remind ourselves of those roots as we celebrate the launching of the European Society for Research in Mathematics Education (ERME), an organization bringing together people from across the continent as well as outside it. The economic ties that European countries are developing, as we are all aware, are part of a general lowering of barriers not only politically, economically, and socially, but also educationally. The "new Europe" presents researchers in mathematics education with challenges and opportunities that you no doubt see more clearly than I do.

In the remarks that follow, I hope to suggest how research in mathematics education has been changing, with some ideas about where it might be headed. Although my emphasis is on research over time, I want to make some observations too about research across borders. I offer some comments on our need for a stronger critical stance toward our own work and that of others and then end by explaining the title I have chosen for these remarks.

1. Retrospect

Although this century began with the prospect of a bright future internationally for research in mathematics education, war, economic depression, political turmoil, colonialism, and other maladies ruined that prospect in many countries of the world much of the time, and perhaps most especially in the decades between 1910 and 1950. As long as a country was not able to educate its citizens to a reasonable standard, it could not afford the luxury of studying the mathematical part of that education. People such as Klein, Godfrey, Hadamard, Montessori, and Smith may have broken ground in the field by conducting investigations, forming organizations, writing books, or starting journals, but it was not until much later in the century that research in mathematics education can be said to have begun in most countries. Still today, there are many countries in which researchers such as ourselves have no opportunity to meet and exchange ideas, no organizations to support journals and other publications.

Mogens Niss (1998) has claimed that the establishment of mathematics education as an academic field on the international scene is a phenomenon of the past three decades—within the professional lifetimes of a few of us. Actually, Niss used the term *academic discipline*, a term that I find somewhat problematic in view of its connotations regarding phenomena and methods of study. But three decades seems about right as the time during which research in mathematics education found a voice and a presence internationally. Certainly the First International Congress on Mathematics Education held in Lyon in 1969 was a watershed for mathematics education and might be seen as one for research as well.

At that congress, E. G. Begle (1969) argued that a large number of the questions debated in mathematics education during the previous decade had been provided with answers, “in many cases with more than one and sometimes with contradictory answers” (p. 233). He observed that the answers almost always had a factual side but that the necessary empirical evidence was generally missing. He not only gave some examples of work he and his colleagues had done to address these questions but also laid out an agenda for research that would deal with other questions. Although we might quarrel today with the way in which he approached research—relying heavily on experimental designs, controlled treatments, standardized instruments, and elaborate statistics—we should give him credit for the problems he identified. Some of the topics he thought worthy of our research are still considered important and being explored: the teaching and learning of specific mathematical concepts, the introduction to formal reasoning in mathematics, how to reach the “proper level” of computational skill, “what makes the effective teacher,” and the effects of the cultural milieu on learning and doing mathematics. Begle’s approach to research and to its role in changing practice might at times have appeared simplistic, but he had an astute grasp of the issues research could and should address.

In Europe in the late 1960s and early 1970s, research in mathematics education seemed to be expanding everywhere, as can be seen by the establishment of institutions where mathematics education would be studied. In the U.K., the Shell Centres for Mathematical Education at Chelsea College and the University of Nottingham were founded, bringing together teams of academics who not only developed some ingenious instructional materials but also conducted careful research into teaching and learning with those materials. In France, the Instituts de Recherche pour l'Enseignement de Mathématiques (IREMs), and in the Netherlands, the IOWO (now the Freudenthal Institute) were founded about the same time, and they too worked on curriculum development and research.

In Germany, the Institut für Didaktik der Mathematik (IDM) was established in Bielefeld, a center where many of today's German researchers in mathematics education got their start. I remember the founders—Hans-Georg Steiner, Heinrich Bauersfeld, and Michael Otte—visiting Cambridge University, where I happened to be on sabbatical during the 1973-1974 academic year. They were working on the twin problems of setting up a research institute and identifying the problems that the institute would investigate. They were excited, and I was excited for them. Although it had not been easy in any of these cases, support from governments and from foundations had allowed these centers and institutes to get off the ground. At last, it seemed, mathematics education had found a place in the university and in academic life generally.

Today, I look back nostalgically to those times, when the future seemed so bright. All of the institutions I have mentioned eventually fell on hard times, not just economically but politically and bureaucratically, as they faced opposition inside and outside the universities that housed them. Research in mathematics education has had difficulty justifying itself to the people who make decisions about supporting it. I am not confident that now, if more money were to become available for research in the countries I have mentioned, mathematics educators would be able to secure a share of it as easily as they could 25 years ago.

I do not, however, look back with nostalgia at every research study that was being conducted in the late 1960s and early 1970s. We have clearly moved beyond the time when experimentation and the control of variation set the standard for our research and statistical analyses drove much of its form. We have widened the scope of our work to include detailed analyses of students' learning individually, in small groups, and in the classroom. We have looked more carefully and analytically at mathematics teaching in ordinary classrooms. We have examined teachers' views of mathematics and teaching as those views are expressed in both their rhetoric and their classroom actions. And we have taken a hard look at the nature of mathematics—how it is created, its functions in society, and its cultural dimensions—as well as its epistemology. Researchers today are much more likely to be seeking an interpretive understanding of mathematics teaching and learning than to be simply quantifying variables and formulating generalizations.

Niss (1998) has nicely summarized many of the findings from this research (for more detailed summaries, see, e.g., Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996). He offers the following list:

- *the astonishing complexity of mathematical learning*
- *the key role of domain specificity*
- *obstacles provided by the process-object duality*
- students' alienation from proof and proving, and
- *the marvels and pitfalls of information technology in mathematics education*

Most of these topics are represented in the conference program for this meeting. I comment here on only the third and the fifth. I recently reviewed the book *Learning and Teaching Mathematics: An International Perspective*, edited by Terezinha Nunes and Peter Bryant (1997). The book offers a nice survey of recent work in which mathematical thinking is viewed as socially constructed and culturally embedded. It well portrays where the field of mathematics education stands vis-à-vis psychology. One thread that I noted in several chapters, although the editors did not characterize it as a theme, was the process-object duality. This duality, discussed in the Nunes and Bryant book in separate chapters by Régine Douady, Carolyn Kieran, and Gérard Vergnaud, has also been noted by Anna Sfard (1991) in her work on reification, by David Tall (1991) in introducing the notion of *procept*, and by Inge Schwank (1993) in her distinction between predicative and functional cognitive structures. The duality is sometimes expressed as a contrast between mathematics as tool and mathematics as object. This idea has been around for some time, but I think there is still much work to be done with it, both in exploring its implications for understanding mathematics and in examining ways in which teachers might deal with the duality in instruction. Much of the work thus far has been done with concepts and processes from advanced mathematics. We still face the challenge of studying the implications of the process-object duality for elementary school mathematics.

Niss's fifth point above deals with information technology. This topic is obviously a site for much current and future research. Niss observes that, far from making mathematics teaching easier, the intelligent use of information technology increases the demands on both the student and the teacher. Much of the research thus far has been conducted by technology advocates and consequently has often been inattentive to the various difficulties, obstacles, and barriers that technology can present to teaching and learning. In particular, students working with information technology may not develop some of the old misconceptions about mathematics that were common when they worked only with pencil and paper. But the students may well develop new misconceptions instead. Moreover, even rather advanced students may transfer the authority they might have invested in the teacher and the textbook to the technology, with sometimes unfortunate results (Lingefjärd & Kilpatrick, 1998). Every technology has its downside. The downside of the new information technology for mathematics instruction has only begun to be given serious consideration.

A notable feature of the items in the list Niss offers us is that they deal much more with what students are doing in learning mathematics than what teachers are doing when teaching it. Although one feature of research in mathematics education over the past decade or so has been the increased attention given to teachers' conceptions, beliefs, and practices, it appears that we still have much more to say about learning than about teaching. I would like to draw attention to a dissertation study recently completed by one of my doctoral students at Georgia, Patricio Herbst (1998). He examined the regulations and norms behind the validation practices he observed in the discourse of the mathematics class, using recordings of secondary mathematics classes in the U.S. as his primary data source. First, I think this type of study—an analysis of the mathematical discourse rather than a psychological analysis of teachers' and students' thinking—may give a productive focus to classroom research. Second, Herbst employed in his work the so-called onion model, as modified by Margolinas (1995), from Brousseau's (1997) theory of didactical situations. In the model, the teacher and the student are each viewed as playing multiple knowledge games within nested situations. Each situation is a layer of the onion and defines a game in which teacher, student, or both, taking on different roles, act within the situation and against a milieu. What is at stake in these games is mathematical knowledge. Herbst's work represents what is likely to be a growing use of didactical situations in the English-speaking world now that Brousseau's work is available in English translation. Third, Herbst saw that in using the metaphor of classroom discourse as a knowledge game, we have often overlooked the symmetry of roles between teacher and student. We tend to think of the teacher as causing learning and the student being the one whose role it is to produce and manage mathematical knowledge. The teacher, however, is also doing mathematics and has an equally important role in the production and management of that knowledge. We need to begin examining more closely the teacher's role in doing mathematics in the classroom.

There are many reasons to be proud of the great body of scholarship we can point to today as excellent research in mathematics education. Not everything is excellent, however. I am particularly disturbed by the thoughtless way in which many researchers have cast aside quantitative methods, even when such methods would be appropriate for the research question they are investigating. There has been, in recent years, a wholehearted and uncritical embrace of qualitative methods that is beginning to cause some problems.

A colleague recently recalled an incident that took place three decades ago in a doctoral research seminar at a major university in the U.S. The purpose was to get the students to think about possible topics for their dissertations. The instructor asked the students to say a few words about their current thoughts about research topics. One student said, "I really haven't thought too much about actual topics, but I *do* know that I will use analysis of variance to analyze the data I'll collect." Some of the other students nodded in agreement, as if to say, "Of course!" The colleague recalling this incident noted that those were the days when statistical significance testing was expected in research. He said, "It is unlikely that any student in a first-year seminar today would make such a statement. … They tend not to be so firmly tied to any particular research technique so early in their programs." I disagree. I know of many recent cases in which students in a first course on research have said something like, "I really haven't thought too much about actual topics, but I *do* know that I will do a case study." They may say that they plan to do a constructivist teaching experiment or a classroom ethnography instead, but the point is that they have a method but not a topic.

Case studies seem to be posing a particular problem because some novice researchers think that all they need to do is to record classrooms for a few hours and interview some teachers, and voilá, they have a case study. Equally disturbing is the researcher who attempts five or ten "case studies" so as to answer questions about how, say, categories of teacher belief are related to categories of classroom performance. To study relationships such as those requires a rather different approach—one that looks across groups of teachers and classes in a systematic way. A case study is meant to explore the uniqueness of the single case. Any attempt to generalize across cases not only introduces severe problems of comparability and data aggregation but also runs the risk of losing the advantages of the case study methodology.

I recall with some dismay a statement I made three decades ago in a review of research on problem solving and creativity in mathematics:

> Much has been said lately about the need for large-scale, complex studies in mathematics education, but the researcher—most likely a doctoral student—who chooses to investigate problem solving in mathematics is probably best advised to undertake clinical studies of individual subjects (children gifted in mathematics, children for whom mathematics is particularly difficult), not only because clinical studies are more commensurate with limited financial and administrative resources, but also because our ignorance in this area demands clinical studies as precursors to larger efforts. (Kilpatrick, 1969, pp. 531-532)

I have lived to regret that advice somewhat, although I still think it makes sense. I guess what salvages it for me is the last few words about the idea of clinical study as precursor. Begle (1969) said much the same thing the same year. Here is how he phrased it:

> I don't want to give the impression that empirical investigations must always involve large numbers. By limiting the size of the population being studied, it is sometimes possible to carry out a much more penetrating and detailed study than can be attempted with the paper and pencil type of instrument that must be

> used when large numbers are involved. Of course the hypotheses developed from such intensive investigation must be considered quite tentative and need to be tested against wider selections of students and teachers. (p. 242)

The selection of a method before one has a topic for research is, I recognize, a problem largely (but not entirely) confined to novice researchers. But the across-the-board rejection of quantitative research methods as unable to provide useful information about the teaching and learning of mathematics is endemic to our field. That rejection is part of the larger malady not simply of taking no interest in alternative approaches to research, assuming there is nothing to be learned from them, but of actively repudiating the work of our peers and forebears. Too often, we make the false supposition that "we now know" how to conduct research and that previous generations did not, and colleagues working in other traditions do not, know what to do. I do not know why we should be so fortunate as to be the possessors of ultimate knowledge as to what research in mathematics education should be, but we certainly act at times as though that were the case. Perhaps it is a sign of the insecurity we feel as mathematics educators.

As a bridge from these observations about past work in our field to a discussion of directions that future work might take, I want to consider the Third International Mathematics and Science Study (TIMSS). This study is not completed. Data analyses are continuing, a new round of data gathering is about to begin, and plans are being made for various follow-up studies. To date, TIMSS has been a largely quantitative study of what students in various countries know about mathematics and science at several points in their school careers, coupled with studies of what school systems expect them to know and of what mathematics and science education look like in different countries. TIMSS is the most massive study of mathematics education ever undertaken. It has been enormously expensive and has attracted much attention. Few countries, even those that did not participate, have been untouched by reports of the results appearing in the public media. In another sense, however, TIMSS is not a study of mathematics education (or science education) at all. Most of the TIMSS analyses have been built on the assumption that mathematics is essentially a black box and that since everyone knows what terms like *algebra* and *geometry* mean in school mathematics, those terms must mean the same thing everywhere for every student and teacher regardless of what instruments are used to assess that meaning. As Gertrude

Stein might have incorrectly observed, “A mathematics test is a mathematics test is a mathematics test.”

TIMSS is actually a study in comparative education conducted primarily by measurement specialists for the benefit of educational politicians. So far, its value for helping us understand mathematics teaching and learning has been minimal. In several countries, in fact, its influence has been quite negative. It has drawn resources away from other types of research that might have been more useful to the field, and people who have dared criticize the TIMSS researchers’ motives, approaches, or claims have found themselves censured and their arguments rejected. To be fair, a number of highly competent researchers in mathematics education from several countries have worked on TIMSS either internationally or in their own countries. But the study as a whole has not been under their control.

TIMSS enthusiasts are fond of pointing out the great technical skill that has gone into such matters as drawing comparable samples, training data gatherers, ensuring comparability of instruments across countries, and analyzing immense bodies of data in a prompt fashion. That skill is indeed impressive. But it does not compensate for the problems posed when countries pick and choose the parts of the study they will participate in, electing to give the cheapest and easiest test questions, for example, and when they fail to meet TIMSS sampling criteria yet are included in the analyses anyway. How can anyone justify conclusions about how a country’s international standing in mathematics performance is improving (or declining), for example, when the comparison is between different students at different grade levels who are not simply taking a different test but are coming from a different set of countries?

Most researchers in mathematics education, thus far, know little about TIMSS other than what they have learned from the media. There is a need for more researchers to look more deeply and with a critical eye at the TIMSS data. Opportunities abound for national and international studies of what TIMSS can tell us about student performance in mathematics and how that might be related to other factors. Perhaps our mistrust or ignorance of quantitative data analysis techniques has kept us from venturing into the TIMSS swamp, but there is certainly much there that deserves our attention. Beyond the need for various so-called secondary analyses by mathematics educators, there is also a need for critical examinations of TIMSS findings. TIMSS is too big and is becoming far too influential to be left to the politicians and scientists who are building careers on it. So far, those people have been able to frame the results as they see them. Researchers in mathematics education need to be heard from on the policy issues arising from TIMSS, which means that at least some of us need to be looking closely at the TIMSS results. TIMSS presents a unique opportunity for people from different countries to work together on one of the most difficult and controversial issues of our time: What can we say about mathematics curricula around the world and the relation of those curricula to the mathematics that students are learning?

2. Prospect

Now let us consider what the future might hold. Our Swedish colleagues in Gothenburg have a slogan that they used for one of their biennial meetings of mathematics teachers: *Mathematik utan gränser*, mathematics without borders. The discipline of mathematics itself may roam freely across borders, but mathematics education as an enterprise is still, and is likely to remain, very much confined within a country's borders. Our countries' educational systems are organized very differently. Even if communication and travel were eventually to allow students to sit in on a variety of classes in other countries, the education they would receive in mathematics would be likely to retain a strong national flavor. The mathematical content may look much the same around the globe, but how that content is taught and what role it plays in the society's educational agenda are set locally.

Several years ago, Ed Silver and I were asked to write an article on the future of research in our field for the twenty-fifth anniversary issue of the *Journal for Research in Mathematics Education*. In preparing to write the article, we talked with colleagues from the U.S. and several other countries and asked them about research, its current status, and its future. Perhaps not surprisingly, we found little agreement on exactly what constitutes research in mathematics education (Silver & Kilpatrick, 1994). We did get some consensus on a few characteristics research might have: It should add to our understanding of mathematics teaching and learning. It ought to connect with both theory and educational reality, drawing on them and adding to them. It should be public and verifiable; its procedures should be checkable, and its results testable. Our informants, however, did not seem to agree on much beyond that, nor did they agree on where research might be headed.

Silver and I were struck by the variety of perspectives researchers in mathematics education have both within a country and across countries. That same diversity had been noted in the discussion document prepared for the International Commission on Mathematical Instruction's study of research in our field (see Sierpinska & Kilpatrick, 1998, pp. 3-8). There does seem to be a difference, however, between within-country and cross-country diversity. Problems that researchers may have some trouble communicating about within a country become magnified when they are communicating across national borders. Not only are there cross-country variations in how education is organized and conducted, there are also variations in research traditions and in how researchers are situated.

The late Efraim Fischbein (1990) noted some years ago that throughout much of the twentieth century, the activity of the International Commission on Mathematical Instruction consisted primarily of comparative surveys and international meetings on curriculum issues. Issues concerning the mathematical content of the curriculum have been easy to discuss because, as in TIMSS, the terms used to talk about curriculum topics are more or less universal. Local, sociopolitical characteristics of a curriculum are easily ignored. Similarly, researchers from quite different traditions can discuss the learning of, say, rational numbers or the use of, say, problem-solving or estimation processes and have little trouble understanding each other's work. Another subject that seems easy to discuss internationally is theory, especially epistemology. Most of the chapters in the book summarizing the research reported at conferences of the International Group for the Psychology of Mathematics Education (Nesher & Kilpatrick, 1990) deal almost exclusively with the learning of content in a subject-matter domain, with epistemological issues, or with a blend of the two.

One cannot deny the progress researchers have made internationally in exploring theoretical issues and how children learn specific mathematical content and processes. At the same time, however, one ought to recognize that many important research issues may have been neglected at the international level "because they do not relate readily to abstractions or universals, requiring instead attention to the nuances of local educational settings" (Silver & Kilpatrick, 1994, p. 750). Researchers should consider the question, “Are all mathematics education research questions able to be considered within the international community?” (p. 750).

In this connection, our colleagues in Portugal may have provided us with a good model for looking at research questions critically. At this year’s meeting of Portuguese researchers in mathematics education, the focus was a document reviewing a decade or more of their work (Ponte, Matos, & Abrantes, 1998). The conference began with reactions from other countries and then from reviewers from within Portugal. It concluded with, apart from reports of additional work, an extended exploration of the document by the participants. It seemed to me a singularly effective way to look at both the national and the international import of local research.

If some researchers are formulating questions that cannot be separated from the local context in which they have been posed, how are they to convey vital background information to their colleagues who do not know that context? Is it possible that some important questions cannot be considered internationally because they do not lend themselves to the kind of forum that organizations such as ERME can provide? Such meta-questions about our field and our international community seem important for at least some of us to explore.

With respect to the immediate future, there are some early signs that the field is recovering from what in retrospect seems to have been an infatuation with what might be called a romantic view of research, teaching, learning, and mathematics itself. We have the familiar theory of “realistic mathematics education” promoted by the Freudenthal Institute. What we may be seeing is the dawn of a new realism about research in mathematics education.

In North America, for example, the efforts of the National Council of Teachers of Mathematics (NCTM) to promote what has come to be called the “standards reform” of curriculum, teaching, and assessment (NCTM, 1989, 1991, 1995) have led to a backlash that reporters are calling the “math wars.” As in so much of what happens in the United States, there is a crazy extremity to the charges and counter-charges flying back and forth that masks what seems to be a reconsideration of rhetorical positions staked out a decade or so ago and, one hopes, a growing recognition of what research can say about reform issues. The writers revising the NCTM Standards (the revision, popularly known as Standards 2000, will be available in a few months in draft form; see http://www.nctm.org/standards 2000) appear to be taking very seriously the need to connect more of their work with the available literature, in part because selective and sometimes distorted readings of research have been used to attack the Standards. The revised Standards document will attempt to restore some of the perceived rejection of proof in the 1989 document by incorporating a standard on “reasoning and proof.” It will also attempt to create a more balanced view of pedagogy. For example, some teachers, researchers, and mathematicians have read the 1991 document as asking mathematics teachers to avoid not only lecturing but even telling students anything about mathematics (Chazan & Ball, 1995; Smith, J. P., III, 1996; Wu, 1998). The received wisdom has become, “Let students construct their own mathematics,” and the implication has been that teachers should not interfere with that process. A more balanced view of mathematics is also needed. Many have been quick to embrace the view that both learning and mathematics are situated within a context without necessarily thinking deeply about what *context* might mean in this case. Even with the new insights that situated cognition and ethnomathematics may have brought to our field, it is helpful to be reminded that there is still a strong sense in which mathematics is both abstract and universal.

In this post-post-modern era, I do not at all claim that societies such as ours that are seeking to be more open and democratic must embrace a realistic epistemology within mathematics education. But I do argue that such a view is worth some consideration. Clearly, the opposing romantic view has its pitfalls. Taken too far, it erodes the possibility of seeking truth—even as we acknowledge that any sort of absolute truth is unattainable. Denial of the possibility of an external reality is integral to totalitarian attempts to control minds, thereby controlling what is both real and true. Recall the claim of the ruling party’s protagonist when he confronts the dissident Winston Smith in Orwell’s *1984*:

> You believe that reality is something objective, external, existing in its own right. You also believe that the nature of reality is self-evident. When you delude yourself into thinking that you see something, you assume that everyone else sees the same thing as you do. But I tell you, Winston, that reality is not external. Reality exists in the human mind, and nowhere else. Not in the individual mind, which can make mistakes, and in any case soon perishes; only

> in the mind of the Party, which is collective and immortal. Whatever the Party holds to be truth *is* truth. It is impossible to see reality except by looking through the eyes of the Party. (Orwell, 1949/1981, p. 205)

We need to understand that even the presumably liberating ideology of children constructing their own knowledge and teachers facilitating that construction is embedded in the nineteenth-century premise that research is intended to serve the social administration of freedom and that the teacher is to be "saved" by academic research (Popkewitz, 1998).

3. Science and Doubt

Most of you know the book *Didactics of Mathematics as a Scientific Discipline* (Biehler, Scholz, Sträßer, & Winkelmann, 1994). The title is provocative. Although I have some problem, as I said above, characterizing our field as a discipline, I have no difficulty with the claim that it is scientific—or at least that it tries to be so. In fact, I would argue that much of our enterprise in mathematics education over the past several decades has not been sufficiently scientific even under a broad interpretation of the term. Recall Klein's call for a unified education in which the scientific and humanistic oppositions are "balanced out" (Rowe, 1985, p. 136) and Begle's (1969) argument that "only by becoming more scientific can we achieve the humanitarian goal of improving education for our children and for everyone's children" (p. 243).

What we often forget about science is that its public, verifiable nature has allowed the tradition of questioning findings to grow and strengthen every field that aspires to be scientific. The fact that scientific findings are always subject to reinterpretation, however, has led too many researchers in mathematics education to reject scientific approaches as faulty and thereby to produce research that is neither public in its procedures nor verifiable, even in principle, in its outcomes. One must salute European efforts in the didactics of mathematics for valiant and productive efforts to do research that is scientific in the best sense.

Where we have all fallen short in developing a tradition of questioning research results, both the ones we ourselves produce and the ones produced by others. George

Polya used to argue that in solving a mathematical problem we needed two kinds of courage: the courage to make a guess at a solution as a means of understanding the problem and the courage to test our guess as a means of moving toward the solution. I would like to argue that we need the same courage in investigating research questions. Researchers have the courage to attempt an investigation, but too frequently they lack the courage either to test their own or others' assumptions and conclusions. Critics everywhere have a bad name, but they are necessary. I see the greatest task ahead for our field as not that of communicating across cultures but that of being sufficiently mature and having enough confidence in the work being done by ourselves and others to criticize it thoroughly and constructively.

Our lack of a critical stance is exacerbated by a pervasive tendency to argue that no one can really understand my situation, which goes back to the view that my reality is my reality; it is not yours. We seem to be caught in the conundrum that you cannot understand me (since you can never know my mind) and you must understand me (if you are to help me learn or teach or do research). You cannot understand me; you can understand me. Both are true. The physicist Niels Bohr once defined a deep truth as a true statement whose opposite is also true. The claim that you cannot know my mind or my reality is a deep truth. It is a truth that we need to carry into our discussions of how different people coming from different cultures do different mathematics and different research. We need to learn how to handle difference within a culture of solidarity.

4. Solidarity

For centuries, people have sought to understand who they are not simply by defining themselves but by understanding others, and not simply by who they are today but by where they come from. In the *Aeneid*, when Aeneas is in the Elysian fields, the shade of his father Anchises shows him Rome's imperial destiny but also observes that the Romans will never surpass the Greeks in such arts and sciences as sculpture and astronomy—and, one might add, mathematics. Love of one's own country is therefore mixed with an acknowledgment of its limitations and of the advantages of other countries. Virgil's "idea of a people transcends biology: significantly it is a Greek, an immigrant, who utters the words 'We Italians.' There is a generosity of imagination here from which we still can learn" (Jenkyns, 1998, p. 50). Europe today is attempting to teach us all that lesson, and we in mathematics education need to be reminded not only of the inadequacies of our own situation but also of the benefits we can gain from other countries.

Thirty-five years ago this summer, on 26 June 1963, John F. Kennedy came to Germany to speak on the fifteenth anniversary of the Berlin airlift and said, "I am a Berliner." The context in which he uttered those words is often forgotten. He was talking not about himself specifically but rather about the honor due those Berliners who had sacrificed so much to maintain their freedom. Here is what he said:

> Two thousand years ago,
> the proudest boast was
> "*Civis Romanus sum*."
> Today,
> in the world of freedom
> the proudest boast is
> "*Ich bin ein Berliner.*"

People around the world, however, understood Kennedy as identifying with those steadfast people who had suffered under the airlift for so many years. I know that is how I heard it. And I think the crowd of Berliners in the square where he was speaking heard it the same way. We heard an expression of solidarity that people in my country, at least, took as including all of us.

That is the sense in which I would like you to understand the title of my remarks today. I do not claim that I am literally a European. I am not, although like many Americans, I trace my ancestry to this continent. But my intention is to express solidarity with you as you begin together to build a European community of researchers in mathematics education. The international community of mathematics educators is not yet a true community. But your example can help all of us take another step in that direction.

Note

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5. References

Begle, E. G. (1969). The role of research in the improvement of mathematics education. *Educational Studies in Mathematics*, 2, 232-244.

Biehler, R., Scholz, R. W., Sträßer, & Winkelmann, B. (eds.). (1994). *Didactics of mathematics as a scientific discipline*. Dordrecht: Kluwer.

Bishop, A., Clements, K., Keitel, C., Kilpatrick, J., & Laborde, C. (eds.). (1996). *International handbook of mathematics education* (Vols. 1-2). Dordrecht: Kluwer.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des mathématiques 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, eds. & trans.). Dordrecht: Kluwer.

Chazan, D., & Ball, D. (1995). *Beyond exhortations not to tell: The teacher's role in discussion-intensive mathematics classes* (Craft Paper 95-2). East Lansing, MI: National Center for Research on Teacher Learning.

Fischbein, E. (1990). Introduction. In P. Nesher & J. Kilpatrick (eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 1-13). Cambridge: Cambridge University Press.

Herbst, P. (1998). *What works as proof in the mathematics class*. Unpublished doctoral dissertation, University of Georgia, Athens.

Jenkyns, R. (1998, 23 April). Cards of identity. *New York Review of Books*, 45(7), 49-52.

Kilpatrick, J. (1969). Problem solving in mathematics. *Review of Educational Research*, 39, 523-534.

Kilpatrick, J. (1992). A history of research in mathematics education. In D. A. Grouws (ed.), *Handbook of research on mathematics teaching and learning* (pp. 3-38). New York: Macmillan.

Lingefjärd, T., & Kilpatrick, J. (1998). Authority and responsibility when learning mathematics in a technology-enhanced environment. In D. Tinsley & D. C. Johnson (eds.), *Information and communications technologies in school mathematics* (pp. 233-236). London: Chapman & Hall.

Margolinas, C. (1995). La structuration du milieu et ses apports dans l'analyse a posteriori des situations [The structure of the milieu and its contributions to the a posteriori analysis of situations]. In C. Margolinas (ed.), *Les debats en didactique des mathématiques* (pp. 89-102). Grenoble: La Pensée Sauvage.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

Nesher, P., & Kilpatrick, J. (eds.) (1990). *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge: Cambridge University Press.

Niss, M. (1998). *Aspects of the nature and state of research in mathematics education* (Tech. Rep. No. 351). Roskilde: Roskilde University, IMFUFA.

Nunes, T., & Bryant, P. (eds.). (1997). *Learning and teaching mathematics: An international perspective*. Hove, UK: Psychology Press.

Orwell, G. (1981). *Nineteen eighty-four*. New York: New American Library. (Original work published in 1949)

Ponte, J. P. da, Matos, J. M., Abrantes, P. (1998). *Investigação em educação matemática e desenvolvimento curricular* [Mathematics education research and curriculum development]. Unpublished manuscript.

Popkewitz, T. S. (1998) The culture of redemption and the administration of freedom as research. *Review of Educational Research*, 68, 1-34.

Rowe, D. E. (1983). A forgotten chapter in the history of Felix Klein's *Erlanger Programm. Historia Mathematic*a, 10, 448-457.

Rowe, D. E. (1985). Felix Klein's "*Erlanger Antrittsrede*": A transcription with English translation and commentary. Historia Mathematica, 12, 123-141.

Schwank, I. (1993). On the analysis of cognitive structures in algorithmic thinking. *Journal of Mathematical Behavior*, 12, 209-231.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 26, 1-36.

Sierpinska, A., & Kilpatrick, J. (1998). Mathematics education as a research domain: A search for identity. An ICMI study (New ICMI studies series, Vol. 4). Dordrecht: Kluwer.

Silver, E. A., & Kilpatrick, J. (1994). E pluribus unum: Challenges of diversity in the future of mathematics education research. *Journal for Research in Mathematics Education*, 25, 734-754.

Smith, J. P., III. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, 27, 387-402.

Tall, D. (1991). Reflections. In D. Tall (ed.), *Advanced mathematical thinking* (Mathematics Education Library, Vol. 11, pp. 251-259). Dordrecht: Kluwer.

Wu, H. (1998, 15 July). *The joy of lecturing—with a critique of the romantic tradition in education writing*. Unpublished manuscript available from <wu@math.berkeley.edu>.

GROUP 1: THE NATURE AND CONTENT OF MATHEMATICS AND ITS RELATIONSHIP TO TEACHING AND LEARNING

FERDINANDO ARZARELLO (I)
JEAN LUC DORIER (F)
LISA HEFENDEHL-HEBEKER (D)
STEFAN TURNAU (PL)

MATHEMATICS AS A CULTURAL PRODUCT

Ferdinando Arzarello[1], Jean-Luc Dorier[2], Lisa Hefendehl-Hebeker[3], StefanTurnau[4]

[1]Dipartimento di Matematica, Università di Torino, via C.Alberto 10, 10123,Torino, Italia.
arzarello@dm.unito.it

[2]Equipe de Didactique des Mathématiques, Laboratoire Leibniz, 46, ave F. Viallet, 38 031 Grenoble Cedex, France.
jean-luc.dorier@upmf-grenoble.fr

[3]University of Augsburg, Universitaetsstrasse 10, 86135, Augsburg, Deutschland.
hefendehl@math.uni-augsburg.de

[4]Pedagogical University, Rijtana 16, 35-959, Rzeszow, Poland.
sturnau@univ.rzeszow.pl

Abstract: *In this short introduction, we report on the main points discussed in our group during the Conference, as well as on some problems that appeared to be important issues for future research. One of the main issues in our group's discussion mathematics is a complex cultural product; so we begin with that.*

Keywords: *enculturation, certainty, argumentation, systemacity*

1. The main issue: mathematics as a cultural product

The emphasis is on both the words: 'cultural' and 'product'; culture, in a wide anthropological sense, influences mathematics products, and conversely such products may have (and generally do have) cultural implications. The issue is addressed in the nine papers of WG1 from different points of view and with respect to different content and contexts: in particular, it is always the case that the epistemological and the didactical aspects, as well as their mutual relationships, are put forward.

A rich set of problematic areas were identified. These included: mathematical concepts, the styles of thinking, the curricula, how working mathematicians devise and examine their products , the relationships between formal and intuitive approaches, pedagogically challenging questions that make a mathematical concept interesting for learners, problems and results that show their usefulness, activities that provide access to different aspects of a concept, activities that enable learners to grasp the common shape of these aspects, the features of modelling activities, etc.

However it was recognised that the rich network of interactions between the items in the list was of greater interest. During the discussion a tentative approach has been made to picture a structure for the network: some key categories have been pointed out and some connections have been stressed. The result is the framework, shown in Tab.I.

MATHEMATICS AS A CULTURAL PRODUCT	
SISTEMACITY	
Balance between complexity and effectiveness	
Coherence	
Choice of challenging situations	
Choice of content, context, tools	
Curriculum	
Modelling	
Representations	
CERTAINTY	
Locality	
	Verifying
Questioning	
	Validating
Formalising	
	Proving
ARGUMENTATION	
Defining	
Representing	
Questioning	
Reasoning	Verifying-Validating
	Making judgements
	Change of Settings
	Justifying

Using dialectic processes
Using intuition
Using consistency
Proving

Tab. I

The table must be read dynamically and systemically; there are interactions between all pairings of the three main categories (Systematicity, Certainty, Argumentation) as well as among their sub-items. Both the content and the dynamic interactions between categories and items substantiate the rich variety of aspects of Mathematics considered to be a cultural product.

2. An example

Let us refer to a very well know example to clarify the three main categories and their relationships. Consider a prospectograph like that illustrated by Piero della Francesca in his book *De prospectiva pingendi* (1482-1487) or that described by Albrecht Dürer in his *Underweysung der Messung mid dem Zyrkel und Rychtscheyd* (1525). It was used by artists during the Renaissance for analysing the way you can represent three-dimensional objects on a two-dimensional plane.

The prospectograph and the picture of the rays which arrive at the artist's eye from the object (called the 'visual pyramid' by Leon Battista Alberti in his book *Della pittura*, 1435) support the arguments given by the masters to illustrate the practical rules which one has to observe in order to reproduce a 'faithful' picture on the canvas: in fact, the drawing consists of the intersection between the visual pyramid and the canvas. In the arguments used by Alberti and still more by Dürer you can find a first systematization of the subject.

The definition of the visual pyramid (or that of the 'transparent door' used by Dürer) is a metaphor which from one side helps in contextualizing the problem within a setting where there is a good balance between the effective representation of the three-dimensional situation and the complexity of the rules which must be used in drawing on a board. From the other side the metaphor of the visual pyramid can rule the reasoning within the setting of Euclidean geometry, in order to verify, make judgements, justify and prove statements concerning the rules of perspective drawing.

But the systematization and the arguments developed by the Renaissance masters are far from giving a safe mathematical basis to the painting recipes for perspective drawings. In particular Euclidean geometry could only partially explain what happened on the canvas: for example what was the meaning of the so called *horizon line*, that is of the ideal point on the canvas which do not correspond, in reality, to any point ?

Fresh ideas were needed in order to give sense to such a phenomenon and, since the XVII century, these have been found by geometers like Desargues, Kepler, Pascal, La Hire etc., who created a new theory, Projective Geometry. It was not only a systematization, but a genuine new theory, canonized as such in the XIX century by people like Poncelet (1822), Gergonne (1824) and others. Now arguments to support its sentences were not any longer based on some intuitive metaphor, but were mathematical proofs. Validating, verifying and proving did not any longer concern practical recipes for drawing but sentences with a different epistemological status, namely that of be[e]ing within a theory, which was the result of de-contextualizing the previous challenging and strongly contextualized situation.

Such a sketch of the origin and development of Projective Geometry illustrates the issue of mathematics as a product of culture and the dialectic between systematicity, argumentation and certainty as an illustration of its complex nature; for more information and an example of a didactical application of such ideas, see Bartolini-Bussi (199?).

3. The game of mathematics: knowing and coming to know

The example above suggests that Mathematics is a kind of Game into which one is supposed to enter between the three main categories. To capture its meaning, one must play the whole game and possibly become conscious of all its aspects; so the main problem for people who learn mathematics is not (only) the issue of 'knowing', but mainly that of ‘coming to know’. Which ways, strategies, theories is one able to develop in order to enter into the game and to play it?

The problem concerns not only people who learn but also the teachers; the issue of looking at ways, strategies and theories is mainly their task. In order that their pupils enter into the game, they themselves must do it and moreover are required to be perfectly conscious of all its features. The teachers are required to build up for the pupils a similar game so that they can grasp the meaning of mathematical ideas, namely this deep and complex intertwining between the issues that have been put into the three major categories of our table.

As far as this issue is concerned, there are two main points for teachers, namely their practices and their theories. As to the former, their main activity should consist in nurturing pupils’ sensibility to the variety of aspects of mathematics; this requires often that pupils’ points of view on mathematics and consequently its understanding must be modified.

The resulting products belong to what is called ‘didactical engineering’ by some people, namely the tools for intervention in the class. But a good didactical engineering requires a good didactical theory; in order to deepen the analysis of how one can enter the game of mathematics, one needs suitable tools for ‘observing’ what happens in the class. This means that theories of didactics are necessary.

4. Some research problems

This section sketches the major themes of research which have been pointed out during the group discussion.

4.1 A list of questions

Both tools (didactical engineering and theories) are present in most of the papers; in fact the authors point out specific tools for observing, analyzing and coaching processes in the class with respect to different contexts and subjects. The main topics relate to the didactical contract, the dialectic ‘outil-object’, ‘fields of experiences’, ‘voices’, epistemology.

Some relevant research questions are raised: they emphasise that the issue of mathematics as a cultural product is the right approach for studying some major didactical questions in a suitable way.

Let us sketch some of them, above all to point out some problems which seem interesting for future investigations, particularly in a research group like this, where many cultures and traditions are present.

- What is the status of proof in the curriculum of different countries? The main points on which to focus are textbooks and practices. In particular, it should be interesting to study students’ mathematical behaviour in the class: i.e., students’ confidence in the utility of proof as an activity and teachers’ declared- practised-epistemology concerning proof.
- How is intuition understood and implemented in mathematics classrooms in different countries? Intuition is a fuzzy concept: its nature with respect to its ‘rigorized counterpart’ is not so clear, neither is it so well defined. It concerns both concepts and strategies and is intertwined with formality to and fro in a very complex and intriguing way. One of the main issues here is that of observation: clinical interviews, analysis of classroom discussions could be used to detect how the above intertwining modalities live and help in developing thinking dynamics in pupils, how teachers model them with their students (for example which is their use of metaphors while doing maths).
- How are ‘voices’ and ‘echoes’ materialised in different classrooms of different countries? The meaning of the words *voice* and *echo* is explained by the following quotation from Boero et al. (1998, p. 2-120):

  > “Some verbal and non-verbal expression (especially those produced by the scientists of the past but also contemporary expressions) [which] represent in a dense and communicative way important leaps in the evolution of mathematics and science. Each of these expressions conveys a content, an organization of the discourse and the cultural horizon of the historical leap. Referring to Bachtin (1968) and Wertsch (1991), we called these expressions ‘voices’. Performing

> suitable tasks proposed by the teacher, the student may try to make connections between the voice and his/her own interpretations, conceptions, experiences and personal senses (Leont'ev, 1978), and produce an 'echo', i.e. a link with the voice made explicit through a discourse".

It is well known that discussions cause different voices to be heard in the classroom: e.g., students' voices as well voices as from the external world, from history, etc. Their detection and subsequent investigation can show the ways pupils come to know as well as the teacher's view on the nature of mathematics; the ways the teachers tackle conflicts in the class can point out how the game of mathematics is developed in different countries.

- What is the ecology of specific knowledge and how does that relate to the didactic culture?

It should be interesting to compare the choices of challenging situations and suitable starting pointS in different countries, as well that of content, context and tools. A similar problem concerns the different ways of defining, representing, questioning, making judgements, changing settings, formaliZing, under the issue of cultural differences.

- Does any teaching of modelling exist as a specific didactical setting, e.g. in the same manner as teaching of algebra exists?

Modelling has some epistemological aspects, which are interesting for the whole teaching of mathematics, namely the discussion of assumption, the judgement of its validity, the change of setting that it implies, etc. How are these features related to the teaching/learning of mathematical concepts and how can they be implemented in such concepts? How, when, why does modelling activity deepen the understanding of mathematics (provided that this effectively happens)?

4.2 An immediate agenda for Working Group 1

It is a concern of this group to provide and debate on:

- case study material concerning proof from specific classrooms working on specific content;
- excerpts and comments from textbooks concerning proof.

The aim is to develop some of the above research questions, comparing situations in different European countries, under the issue of mathematics as a cultural product. Of course all people interested in developing and working on such a project are asked to contact one of the leaders of Working group 1.

5. References

Bartolini Bussi M. (199?): Multimedia: Instruments. In J. Fauvel and J. van Maanen (eds.), *The role of the history of mathematics in the teaching and learning of mathematics*. Dordrecht: Kluwer. To appear.

Bachtin, M. (1968): *Dostoevskij, poetica e stilistica*. Torino: Einaudi (Italian translation from Russian).

Boero, P.; Pedemonte, B.; Robotti, E. and Chiappini, G. (1998): The “Voices and Echoes Game” and the interiorization of crucial aspects of theoretical knowledge in a Vygotskian perspective: ongoing research. *Proceedings of PME-XXII*, Stellenbosch, vol. 2, pp. 120-127.

Leont`ev, A.N. (1978): *Activity, Consciousness and Personality*. Englewood Cliffs: Prentice-Hall.

Wertsch, J. (1991): *Voices of the Mind*. Harvester: Wheatsheaf.

SUCCESS AND FAILURE: FINDINGS FROM THE THIRD INTERNATIONAL MATHEMATICS AND SCIENCE STUDY (TIMSS) IN BULGARIA

Kiril Bankov
Faculty of Mathematics and Informatics University of Sofia, James Boucher bul. 5, 1164 Sofia, Bulgaria
kbankov@fmi.uni-sofia.bg

Abstract: *The Third International Mathematics and Science Study (TIMSS) is the most ambitious and complex comparative study conducted by the International Association for the Evaluation of Educational Achievement (IEA). During May and June 1995 the study tested the mathematics (and science) knowledge of students from 41 countries. The International results were released in November 1996. This paper is presented by the National Research Coordinator for Bulgaria in TIMSS, and deals with the achievement in three of the mathematics content areas of the Bulgarian 7-th and 8-th grade students in the study.*
Keywords: *testing, mathematical abilities, teaching vs performing*

1. Study and test design

The TIMSS study was designed to test the achievement of the students in the pair of adjacent grades (lower grade and upper grade) containing most of the 13-year-olds at the time of testing. In Bulgaria and most other countries these grades are 7-th and 8-th.

Item Response Theory was used to calculate the students scores in the mathematics scale having means of 500 and a standard deviation of 100 (Adams, Wilson & Wu, 1997). The sample design was a two-stage sample of intact classes. Therefore, the intraclass correlation coefficient and the design effect had to be taken into account (Foy & Schleicher, 1994). The standard error of sampling of the mean (or percentage) was calculated using the Jack-knife method (Rust, 1985).

The TIMSS test was designed to allow sub-scores on various aspects of mathematics (and science). The mathematics part of the test covered six content areas:

Fractions and number sense (FNS), Geometry (G), Algebra (A), Data representation, analysis and probability (DRAP), Measurement (M), Proportionality (P).

2. Student achievement

The international average (the average of all country means) in the mathematics scale is 484 for grade 7, and 513 for grade 8. The mean score of the Bulgarian students is 514 for grade 7, and 540 for grade 8, i.e. above the international average.

Countries with average scores significantly higher then the Bulgarian are Singapore, Korea, Japan, Hong Kong. Countries with average scores not significantly different ftom the Bulgarian are Czech Republic, Slovakia, Switzerland, the Netherlands, Slovenia, Austria, France, Hungary, Russia, Canada, Thailand, Ireland. Countries with average scores significantly lower then the Bulgarian are Germany, England, New Zealand, Norway, Denmark, the USA, Scotland, Latvia, Spain, Iceland, Greece, Romania, Lithuania, Cyprus, Portugal, Iran, Kuwait, Colombia, South Africa.

Table I presents the average percent correct by mathematics content areas (Beaton A. et al., 1996).

		Mathematics overall	FNS	G	A	DRAP	M	P
International percent correct	**Lower grade**	**49**	**53**	**49**	**44**	**57**	**45**	**40**
	Upper grade	55	58	56	52	62	51	45
Bulgaria	**7 grade**	**55**	**56**	**61**	**58**	**56**	**52**	**44**
	8 grade	60	60	65	62	62	54	47

Tab. I

The most difficult content area for Bulgaria, and on the international level as well, is Proportionality. The international percent correct is highest on Data representation, analysis, and probability. The Bulgarian students' performance in this content area is also very good.

A major concern in a study like TIMSS is what students have actually learned from their mathematics curriculum. It could not be expected that in an international study like TIMSS the items precisely cover the mathematics curriculum for each country. It is not unusual that, for each country, there are some items less covered or even not covered at all. This must be taken into account when analyzing the results.

Taking into account the curriculum coverage of the items, the researchers in each country can make a hypothesis about the performance of their students. If a particular topic is of major importance in the curriculum, and students perform well (according to the hypothesis) on it, this is considered a success. If they perform poorly, this is considered a failure. On the other hand, we cannot blame students for poor performance on a topic that is barely mentioned in the curriculum, but their good performance on such a topic can be considered a success.

The aim of the following pages is to discuss the success and failure of the Bulgarian students on three of the TIMSS content areas, in relation to the mathematics curriculum.

2.1 Proportionality

For many years school students seem to have had difficulty in adopting methods for solving proportional tasks. For example, Hart (1981) examined 2000 English students, and reported that almost none used the rule-of-three. This is not because the mathematics textbooks do not provide students with enough examples of proportionality. Kuchemann (1989) gives examples from different textbooks that introduce and emphasis the topic, and also offer methods of solution.

Traditionally in Bulgaria, proportionality is taught in grades 5-6. Fractions as quotients are a major topic in grade 5. Proportionality is a separate topic in the grade 6 mathematics textbooks. The Bulgarian mathematics curriculum recognizes the great importance of proportionality, and also the fact that it is often used in subjects other than mathematics. The textbooks emphasis this role of the topic. This is why the

Bulgarian students were expected to perform well in proportionality. However TIMSS findings show that the Bulgarian students demonstrate limited understanding when solving proportionality items. To be more concrete, here are two examples.

Example 1. To mix a certain color of paint, Alana combines 5 liters of red paint, 2 liters of blue paint, and 2 liters of yellow paint. What is the ratio of red paint to the total amount of paint?

A. 5/2 B. 9/4 C. 5/4 D. 5/9

@IEA TIMSS

International percent correct for this item is 37 for the lower grade and 42 for the upper grade. The Bulgarian students perform less well: 28 for grade 7 and 37 for grade 8.

Actually, they should have known how to answer this item from grade 6. In the textbooks students may find an explanation of ration in this context. Even it is not in the Bulgarian curricula, to begin the learning of proportionality with exercises on ratio of mixtures is a good idea. These exercises are simple to understand and can be expressed with everyday language. Kuchemann (1989) points out that mixtures are used for introducing the proportionality concepts in different textbooks. For example, the School Mathematics Project textbook SMP (1983) uses the idea of paint mixture, which is intended for year 1 or 2 of the secondary school. Probably this is one of the reasons for the better achievement of the English students on this item (percent correct 34 for the lower grade and 39 for the upper grade).

Example 2. A class has 28 students. The ratio of girls to boys is 4:3. How many girls are in the class?

Answer:________________________

@IEA TIMSS

International percent correct for this item is 30 for the lower grade and 37 for the upper grade. Although the Bulgarian students perform better, 46 for grade 7 and 54 for grade 8, the Bulgarian researchers are not satisfied. The importance of questions like this is fully recognized both in the mathematics curriculum and in the textbooks for grade 6 in Bulgaria but in mathematics lessons proportionality is usually taught as a formal mathematics relation. Exercises on some very standard cases are made. This does not lead to a deep understanding of the meaning of proportionality. It is a pity that the tendency to tell students about proportionality and to start with descriptive tasks is not considered in the Bulgarian curriculum.

2.2 Algebra

This topic plays a major role in Bulgarian school mathematics. In grade 7 the transition from arithmetic to algebra begins with a formal study of algebraic expressions, equations and inequalities. Filloy and Sutherland (1996) discuss two typical ways of making this transition: algebra as a natural development of arithmetic from early years, and algebra as a separate course for older students. The Bulgarian mathematics curriculum follows the first way. This is why there is no need to begin algebra with learning the appropriate syntactical rules, but just to change students' conceptions of operations performed on "new" objects (unknowns) and the concept of these "new" objects themselves. In these terms, an equation such as

(a, b and c are numbers) can be solved using arithmetical notions (i.e. by undoing the operations on the left-hand side), no operations on the unknown are needed. Therefore, the high performance of the Bulgarian 7-graders (and also 8-graders) on example 3 is not accidental.

Example 3. If $3(x + 5) = 30$, then $x=$

A. 2 B. 5 C. 10 D. 95

@IEA TIMSS

International percent correct for this item is 62 for the lower grade and 72 for the upper grade. The Bulgarian students perform better: 82 for grade 7 and 84 for grade 8.

Some researchers relate the difference between arithmetical and algebraic thinking to the difference between more informal and more formal approaches to word problems (Sutherland & Rojano, 1993). Word problems always reflect local style of life. For this reason the students learn to represent their experience from everyday life by a mathematical sign system. Traditionally, in Bulgaria the teaching of algebra makes little reference to practical knowledge. But there is a great emphasis in grades 6 and 8 for the Bulgarian mathematics curriculum on solving word problems. The most popular method is the Cartesian Method. It translates the story of the problem in an equation whose solution leads to the solution of the problem. This is why the Bulgarian students are familiar with the process of representation of some of the unknown elements of the problem by algebraic expressions. This is the task of example 4.

Example 4. Juan has 5 fewer hats than Maria, and Clarissa has 3 times as many hats as Juan. If Maria has n hats, which of these represents the number of hats that Clarissa has?

A. $5-3n$ B. $3n$ C. $n-5$ D. $3n-5$ E. $3(n-5)$

@IEA TIMSS

International percent correct for this item is 37 for the lower grade and 47 for the upper grade. Bearing in mind the section before the example, it is not a surprise that the Bulgarian students perform very well on this item: 64 for both grades 7 and 8, which may be considered a success.

2.3 Data representation, analysis and probability

Over the past two decades the importance of this topic has been growing in the school mathematics curricula in many countries. Worldwide documents such as the Cockroft Report (Cockroft, 1982), NCTM Standards (1989), etc. consider data representation, analysis and probability equally as important as arithmetic, algebra, geometry, and measurement in the school mathematics. Shaughnessy, Garfield and Greer (1996) point out that, over recent years, there has been more curriculum development activities in data representation, analysis and probability for middle school students than all the other levels put together.

From this position, it seems strange that there is no emphasis on data representation, analysis and probability in the secondary mathematics school curriculum in Bulgaria. Students may find a few references to data representation and analysis, not systematically written, in some parts of the textbooks. Teaching of elementary probability theory is usually postponed until grade 10 and both teachers and students consider it to be a very hard topic, probably because it is presented on a formal mathematical way. Hence, it was not expected that Bulgarian students would well perform on this topic.

Some researchers (Begg, 1997) consider the essential skills dimension of the curriculum framework related to communication, numeracy, information handling, problem-solving, etc. These skills are generic and can be learned through all subjects. TIMSS Data representation, analysis and probability items require such skills: reading data presented in tables and graphs, and understanding the meaning of chance in real life situations. Since the Bulgarian students had acquired similar skills from other subjects, they answered the items using their general knowledge, but not by applying anything that they had been taught in the mathematics lessons.

The next two examples support this argument.

Example 5. This chart shows temperature readings made at different times on four days.

TEMPERATURES

	6 a.m.	9 a.m.	Noon	3 p.m.	8 p.m.
Monday	15°	17°	20°	21°	19°
Tuesday	15°	15°	15°	10°	9°
Wednesday	8°	10°	14°	13°	15°
Thursday	8°	11°	14°	17°	20°

When was the highest temperature recorded?

A. Noon on Monday B. 3 p.m. on Monday
C. Noon on Tuesday D. 3 p.m. on Wednesday

International percent correct for this item is 85 for the lower grade and 87 for the upper grade. The Bulgarian students perform less well: 82 for grade 7 and 81 for grade 8 but the difference is not substantial. Surely this item demonstrates that it is not taught mathematics which is being tested.

Example 6. Each of the six faces of a certain cube is painted either red or blue. When the cube is tossed, the probability of the cube landing with a red face up is 2/3. How many faces are red?

A. One B. Two C. Three D. Four E. Five

@IEA TIMSS

International percent correct for this item is 41 for the lower grade and 47 for the upper grade. The Bulgarian students perform almost similarly: 38 for grade 7 and 46 for grade 8. This is one of the hardest items in this topic. One may argue that the word "probability" sounds uncommon for those who have not learned probability theory yet but students correctly understood "probability" as the common everyday word "chance".

3. Summary

Coming back to the words "success" and "failure", we have seen that the Bulgarian performance on the topic Proportionality is a failure. On the other hand, the Bulgarian students achieved an expected success on Algebra, and an unexpected success on the topic Data representation, analysis and probability.

This is one of the ways to look at TIMSS results. I hope it may be interesting for more countries that participated in the study to analyse their success and failure in the TIMSS mathematics topics. This analysis will make them possible to compare students' performance to the curriculum emphasis on the topics.

4. References

Adams, R. J., Wilson, M. R. & Wu, M. L. (1997): Multilevel Item Response Models: An Approach to Errors in Variables Regression. *Journal of Educational and Behavioral Statistics*, (22)1, 47-76.

Beaton A. et al. (1996): *Mathematics Achivement in the Middle School Years*, TIMSS International Study Center, Boston College.

Begg, A. (1997): Some Emerging Influences Underpinning Assessment in Statistics. In *The Assessment Challenge in Statistics Education*, IOS Press, Amsterdam,.

Cockroft, W. H. (1982): *Mathematics Counts*, Report of the Committee of Inquiry into the Teaching of Mathematics in School. Her Majesty's Stationary Office, London.

Filloy, E. & Sutherland, R. (1996): Designing Curricula for Teaching and Learning Algebra. In *International Handbook of Mathematics Education*. Kluwer Academic Publishers.

Foy, P. & Schleicher, A. (1994): *Third International Mathematics and Science Study: Sampling Manual* IEA. Vancouver.

Hart, K. (1981): Ratio and Proportion. In K. Hart (Ed.), *Children's Understanding of Mathematics*, 11-16. London: John Murray.

Kuchemann, D. (1989): Learning and Teaching Ratio: A Look at Some Current Textbooks. In P. Earnest (Ed.), *Mathematics Teaching. The State of Art*. London: The Falmer Press.

National Council of Teachers of Mathematics (1989): Curriculum and Evaluation Standards for School Mathematics, NCTM, Reston, VA.

Rust, K. (1985): Variance Estimation for Complex Estimators in Sample Surveys. *Journal of Official Statistics*,1(4), 381-397.

Shaughnessy, J. M., Garfield, J. & Greer, B. (1996): Data Handling. *In International Handbook of Mathematics Education*, Kluwer Academic Publishers.

SMP (1983): *SMP 11-16, Ratio 1*. Cambridge: Cambridge University Press.

Sutherland, R. & Rojano, T. (1993): A Spreadsheet Approach to Solving Algebra Problems. *Journal of Mathematical Behaviour*, 12, 351-383.

MATHEMATICS AND THEIR EPISTEMOLOGIES - AND THE LEARNING OF MATHEMATICS

Leone Burton
School of Education, University of Birmingham, Edgbaston, B15 2TT Birmingham, UK
l.burton@bham.ac.uk

Abstract: *This paper reports on the results of a study of the epistemologies of seventy research mathematicians utilising a model containing five categories, socio-cultural relatedness, aesthetics, intuition, thinking style and connectivities. The perspectives of the mathematicians demonstrate extreme variability from one to another but certain persistent themes carry important messages for mathematics education. In particular, although mathematicians research very differently, their pervasive absolutist view of mathematical knowledge is not matched by their stories of how they come to know, nor of how they think about mathematics.*
Keywords: *epistemology, researching, learning*

1. Introduction

Mathematics is not accepted by everyone as the inevitable and only possible product of an inexorable process which is independent of those mathematicians who pursue it and of the time and place where it is pursued. There is a literature (for example, see Bloor, 1991, Davis & Hersh, 1983, Lakatos, 1976) which critiques philosophical and epistemological perspectives on the 'objectivity' of mathematics, rejects absolutism (Ernest, 1991) and attempts to build links between the socio-cultural positioning of a subject as powerful as mathematics and its epistemological stance (for example, see Lerman, 1994, Restivo et al., 1993, Skovsmose, 1994). In many ways, this literature matches similar critiques made by philosophers and sociologists of science (for example, Harding, 1991, Rose, 1994) of reductionism and claims of value-free 'objectivity'. In Burton, 1995, I used the critical literature to derive an epistemological model built upon five categories. I conjectured that this model could offer a framework for exploring and possibly explaining links between coming to know mathematics, as a researcher, and as a learner. The five categories were:

- person- and cultural-social relatedness;
- aesthetics;
- intuition and insight;
- styles of thinking;
- connectivities.

I pointed out that:

> Knowing mathematics would, under this definition, be a function of who is claiming to know, related to which community, how that knowing is presented, what explanations are given for how that knowing was achieved, and the connections demonstrated between it and other knowings. (Ibid: 287)

In 1997, I undertook a study with thirty-five women and thirty-five men in career positions as research mathematicians in universities in England, Scotland, Northern Ireland and the Republic of Ireland in order to explore how robust was my model as a way of describing how they came to know mathematics. I was also particularly interested in comparing the research practices of these mathematicians with the teaching practices used with learners. There has been an emphasis in the international mathematics education literature on the importance of coming to know through a process of socio-cultural construction and on problem-solving and investigational-style activities to encourage this process (see, for example, Resnick et al, 1991, Steffe & Gale, 1995 and much of my own writing). How close to this model of learning mathematics would be the practices of research mathematicians as they came to know?

The study was interpretive, relying upon data collected by tape-recording and by taking detailed notes of 64 face to face interviews and 6 telephone interviews. Interviews averaged an hour and a half in length. Since I was interested in interviewing equal numbers of females and males, I approached female mathematicians. When they expressed interest in joining the study, I asked them to find a male "pair" preferably in their own institution. I did not dictate what constituted a "pair" but asked that they would indicate to me what governed their choice. In no case did this appear to present a problem to the research. The majority chose someone at a similar level of the hierarchy to themselves and/or in the same mathematical speciality. I have three marital pairs. The table below shows the distribution of the participants, by status and sex:

	Post-doc	Lecturer	Senior Lecturer	Reader	Professor	Snr Research Officer	Research Fellow
Females	1	19	7	3	3	1	1
Males	1	17	9	2	6		

Tab. I

I interviewed in 22 institutions, of which 4 were new universities, in England, Scotland and Ireland, North and South. Prior to the interview, participants were provided with a one-page outline of the topics of interest (see appendix). These related to their "history"and their current research practices, and to how they come to know mathematics. They were offered the option, which none used, of deleting anything they did not wish to discuss and were assured that typed notes of the interview would be returned to them so that they could agree their contents, amend, change or delete. They were guaranteed anonymity. The data were entered into NUDIST, a qualitative analytical computer-based tool and quantitative data were collected, computed and tabulated on EXCEL.

The interviews were discursive in style and the participants were free to introduce and explore issues of their choice. Although teaching and learning were *not* the subject of the interviews, in the course of describing their experiences, participants inevitably talked of themselves as learners and as teachers. The resulting analysis is mine as are the interpretations although these are made on the basis of an extensive data-base.

In this paper, I am reporting, briefly, upon how these research mathematicians talk about the five epistemological categories in my model and what, as mathematics educators, we can learn from this with respect to the learning and teaching of mathematics. Other papers in preparation deal with themes which emerged from the research.

2. The person- and cultural/social relatedness of mathematics

I began the study with a conjecture that the process whereby mathematicians are encultured into their discipline during their period of study would be strong enough to overlay any other differences, such as sex, with respect to how they understand and practise mathematics. While it was certainly the case that these research mathematicians were predominantly Platonist in their epistemological view, the variation between them was startling. Many, like this male Professor,

> "*don't think philosophically about mathematics*".

A female Reader said:

> "*I don't think about it. I am just curious and I do it.*"

However, a female statistics Professor said:

> "Some of the time I am a Platonist. As mathematicians we want to believe that it is objective but I guess rationally I think it is socio-cultural, and emotionally I want to believe it is objective."

A female pure mathematics lecturer said:

> "*I think mathematics is a product of people. It does turn out to be extremely useful. But even utility is a cultural product*"

whereas a male pure mathematics lecturer provided an explanation for the contradictory positions in which some participants found themselves:

> "*There is a desire to be a Platonist, to accept these ideas as 'truths' but there is a contradiction that I cannot really do that. On the one hand, I agree with the cultural dependency argument, but on the other hand, I do find appealing the idea that it might be possible to build something which would be meaningful to some other race of people or creatures.*"

For these mathematicians, as he explained, some of the excitement of mathematics lies in its power to provide a Big Global Picture. Another pure mathematician, in claiming that:

> "*the only thing mathematicians can do is tell a good story but those stories do uncover mathematical truths*"

tapped, at the same time, into narrative relativism and mathematical positivism.

It was possible to discern a distinction between those in pure mathematics and those in applied mathematics or statistics. Applied mathematicians emphasised utility:

> "*you can do things with it, you can model real things, you can make predictions, you can compare experiments*"

and were mainly concerned about

> "*describing physical things using formulae that work, modelling the real world*".

But even this very practical approach did not preclude this mathematician from acknowledging the socio-cultural basis by

> "*having a set of rules, a language, which allows you to do that*".

While these research mathematicians believed, or at least hoped, that mathematics was objective, their perspectives on how they came to know mathematics were exceedingly personal. So they presented an epistemological dichotomy between knowledge in mathematics, one said:

> "*knowledge is a question and an answer*"

and their own mathematical knowing which they mostly described both more personally and more lyrically. In the main, coming to know was explained as a process of, either, inserting the last piece in a jigsaw puzzle or a geographical journey, a map, a view. A male Reader claimed that:

> "*The geographical metaphor is an intellectually reasonable metaphor because you are faced with a problem so you are here and you solve the problem and you are there - it is a sequence of journeys in that sense - and along the way you have seen some new country.*"

Their responses, as reported above, came from questions about what, for them, constituted mathematics and how they knew when they "knew" something new. Coming to know, and reflecting upon that process was not, they claimed, something to which they gave much thought, neither for themselves, nor for their (research) students.

3. The aesthetics of mathematical thinking

Philip Davis and Reuben Hersh identify "aesthetic delight" as lying in the revelation of "the heart of the matter" instead of in a "wiseguy argument" (op.cit: 298/301) Many learners might wish that they could encounter aesthetic rather than "wiseguy" arguments in mathematics but to do so there has to be a cultural agreement about what constitutes an aesthetic argument and this must be shared with learners. The status of a "wiseguy" argument has to be able to be challenged, especially by students. Some of the research participants introduced 'beauty' or 'elegance' when asked to talk about what made a result publishable. Others, when asked why they had failed to mention aesthetics, once again took up very different positions along a continuum from

> "*I think words like beauty and elegance are over used*"

through to

> "*Mathematics is closely related to aesthetics*"

so there was no need to mention it. The, by now, familiar contradictions between these extremes appeared along the way: For example:

> "*Beauty doesn't matter*"

but later in the same interview:

> *"This is a three-dimensional generalisation of the two-dimensional problem I have already done. I expected it to be the same and it is not the same. It is annoying because if it were the same it would be beautiful..."*.

A female pure mathematician said:

> "*I think beauty probably requires a culture before you can appreciate it*"

while another drew a distinction:

> "*Elegance is more precise. It has to do with structure. Beauty is a reaction, you can't necessarily analyse it. You just feel it.*"

A third explained:

> "*It is the chain of logic, the structure connecting the beginning to the end, the completeness, the structure of the proof appealed to me aesthetically. When I chose a branch of mathematics, it had to appeal to me in that particular kind of way.*"

These factors to which she was drawing attention *do* constitute a culture which is identifiable and teachable, a culture which is not only about acceptable mathematical dialogue, but also about feelings. The feelings of delight and joy which many of the mathematicians expressed when talking of their research, that is the aesthetic pleasure they gain, could be set against their reactions against the "wiseguys", those in their community who attempt to obfuscate:

> "*So often they can put up hundreds of equations that you can't take in, you might have no idea what the person is talking about after the first five minutes. I think this is a waste of time.*"

4. The nurturing of intuition and insight

If, in the course of describing their search processes, they failed to mention intuition, insight or a similar construct, I asked them why. There were a few participants who said something like:

> "*I don't think intuition plays a part.*"

However, most, whether they called it intuition or insight, recognised when

> "*a light switches on when I look at a problem.*"

But this was not a mystical thing - they were clear that experience helped to show

> "*what works and what doesn't.*"

For the majority, their intuitions (or insights) played a major role in how they conducted their research. Without being able to describe what these were, or how they came by them, they 'knew' their importance. But many claimed their students were different:

> "*one of the things I find about students, undergraduates in particular, is that they seem to have very little intuition. They are dependent on being spoon-fed. The ability to look at a problem from different angles is crucial.*"

This seems to me to be a devastating comment on the teaching style experienced by students. If intuition, or insight, is as important to mathematics as many claimed to me (and see Hersh, 1998), and as important to mathematics education as is claimed by Efraim Fischbein (1987), why do teachers do so little to acknowledge and nurture it?

5. Styles of thinking

I set out on the study with two conjectures with respect to thinking style neither of which was confirmed. One was that I would find two different thinking styles well recorded in the literature, the visual and the analytic (for example, see Hadamard, 1945: 86). The other was that research mathematicians would move flexibly between the two. When talking me through how they solved problems, I found that the participants offered evidence on three, not two, different thinking styles which I am now calling

Style A: Visual (or thinking in pictures, often dynamic),
Style B: Analytic (or thinking symbolically, formalistically) and
Style C: Conceptual (thinking in ideas,classifying).

Differentiated between pure, applied and statistics, my participants spread across the three styles in the following ways (female/male shown in brackets):

	Pure	Applied	Statistics	Totals
A	22 (12/10)	14 (5/9)	10 (5/5)	46 (22/24)
B	11 (7/4)	11 (5/6)	7 (2/5)	29 (14/15)
C	11 (8/3)	15 (6/9)	7 (4/3)	
Totals	44 (27/17)	40 (16/24)	24 (11/13)	33 (18/15)

TAB. II

The numbers do not sum to 70 since, although 25 mathematicians claimed to make use of just one style of whom, for 15, it was Style A (9/6), for 3 it was Style B (2/1) and for 7 it was Style C (4/3), the majority, 42, used a combination of two out of the three and 3 claimed to use all three styles (2/1). Examples of the 3 possibilities of two style combinations follow:

Style A/B: *"I think very visually and I mostly only have recourse to algebra when I have to work out a proof rigorously."*

Style A/C: *"I always see the set rather than the group that acts on it. It is a mental picture. I cannot imagine thinking in a way that is completely non-visual...Depending on what the problem is but usually I am trying to count how many different orbits there are so things fall into classes so you can move within a class."*

Style B/C: *"I am not a very visual person. I think in terms of equations but I think it is problem driven. I don't think I can ever deal with solid geometry problems. If I am looking for a taxonomy, I use hierarchies, classificatory pictures."*

66% of the 70 participants claimed to be using Style A alone or in combination, 37% Style B and 47% Style C. The breakdown between female and male does not justify the assertion of difference and certainly does not substantiate the stereotype of the visual male.

From the perspective of a mathematics educator, a number of things are worrying about this emerging picture. First, those who claim *not* to use a visual style of thinking appear to be aware that there are other thinking styles but for many of the others there appeared to be an assumption that everyone thought about mathematics in the same way as they did. Second, most of my participants did not appear to realise that thinking and learning styles go very closely hand-in-hand - a lecturer with one dominant style is likely only to be communicating fluently with those students who share that style. Third, teaching materials are not constructed to exploit differences between thinking styles and offer alternative views, pathways, for reaching a particular mathematical goal.

6. Connectivities

I asked the mathematicians how important it was that their mathematics connected either to other mathematics or to other so-called real-world data. Their answer was, very important. Few of my participants were not interested in connectivities whatever their area of speciality. Sometimes these connections were to real world phenomena. A female Reader said:

> "*Sometimes areas overlap and that is important. I am very interested in connecting up areas. Applicability across fields is important*"

Another participant linked epistemological areas in saying

> "*The aesthetic is the aesthetic of the connections*".

Yet another pointed out that

> "*your work is not geared immediately to applications although it would be excellent if one could establish the bridgehead which allowed that. One is always keen to try that.*"

A lecturer summed up her ambition to put another piece into the jigsaw puzzle

> "*because it is linking in with something else you feel that what you have been doing is part of something bigger.*"

The drive to establish connectivities was spoken about, by my participants, with great passion. And yet mathematics is presented to learners in disconnected fragments and the individual is usually left to achieve any connection that might be made on their own.

7. Conclusions

> "*I am still fascinated by the phenomena of quantum mechanics.*" (Female Reader in applied mathematics)

> "*When I think I know, I feel quite euphoric. So I go out and enjoy the happiness. Without going back and thinking about whether it was right or not, but enjoy the happiness.*" (Male pure mathematics lecturer)

> "*It is just fun.*" (Female Reader in pure mathematics)

> "*You can do all these interesting and exciting things without having to go out and do things with them. Whether what you are thinking about is new, research, known things or not, for you it is all new.* (Male Professor of pure mathematics)

There is a world of difference between the excitement, fascination and fun expressed by these research mathematicians, and the examples of pupils' comments about learning mathematics in a transmissive style given by Jo Boaler (1967):

> "*Normally, there's a set way of doing it and you have to do it that way. You can't work out your own way so that you can remember it." (Carly, year 11)* (p.23)

> *"Yeah, he gives you the answere, you write the answer down and that's it."(Helen, year 11)* (p.22)

> *"In maths, there's a certain formula to get to, say from a to b, and there's no other way to get to it, or maybe there is, but you've got to remember the formula, you've got to remember it." (Simon, year 11)* (p.36)

But, the pupils she interviewed who had enquired into and explored mathematics, possibly using approaches not too dissimilar from the research mathematicians, were clear about the benefits of such project or coursework:

> *"It's more the thinking side to sort of look at everything you've got and think about how to solve it." (Jackie, year 10)* (p.100)

> *"Yeah, and you understand how they got it, when you're working from a book you just know that's the thing and that you just stick to it, you tend to understand it more from the activities." (Lindsey, year 11)* (p.100)

> *"Well, when we used to do projects, it was like that, looking at things and working them out, solving them - so it was similar to that, but it's not similar to this stuff now, it's, you don't know what this stuff is for really, except the exam." (Sue, year 11)* (p.103)

In this paper, I have used a five category model to describe the epistemological positions of research mathematicians. The model matched well with how they discussed coming to know mathematics but not at all in the ways that I had expected. They all agreed on the importance of connectivities, although not necessarily to themselves. On the other four categories what was underlined was their great variety. Responses lay along a continuum between opposites (there is no such thing - to - it is the most important thing) or offered more than one possibility. When talking about their research, these mathematicians talked, as above, in very emotive terms, describing frustration and difficulty but always in the context of (sometimes rare) pleasurable reward. There were two quite major surprises from the study. The first was with respect to the mathematicians' thinking styles, described above. The second lay in the pervasive collaborative organisation with which they described how they did their research. Unlike the model of the isolated mathematician, alone in his (sic) study, these mathematicians were either working with colleagues in other disciplines (which I describe as co-operation, see Burton, forthcoming) or were collaborating with others in the same disciplinary speciality to a degree that the papers that were produced were “seamless”, that is they could not recognise who had contributed what. When asked why they were collaborating so extensively, they produced all the same reasons which can be found in the mathematics education, and the general education, literature, as to the benefits of collaborative group work. These included benefitting from the experience of others, sharing the work, increasing the quantity and quality of ideas and enhancing the range of skills (ibid).

These outcomes lead me to ask a number of questions that are relevant to mathematics education.

Why do learners tend to encounter mathematical knowledge without the exciting experiences of making personal and socio/cultural connections through their very varied styles of coming to know that mathematics? If we are honestly worried about students turning away from mathematics, we have a responsibility to make it’s study more akin to how mathematicians learn and to be less obsessed with the necessity to teach “the basics” in the absence of any student’s need to know.

Why is mathematics presented in such a way as to deny learners the thrill and experience of perceiving its aesthetics?

> Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book, as remote as the laws of infangthief of fifteenth century Scotland. Contrariwise, appreciation of this element makes the sugject live in a wonderful manner and burn as no other creation of the human mind seems to do. (Davis & Hersh, op.cit., p.169).

As we all know that the learning of the telephone book does not fill anyone with enthusiasm, it is unsurprising that this style empties our classrooms, mentally and physically, of students. The mathematicians above would not dream of trying to learn the telephone book but many of them are as much engaged in its dissemination as school teachers.

How might we accord respect to young people's intuitions or insights in mathematics, nurture them and help them to make links between these insights and the power and function of proof? I am convinced that a teaching style which encouraged learners to explore their intuitive reactions to a situation, whether these were apparently consistent with the 'required' knowledge or not, would be more likely to provoke learning than denial of their possibilities.

What curriculum developments need to be encouraged in order to alert teachers to differences in thinking style and to provide opportunities for learners to exploit their dominant style and learn from its distinctiveness? In the development of an agenda for teacher education, this would seem to me to be far more important than the specifics of the mathematics curriculum.

Finally, how can we avoid the fragmentation of the curriculum and help teachers and learners to incorporate connections as a necessary part of the learning scene?

To sum up, it is apparent from the data that, working as these mathematicians do at the cutting edge of their disciplinary areas, they are engaged on a creative endeavour which demands a very different epistemological stance from the one which pervades the teaching and learning of mathematics. I believe identifying their practices and demonstrating their validity as ways of learning is enormously helpful to opening with them a long overdue pedagogical debate.

8. References

Boaler, J. (1997): *Experiencing School Mathematics*, Open Uni. Press, Buckingham.

Bloor, D. (1991): Knowledge and Social Imagery (2nd Ed.), Uni. of Chicago Press, London.

Burton, L. (1995): 'Moving Towards a Feminist Epistemology of Mathematics', *Educational Studies in Mathematics*, 28, 275-291.

Burton, L. (199?): '*Mathematics-Communities of Practices?*', forthcoming.

Davis, P.J. & Hersh, R. (1983): *The Mathematical Experience*, Harmondsworth, Mx.: Penguin Books.

Ernest, P. (1991): *The Philosophy of Mathematics Education*, London: McGraw-Hill, 1991.

Fischbein, E. (1987): *Intuition in Science and Mathematics*, D. Reidel, Dordrecht.

Hadamard, J. (1954): *The Psychology of Invention in the Mathematical Field*, Dover, New York.

Harding, S. (1991): *Whose Science? Whose Knowledge?* Milton Keynes: Open University Press.

Hersh, R. (1998): *What is Mathematics, Really?* Vintage Books, London.

Lakatos, I. (1976): *Proofs and Refutations*, Cambridge: Cambridge University Press.

Lerman, S. (ed., 1994): *Cultural Perspectives on the Mathematics Classroom*, Kluwer Academic Publishers, Dordrecht:.

Resnick, L.B., Levine, J.M. & Teasley, S.D. (eds., 1991): *Perspectives on Socially Shared Cognition*, American Psychological Association, Washington, DC.

Restivo, Sal et al. (eds., 1993): *Math Worlds*, SUNY Press, New York.

Rose, H. (1994): *Love, power and knowledge*, Polity Press, Cambridge.

Skovsmose, O. (1994): *Towards a Philosophy of Critical Mathematics Education*, Kluwer Academic Press, Dordrecht.

Steffe, L.P. & Gale, J. (eds., 1995): *Constructivism in Education*, Lawrence Erlbaum assoc, Hillsdale, NJ.

Appendix

Interview

The questions below are meant to give direction if we need it but not to be a requirement of how our conversation must develop.

About you:

1. Can we chart the historical trajectory of your becoming a mathematician?
2. From your experiences, what would you say you have learnt about mathematics?
3. What would you say you have learnt about yourself?
4. What would you say about how you come to know maths?
5. Is there anything you would like to say about your undergraduate or postgraduate experiences of coming to know maths?

6. How would you describe your experiences of research supervision and yourself as a supervisor of research students?
7. Of which mathematical community would you claim membership? Is that membership important and in what ways?
8. Do you have experience of collaborating on any research projects? Will you describe that experience and say what you have learnt from it or explain why you think collaboration is not helpful to you?

About how you come to know mathematics:

1. What do you now believe mathematics is? That is, how would you describe the focus of your work?
2. So what IS a mathematician, do you think?
3. Who are the mathematicians?
4. When you are acting as a mathematician, can you explain what you do, what choices you have, what leads you to make one choice rather than another?
5. Do you always know when you have come to know something new? How? Have you been justified/unjustified in this confidence?
6. Has it always been like this?
7. Do you know whether a result will be considered important, interesting or rejected by your community?
8. Do you share their criteria? What are they?
9. Where do you find the problems on which you work and what makes them something which engages you?

Conduct:

Anything from the above list which you do not want to discuss we can delete before the interview begins. The interview will be audiotaped and I will also take handwritten notes. Typed notes of the interview will be returned to you afterwards so that you can agree their contents, amend, change or delete. These agreed notes will be the basis of the analysis which I do, backed up by the audiotapes where necessary. All information will be maintained as confidential. Copies of any subsequent publications will be offered to you.

TEACHING AND LEARNING LINEAR ALGEBRA IN FIRST YEAR OF FRENCH SCIENCE UNIVERSITY

Jean-Luc Dorier[1], Aline Robert[2], Jacqueline Robinet[2], Marc Rogalski[2]
[1]Équipe de didactique des mathématiques, Laboratoire Leibniz, 46, ave F. Viallet, 38 031 Grenoble Cedex, France
Jean-Luc.Dorier@imag.fr
[2]Équipe DIDIREM, Paris

Abstract: -
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1. General presentation of the research

The teaching of linear algebra in France has undergone great modifications within the last thirty years. Today, linear algebra represents more or less a third of the mathematical contents taught in the first year of all French science universities. Traditionally, this teaching starts with the axiomatic definition of a vector space and finishes with the diagonalisation of linear operators. In a survey, Robert and Robinet (1989) showed that the main criticisms made by the students toward linear algebra concern the use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics. It is quite clear that many students have the feeling of landing on a new planet and are not able to find their way in this new world. The general attitude of teachers consists more often of a compromise: there is less and less emphasis on the most formal part of the teaching (especially at the beginning) and most of the evaluation deals with the algorithmic tasks connected with the reduction of matrices of linear operators. However, this leads to a contradiction which cannot satisfy us. Indeed, the students may be able to find the Jordan reduced form of an operator, but, on the other hand, suffer from severe misunderstanding on elementary notions such as linear dependence, generators, or complementary subspaces.

In response to this situation, we have developed a research program on the learning and teaching of linear algebra in the first year of French science universities. This work, which started some ten years ago, includes the elaboration and evaluation of experimental teaching based on a substantial historical study and a theoretical approach within the French context of “didactique des mathématiques”, and in collaboration with other teams working on the teaching of linear algebra, in France and abroad (most of the results of these collaborations are gathered in Dorier 1997). At first, we made several analyses of students’ works in ordinary teaching; we also created tests and interviewed students and teachers in order to better understand the meaning of what we have called the “obstacle of formalism” (Dorier 1990 and 1991). We also investigated the history of linear algebra through the epistemological analysis of original texts from the 17th century up to recent developments (Dorier 1995a and 1997, Part 1). From this first stage of our research, we have drawn a first conclusion, which we will call the *fundamental epistemological hypothesis*:

- The theory of vector spaces is a very recent theory that emerged in the late 19th century but only spread in the 1930s. At this stage, it became widely used not so much because it allowed new problems to be solved (for instance linear functional equations were solved with use of the theory of determinants generalized to infinite dimensions) but mostly because it was a way of unifying different methods and tools used in different contexts and generalizing them (this was necessary because of the rapid increase in the number of new mathematical results). Therefore linear algebra is a unifying and generalizing theory. As a consequence, it is also a formal theory; it simplifies the solving of many problems, but the simplification is only visible to the specialist who can anticipate the advantage of generalization because he already knows many contexts in which the new theory can be used. For a beginner, on the other hand, the simplification is not so clear as the cost of learning many new definitions and theorems seems too great with regard to the use she or he can make of the new theory in contexts in which solving systems of linear equations is usually quite sufficient. In other words, the unifying and generalizing nature of linear algebra has a didactical consequence : it is difficult to motivate the learning of the new theory because its use will be profitable only after it may have been applied to a wide range of situations.

In order to make the introduction of the theory of vector spaces more meaningful for students, we have developed a strategy based on two ideas :

- before the introduction of the formal theory, we make the students work on linear situations in three or four different contexts (linear systems, geometry, magic squares, equations of recurrence, …). This phase is mainly experimental, even specific vocabulary is avoided if superfluous, but the results established are capitalized in a second phase, after the general theory has been introduced, in order to show the possible unification and generalization.
- we use what we called the “meta lever”. “Meta” means that a reflexive attitude from the student on his mathematical activity is expected, and “lever” points out something which has to be used at the right moment in the right place to help the student get into this reflexive attitude while achieving a mathematical task which has been prepared carefully. The general idea is to put the students in a mathematical activity that can be solved by him and from there, to make him analyze, in a reflexive attitude, some possibilities of generalization or unification of the methods he has developed by himself. Such a strategy induces flexibility in the means of access to knowledge. Therefore our teaching experiment pays much attention to changes in mathematical frameworks, semiotic registers of representation, languages or ways of thinking. Indeed, the historical analysis shows that linear algebra comes from very varied sources and that the interactions between different contexts and ways of expressing similar ideas was essential in its development.

On another hand, we have used our epistemological analysis of the historical context as a means of better understanding the mistakes of the students, not only through Bachelard’s epistemological obstacles but also in order to give more significance to the bare simplicity of formal modern definitions. The historical context provides us with possible ways of access to the knowledge. Yet, none is neither the best one, nor the shortest in terms of psychological cost, and if history is a privileged source of inspiration for building teaching situations, it is not sufficient by itself, the historical context has to be adapted to the didactical context which is usually very different. However, the historical source has to be as complete as possible (we do not trust second hand summaries). In our work, we have explored in particular the role of the study of systems of linear equations and of geometry.

In this paper we cannot go into all the details of this work, but in order to make things more tangible, we would like to present one of the most recent part of our work on the concepts of linear dependence. It will put forward the specific use made of historical data in order to analyze students' misunderstanding and to elaborate didactical situations.

2. The case of linear dependence

2.1 Introduction

Linear dependence and independence, generators, basis, dimension and rank are the elementary concepts which constitute the foundations of the theory of vector spaces. For any mathematician, they appear to be very simple, clearly interrelated notions. Indeed, in the formal language of modern algebra they correspond to easily expressible definitions. Moreover, the logic of a hypothetico-deductive presentation induces a "natural" order between them (more or less the order given above) which reflects their intrinsic network of relations.

Let a system of linear equations have as many equations as unknowns (n). A dependence between the equations of the system may be understood in different ways. If one is not familiar with the notion of linear dependence, but more familiar with the solving of the system, the dependence reflects an undetermination in the solutions of the system. Practically, it means that, in the process of resolution, one (or more) unknown(s) will be left undetermined. Therefore n dependent equations in n unknowns will be characterized by the fact that they determine less than n unknowns and therefore act as if they were less than n. With regard to the solving of equations, dependence is therefore an incident in the solving that results in the vanishing (within the process of solving) of at least one equation and the undetermination of at least one of the unknowns. It is an incident because n equations usually determine n unknowns exactly. If the method for solving the system uses linear combinations, this incident may be related to the fact that a linear combination of the equations is zero. If the dependence is "obvious", one may even see directly that one equation is a linear combination of the others, although this will not be the central characteristic of the dependence.

2.2 Historical background

This might be difficult to admit for a modern mathematician so familiar with the vocabulary and basic notions of linear algebra. But such a way of considering dependence between equations may be found (with more or less the same words) in a text from 1750, by Euler, and still prevailed in most of the texts about linear equations up to the end of the 19th century (see Dorier 1993, 1995, 1996). Euler's text is the first one in which the question of dependence is discussed. The general idea that *n* equations determine *n* unknowns was so strong that nobody had bothered to discuss the odd case, until Euler was confronted with Cramer's paradox and pointed out this particularity. 1750 is also the year Cramer published the treatise that introduced the use of determinants which was to dominate the study of linear equations until the first quarter of the 20th century. In this context, dependence was characterized by the vanishing of the determinant. The notion of linear dependence, now basic in modern linear algebra, did not appear in its modern form until 1875. Frobenius introduced it, pointing out the similarity with the same notion for *n*-tuples. He was therefore able to consider linear equations and ^{n}-tuples as identical objects with regard to linearity. This simple fact may not seem very relevant but it happened to be one of the main steps toward a complete understanding of the concept of rank. Indeed in the same text, Frobenius was able not only to define what we would call a basis of solutions but he also associated a system of equations to such a basis (each *n*-tuple is transformed into an equation). Then he showed that any basis of solutions of this associated system has an associated system with the same set of solutions as the initial system. This first result on duality in finite-dimensional vector spaces showed the double level of invariance connected to rank both for the system and for the set of solutions. Moreover, Frobenius' approach allowed a system to be seen as part of a class of equivalent systems having the same set of solutions: a fundamental step toward the representation of sub-spaces by equations. This shows how adopting a formal definition (here of linear dependence and independence) may be a fundamental step in the construction of a theory, and is therefore an essential intrinsic constituent of this theory.

2.3 Didactical implications

Anyone who has taught a basic course in linear algebra knows how difficult it may be for a student to understand the formal definition of linear independence, and to apply it to various contexts. Moreover, once students have proved their ability to check whether a set of *n*-tuples, equations, polynomials or functions are independent, they may not be able to use the concept of linear independence in more formal contexts. Robert and Robinet (1989) have tested beginners on these two questions :

1. Let U, V and W be three vectors in $\mathbb{R}^3$, if they are two by two non collinear, are they independent question

2.1 Let U, V and W be three vectors in $\mathbb{R}^3$, and f a linear operator in $\mathbb{R}^3$, if U, V and W are independent, are f(U), f(V) and f(W) independent question

2.2 Let U, V and W be three vectors in $\mathbb{R}^3$, and f a linear operator in $\mathbb{R}^3$, if f(U), f(V) and f(W) are independent, are U, V and W independent question

These questions were generally failed by beginners. In the three cases, they used the formal definition of linear independence and tried different combinations with the hypotheses and the conclusions, so that, to the first question they answered yes, and to the last two they answered respectively yes and no, despite coming close to writing the correct proof for the correct answers. In their initial analysis, the authors concluded that the main difficulty was related with the use of the implication and a confusion between hypothesis and conclusion. This is indeed an obvious difficulty in the use of the formal definition of linear independence. A few years later, Dorier (1990) used the same questions. Previously, he had set up a test, to evaluate the students' ability in elementary logic and particularly in the use of implication. The result was surprisingly that the correlation between the questions on the use of implication and the three questions above was insignificant, in some cases it was even negative. Yet, on the whole, there was quite a good correlation between the two tests. This shows that if a certain level of ability in logic is necessary to understand the formalism of the theory of vector-space, general knowledge, rather than specific competence is needed. In other words, if some difficulties in linear algebra are due to formalism, they are specific to linear algebra and have to be overcome essentially in this context.

2.4 A proposition

In response to this analysis, we set up a teaching experiment based on the following scheme:

- after the definition of a vector space and sub-space and linear combination, we defined the notion of generator. A set of generator concentrates all the information we have on the sub-space, it is therefore interesting to reduce it to the minimum. Therefore, the question is to know when it is possible to take away one generator, the remaining vectors being still generators for the whole sub-space. The students easily found that the necessary and sufficient condition is that the vector to be taken away is a linear combination of the others. This provides the definition of linear dependence : “a vector is linearly dependent on others if and only if it is a linear combination of them”. This definition is very intuitive, yet it is not completely formal, and it needs to be generalized to sets of one vector. It induces without difficulty the definition of a set of independent vectors as a set of which no vector is a linear combination of the others. To feel the need for a more formal definition, one just has to reach the application of this definition. Indeed, suppose one has to answer the question: “are these vectors independent or not ?”. With the definition above, one needs to look for each vector, one after the other, to see if it is a linear combination of the others. After a few examples, with at least three vectors, it is easy to explain to the students that it would be better to have a definition in which all the vectors play the same part (it is also interesting to insist on the fact that this is a general statement in mathematics). One is now ready to transform the definition of linear dependence into : “vectors are linearly independent if and only if there exists a zero linear combination of them, whose coefficients are not all zero.” The definition of linear independence being the negation of this, it is therefore a pure problem of logic to reach the formal definition of linear independence. A pure problem of logic, but in a precise context, where the concepts have made sense to the students previously.

This approach has been proved to be efficient with regards to the student’s ability to use the definitions of linear dependence and independence, even in formal contexts such as in the three questions quoted above.

Moreover, it is quite a discovery for the students to realize that a formal definition may be more practical than an “intuitive” one. Indeed, most of them keep seeing the fact that a vector is a linear combination of the others as a consequence of the definition of linear dependence. Therefore they believe that this consequence is the practical way of proving that vectors are or are not independent, even if that is against their use of these definitions.

2.5 Conclusions

This example is relevant with regard to the question about the role of formalism in linear algebra. Formalism is what students themselves confess to fear at most in the theory of vector spaces. One didactical solution is to avoid formalism as far as possible, or at least to make it appear as a final stage gradually. Because we think that formalism is essential in this theory (our historical analysis has confirmed this epistemological fact), we give a different answer: formalism must be put forward in relation to intuitive approaches as the means of understanding the fundamental role of unification and generalization of the theory (see Freudenthal 1983). This has to be an explicit goal of the teaching. This is not incompatible with a gradual approach toward formalism, but it induces a different way of thinking the previous stages. Formalism is not only the final stage in a gradual process in which objects become more and more general, it must appear as the only means of comprehending different previous aspects within the same language. The difficulty here is to give a functional aspect to formalism while approaching it more intuitively.

We have explained above that dependence for equations could not mean linear dependence, although it is logically equivalent to it. Moreover, in any context, dependence is very different whether it applies to fewer or more than two vectors. It is important to understand how dependence is the generalization to more than two vectors of the notions of collinearity or proportionality. Question 1, quoted above, is an example of the difficulty students may encounter in this matter. In geometry, dependence may have different aspects too. “Three vectors are dependent if they are on the same plane” is the most intuitive way of imagining the dependence. What is the relation between this point of view and the fact that one vector is a linear combination of the others ? Students have a conception of coplanar points prior to their knowledge of linear algebra. Is this conception compatible with the notion of linear combination?

Linear dependence is a formal notion that unifies different types of dependences which interact with various previous intuitive conceptions. It has been shown above how in the historical development of linear algebra the understanding of this fact was essential for the construction of the concept of rank and partly of duality. In the teaching, this questioning has to be explicit, if we do not want misunderstanding to persist. Therefore even at the lowest levels of the theory the question of formalism has to be raised in interaction with various contexts where previous intuitive conceptions have been built by the students. The construction of a formal approach right from the beginning is a necessary condition to the understanding of the profound epistemological nature of the theory of vector spaces. In this sense, formalism has to be introduced as the answer to a problem that students are able to understand and to make their own, in relation to their previous knowledge in fields where linear algebra is relevant. These include at least geometry and linear equations but may also include polynomials or functions, although in these fields one may encounter more difficulties.

3. References

Dorier J.-L. (1991): Sur l'enseignement des concepts élémentaires d'algèbre linéaire á l'université. *Recherches en didactique des mathématiques*. 11(2/3), 325-364.

Dorier J.-L. (1992): Illustrer l'aspect unificateur et généralisateur de l'algèbre linéaire. *Cahier DIDIREM*, n° 14, IREM de Paris VII.

Dorier J.-L. (1993): Emergence du concept de rang dans l'étude des systèmes d'équations linéaires. *Cahiers du Séminaire d'Histoire des Mathématiques*, 2ème série, 3, 159-190.

Dorier J.-L., Robert A., Robinet J., Rogalski M. (1994a): L'enseignement de l'algèbre linéaire en DEUG première année, essai d'évaluation d'une ingénierie longue et questions. In Artigue M. et al. (eds.), Vingt ans de Didactique des Mathématiques en France, La Pensée Sauvage, 328-342, Grenoble.

Dorier J.-L., Robert A., Robinet J. and Rogalski M. (1994b): The teaching of linear algebra in first year of French science university. *Proceedings of the 18th conference of the international group for the Psychology of Mathematics Education*, Lisbonne, vol. 4, 137- 144.

Dorier J.-L. (1995a): A General Outline of the Genesis of Vector Space Theory. *Historia Mathematica*, 22(3), 227-261.

Dorier J.-L. (1995b): Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational Studies in Mathematics*, 29(2), 175-197.

Dorier J.-L. (ed.) (1997): L’enseignement de l’algèbre linéaire en question, La Pensée Sauvage, Grenoble.

Dorier J.-L. (1997): The role of formalism in the teaching of the theory of vector spaces. *Linear Algebra and its Applications*, 275/276, 141-160.

Douady R. (1986): Jeu de cadres et dialectique outil/objet. *Recherches en Didactique des Mathématiques*, 7/2, 5-31.

Freudenthal H. (1983): *Didactical phenomenology of mathematical structures.* Dordrecht: D.Riedel.

Robert A., Robinet J. (1989): Quelques résultats sur l’apprentissage de l’algèbre linéaire en première année de DEUG. *Cahier de Didactique des Mathématiques,* 53, IREM de Paris VII.

Robert A., Robinet J. (1996): Prise en compte du méta en didactique des mathématiques. *Recherches en Didactique des Mathématiques*, 16(2), 145-176.

Rogalski M. (1996): Teaching linear algebra : role and nature of knowledge in logic and set theory which deal with some linear problems. In L. Puig et A. Guitierrez (eds.), *Proceedings of the XX°International Conference for the Psychology of Mathematics Education*, Valencia : Universidad, vol 4, 211-218.

RELATION FUNCTION/AL ALGEBRA: AN EXAMPLE IN HIGH SCHOOL (AGE 15-16)

Régine Douady
University Paris7/IREM-CP7018U, 2, place Jussieu, 75251 Paris Cedex 05, France
rgdouady@gauss.math.jussieu.fr

Abstract: *In this proposal, we present some elements of a didactical engineering concerning an algebraic question from a double algebraic and topological viewpoint. Teachers know that " the study of sign of polynomials" is difficult in spite of the simplicity to build and use the array of signs. We think that a reason of this difficulty is in the following fact : impossibility for pupils to implement personal knowledge which is not algebraic but which can be pertinent to tackle the question. For instance, the intermediate value principle is a result which is very easily available as an implicit tool to draw graphics and exploit them.According to our analysis, the pupils have to deal with problems where functions, graphics and algebraic expressions are put in stage dialectically.*
Keywords: *functions, algebra, graphics, interplay between frameworks*

1. Problematic and methodology of the research

The research is concerned with the learning of polynomial fonctions for pupils 15-16 years old. We examine the conditions for a teaching based on polynomial functions in order to develop the knowledge of the pupils about such functions in the algebraic setting, and also to make easier conceptualization of and dealing with other functions : homographic, trigonometric functions.
Our aim is to study the consequences of a topological approach of algebraic questions on the knowledge of the pupils. We use a method of Didactical Engineering.

1. Choose a teaching object in the current program
2. Place the mathematical context in relation with the teaching tradition
3. Bring out hypotheses about the difficulties of the pupils and set the basis for a didactical engineering

4. Develop such an engineering, proceed to the a-priori analysis
5. Implement it and make an a-posteriori analysis of the collected data
6. Reproduce the implementation, under experimental control, after possible modification in view of the previous analysis
7. Test the supposedly acquired knowledge of the pupils in questions for which they are adapted tools
8. Compare the output of the pupils and their skill with expectations, and conclude about the relevance of the didactical hypotheses.

We present briefly below some of the issues listed above. This is work in progress.

2. Successive steps of the engineering

2.1 Study of polynomial functions (age 15-16)

The study of sign of polynomials, viewed as real functions of one real variable, is part of the curriculum in high school at age 15-16. It comes after the study of computations with algebraic expressions in the previous years. In particular the pupils have learned to expand expressions given as product of factors, and also in some "very simple" cases to transform expanded expressions into products of factors.

From a mathematical viewpoint, one form or the other is preferable according to the question dealt with. For instance, it is easy to determine the sign of a product of factors on an interval where each factor has a constant sign : one just has to know these signs, and the multiplication rule for signs. This study can be performed in a purely algorithmic way, ignoring the continuity aspect of the question. But is this really an advantage ?

2.2 The teaching tradition

From the teacher's viewpoint, it is customary to familiarize the pupils (age <15) with conversions of expressions, regardless of their mathematical relevance. At age 15-16, they are trained to form arrays of signs, a representation of data which allows an economical algebraic treatment. The method is, given a polynomial expressed as a product of factors of degree 1 or 2, to look for the zeroes, to determine the sign of each factor on each interval, and then apply the rule of signs to get the sign of the product. It is then efficient to organize information in an array.

However the economical feature is not an issue of the teaching. Though the mathematical object to be studied is *polynomial functions*, the stress is put on *polynomial* and very little on *functions*. The prevalent idea is that the teaching of algorithms will enable the pupils, in due time, to use such techniques properly, in situations which are possibly new for them, under their responsibility and control. It would be so for the study of functions, in which the sign of the derivative plays an important role.

On the pupils side, the availability of such algebraic tools is in fact not that clear. Low cost and efficiency are not perceived. Often you hear the pupils one year older saying: “we have been taught the array of signs last year, but we understood nothing” or “I have known how to do, but I forgot”. Moreover, even if they have good will, they are not always in a context of direct application. Here for instance, what can they do if they are given a polynomial function in an unusual way ? Have they a mean to test the validity of their possible results ? Would that be legitimate anyway, as the most widespread conviction is that the teacher is in charge of such controls.

2.3 An epistemological difficulty

Putting aside the problems of contract, the uneasiness of pupils in the algebraic study of polynomial functions may have several reasons. According to us, part of this uneasiness is due to the fact that the function aspect is conceptually underestimated or ignored, behind more algebraic features : number and values of zeroes, viewed in an algorithmic way. The situation in the neibourhood of a point where the function vanishes is not considered. The fact that the sign is constant between two consecutive zeroes is admitted without justification nor discussion. This has consequences on the signification given by pupils to the array of signs, and consequently to its use in contexts different from those who allowed introduction.

2.4 A didactical hypothesis

A hypothesis we are working on is the following : in order to be able to master easily the polynomial functions, the pupils need to conceive them both from an algebraic and topological viewpoint, in spite of the fact that the topological approach is not strictly necessary for the algebraic study.

In terms of teaching, the expression is that the study of zeroes and their multiplicity must be performed in relation with properties of continuity and derivability, at least implicitly. In this view, the cartesian graphical representation can play an important role as a tool. This study must be an issue in the learning of such functions.

2.5 The role of the cartesian graphical representation

This representation plays a multiple role :

- *in the conceptualization of functions, and in particular of polynomial functions.* It allows one to *see* a set of pairs of numbers (x,f(x)) coming from a computational program performed or evoked, as one new mathematical object. Many mathematical questions can be asked to specify the object. The study of these various questions will probably lead to get out of the framework in which the initial problem is given.
- *in the search of zeroes* For a polynomial function $x \to f(x)$, a value of x for which $f(x) = 0$ is easily perceived as a frontier between the values for which $f(x) > 0$ and those for which $f(x)<0$ after priming the process by numerical computation if necessary. Such a situation induces the pupils to use the intermediate value principle, a result which is very easily available as an implicit tool.
- *in the notion of multiplicity of roots* As soon as the pupils are able to conceive that a polynomial function is represented by a curve, the way the curve meets the horizontal axis, transversally or tangentially, more or less pressed, expresses the multiplicity of the root.

- *in the determination of signs*
  With the usual representation convention (positive above the x axis, negative below), the sign of the function on a given interval can be read on the graphic. Beyond this, reasoning has to take place. One also has to reason in order to check the validity of graphical information.

The graphic enables one to organize the interplay between frameworks, each one being used as a heuristic tool and mean of control for the other.

2.6 A didactical engineering based on frameworks interplay: algebraic-graphical-functions

Starting from the previous analysis and our didactical hypothesis, we propose to the pupils 3 types of problems, with different aims. The concept of function is the leading strand for the graphical/algebraic interplay. The issue is:

1. to realize that the *developed* and the *factorized* expressions allow an easy treatment of *different* mathematical questions.
2. to get acquainted with graphic as a tool for research, solution and control of results. Develop graphical/numerical interplay with graphic as the starting framework. Approach graphically the question of the sign of a polynomial and the relation between sign and the multiplicity of roots. In that phase of the work, the graphical medium is paper.
3. to work out algebraically on polynomials of various degree (up to 6 or 7) the role of zeroes and their multiplicity (in particular its parity) in the sign of polynomials, use the graphic as a tool to progress in the search of the sign. The medium can be paper or graphical calculator, these two being used in alternation.

2.7 Statements: choices and reasons for the choices

The sign of a polynomial function f depends on its roots and their multiplicity. Number, value and multipliciy of the roots are variables the teacher can play with.

The study relies on *continuit*y, which is used as an *implicit tool*. Graphic provides an approach to this issue and leads to admit explicitly the *principle of intermediate value*s.

Choice of the variables

- If the aim is the teaching of arrays of signs as an economical and efficient way of representing the relevant data on f in order to determine its sign, there should be several roots, but not too many so that the pupils can deal with the problem.
- If it is to establish the idea that f changes sign when x crosses a zero, but not always, then it is preferable to choose functions which display the two cases : simple and double roots.

Choice of functions

- One should choose a polynomial of degree at least 3 , preferably 4 or 5 , but not much more. As the attention is to be drawn on the unavoidable character of the study of roots, one shall choose to have f expressed as a product of factors of degree 1, possibly degree 2 with no real roots. Coefficients are chosen to be small integers, so as not to mislead the pupils by technical work.

Choice of the statement

It is worked out with reference with the theory of *Tool-Object Dialectic* and *Interplay between frameworks*.

- In a first time, the pupils can use their old knowledge, here compute the value f(x) corresponding to various choices of x .
- But their knowledge is not sufficient to solve completely the problem, here to give the sign of f(x) for any value of x assigned by the teacher. They would have to know the sign of f(x) for all values of x, i. e. for infinitely many values. The graphical representation acts as a lever to guide the research, in particular to determine the sign of f at the neighbourhood of a zero of the function, or between two consecutive zeroes. It should allow to formulate conjectures to be studied algebraically, for instance on the conditions under which the sign changes.
- An exposition of the results (some pupils, in turn, expose for the rest of the class the state of the work of their team) is an opportunity to make explicit what has

been solved and what remains open. It is also an opportunity to ask questions arising from the first studies (how many zeroes for f). To come back on known notions in new contexts, for instance on what is a function, on how to find the sign of a product of numbers when the sign of each one is known. It is an opportunity to organize in an array the information on the sign of the various factors which are involved in the expression of f .

- The analysis of the different examples should lead to an institutionnalization of the array of signs as a method.
- Familiarization with arrays of signs is planned, through the study of various examples.
- The study of well chosen non polynomial functions should allow to test the ability to reinvest as a tool what has been learned with polynomial functions.

2.8 Statement given in 2 times

1. *Calculators forbidden*

Giving x numerical values, you will get numerical values for the following expression:

$$f(x) = (x - 2) (2x - 3) (x + 5) (4x + 1) (1 - x)$$

Are they always positive ? Are they always negative question
Are they sometime positive, sometime negative, sometime zero ? Compute.
When you have an answer, call your teacher.

2. *Calculators forbidden*

$$f(x) = (x - 2) (2x - 3) (x + 5) (4x + 1) (1 - x)$$

Find a way which enables you to tell, very fast and reliably, when your teacher gives you a numerical value for x , whether the expression is > 0, < 0 or $= 0$.
Orders : *Only one answer accepted. Computing the expression is not allowed : it is too long.*
When you think you have a method, call your teacher.

2.9 Organization of the class and unfolding

At the end of the first session, the assessment brings out the difficulties of the pupils with the second question. Pupils repeating the class have looked for the places where f vanishes, and applied a principle of alternating sign each time the function passes by a zero. But they are unable to satisfy the requests for checking or justifying from their schoolmates who think this method just falls from heaven. Some pupils tried to use a calculator “just to see if it gives ideas”, but they ran into trouble : with the window they use, the graph is in several pieces, and they don’t know how to match the graphical information with the algebraic study. In particular they are not sure to have found all the vanishing values for f .

In the exposition of the results, a conviction graphically conforted is expressed : when the function passes from a positive value to a negative one (or the other way around), it has to go through the value 0. So between the two values, there is a value of x where f vanishes.

Hence a *method*: Look for the vanishing values. But a method which gives all of them.

A *question*: each time f vanishes, it changes sign. True or false ?

In the implementation, in one class this study has taken a long time. In another one where the rythm was much faster, the mere recording of cases where the statement was valid and others where it was not, was enough to focus interest on the reason for the difference in the behaviour of the functions.

We have decided to ask a problem where the issue was the study of the conditions of validity for the *conjecture: whenever* f *vanishes, it changes sign*.

For that study, we use an *interplay between functions and graphics.*

2.10 Statement 2

The conjecture was coming by graphical observation. In order to let the graphical representation play completely its heuristic function and its control function too, we have chosen a problem in which the *objects concerned are functions, but given by their graph* in cartesian frame (not by an equation). *The expected answer is graphical*. But in order to get it, one has to *work numerically* on the given functions.

Functions are given by their graph D_1 , D_2 , D_3 , D_4 (see Fig. 1 and Fig. 2). In each case, one has to consider two of them representing two functions f and g.

1. Determine the sign of the product h of the functions f and g .
   Definition of h: for each value of x , h(x) = f(x).g(x)
2. Propose a reasonable drawing of the product function h .
3. One of the lines being fixed, can one move the other one parallelly to itself in such a way that the product h has a constant sign when x varies question
   No measure with a ruler.

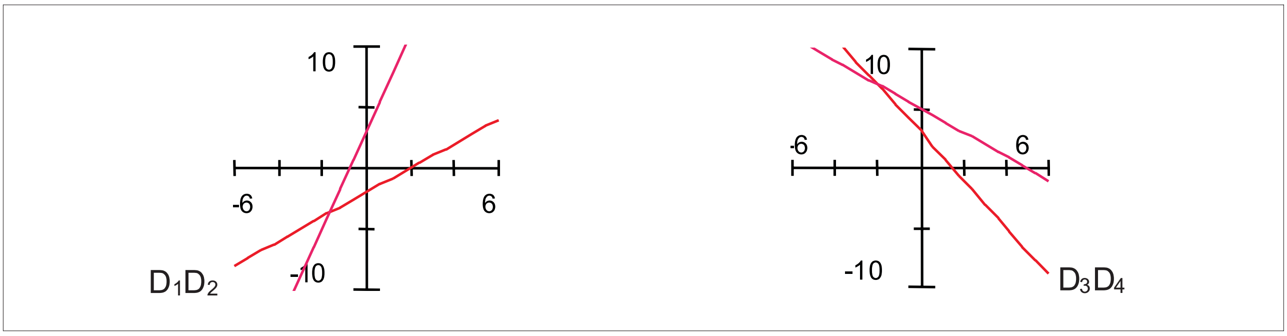


Fig. 1

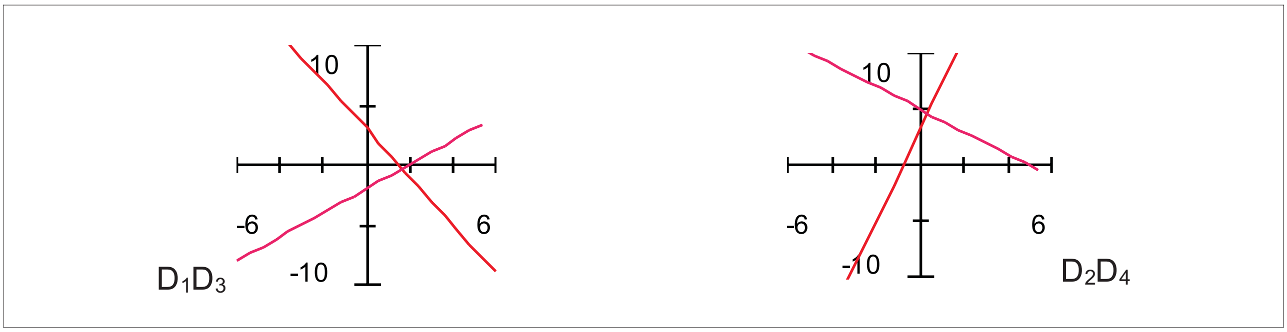


Fig. 2

Several variables are at the disposal of the teacher and are involved in the choice of the statement and the choice of the lines. We have chosen straight lines in order to be able to get without measuring 6 distinct certain points in the unknown graphic: those corresponding to $f(x) = 0$, 1 or -1 , and similarly for g . We have chosen the slopes : two examples with the same sign and two examples of opposite sign.

2.11 Realization and comments

During the implementation of the study, one observes that the pupils are bewildered by a practically “mute” graphic. In their practice, a function is given by its equation. How could they multiply graphics ?

It is clear that in this problem, the pupils have to distinguish between the object *function* and its representations. They have to operate on the object and give their answer in an assigned register. For this, the auxiliary working framework is the numerical part of the framework of functions. Only the use of the properties of the numbers 0 , 1 , -1 , and the relations between order and operations make it possible to make relevant selections and to get qualitative results consistent with the graphical data.

The pupils see no relation with the previous study on polynomial functions, their zero, their sign.

An intervention by the teacher unjams the situation and makes the research process start promptly. After recalling the definition of h , the teacher asks the question :
Are there points which are certain, points for which we are sure that they belong to the graph we look for?

The rumor on the interest of the zeroes of f and g, i. e. the points of intersection of the given lines with the x axis, sprays rapidly. In less than 10 minutes, several groups propose 4 certain points, and then those corresponding to $f(x) = -1$ and $g(x) = -1$ together with their geometrical construction.

The two other questions are also solved rapidly. The conclusions on which the pupils all agree are the following : we read on the graphic $f(x_1) = 0$, $g(x_2 = 0$, then

- $h(x_1) = 0$, $h(x_2) = 0$.

- If x_1 differs from x_2, the function h changes sign when x crosses x_1, and again when it crosses x_2.
- In order for h not to change sign, the interval $[x_1,x_2]$ must be reduced to a point. The functions f and g must vanish for the same value of x.

Their conclusion is still contextualized. But it is not far from a general statement which would have some meaning for them.

2.12 Conclusion, reinvestment

We have reproduced this engineering starting with the graphical problem, and later inserting in it a work which had been decided at national level by the commision inter IREM for computer science : *Draw a fish and send by internet a message without drawing to another class so that the pupils can draw the same fish, or one as much alike as possible*.

Students split their drawing in several parts and use functions to describe each of them. We have ended up with the study of the sign of a polynomial. The results show the process is very efficient.

The test performed on June confirms a good reinvestment of the array of signs in the study of the sign of polynomial functions, of homographic functions.

Familiarity with the graphic allows to propose graphical representations for the functions $\sin^2$, $\cos^2$, $\sin.\cos$ with their period, sign and some symmetries, starting from the known graphical data on $\sin$ and $\cos$. Then the question arises of whether the curves obtained are actually sine curves, and this leads after some algebraic-geometric work to conjecture trigonometric formulas which can later be proved.

3. References

Artigue M. (1988): Ingénierie didactique. *Recherches en Didactique des Mathématiques*, 9(3), 281-308; English version: Didactic engineering. In R. Douady & A. Mercier (eds.), *Research in Didactic of Mathematics*, La Pensée Sauvage, Grenoble, 41-65.

Artigue M. (1989): Une recherche d'ingénierie didactique sur l'enseignement des équations différentielles en 1er cycle universitaire. *Cahier du Séminaire de didactique des mathématiques et de l'informatique*, IMAG Université de Grenoble, 107, 183-209.

Artigue M. & Dias M. (1995): Articulation problems between different systems of symbolic representations in linear algebra. *Proceeding of PME XIX*, vol. 2, 34-41.

Artigue M. & Perrin-Glorian M.J. (1991): Didactic engineering, research and development tool: some theoretical problems linked to this duality. *For the learning of Mathematics*, 11(1), 13-17.

Dias M. (1998): *Les problèmes d'articulation entre points de vue "cartésien" et "paramétrique" dans l'enseignement de l'algèbre linéaire*. Thèse de Doctorat, I.R.E.M Université de Paris VII.

Douady R. (1987): Jeux de cadres et Dialectique outil-objet. *Recherches en Didactique des Mathématiques*, 7(2), 5-32.

Douady R. & Perrin-Glorian M.J. (1989): Un processus d'apprentissage du concept d'aire de surface plane. *Educational Studies in Mathematics*, 20, 387-424.

Douady R. (1991): Tool, Object, Setting, Window …. In A. Bishop, S. Mellin-Olsen & Van Dormolen (eds.), *Mathematical knowledge: its growth through teaching*, 109-129. Dordrecht, Boston, London: Kluwer Academic Publishers.

Douady R. (1997): Didactic engineering. In T. Nunes & P. Bryant (eds.), *Learning and teaching Mathematics, an International Perspective*, 373-401.

Douady R. and Parzysz B. (1998): Geometry in the classroom. In C. Mammana & V. Villani V. (eds.), *Perspectives on the teaching of Geometry for the 21st Century*, 159-192. Dordrecht, Boston, London: Kluwer Academic Publishers.

Duval R. (1993): Registre de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactiques et de Sciences Cognitives*, 5, I.R.E.M. de Strasbourg, 37-65.

Duval R. (1995): *Sémiosis et pensée humaine.* Paris: Peter Lang.

Grugeon B. (1997): Conception et exploitation d'une structure d'analyse multidimensionnelle en algèbre élémentaire, *Recherches en Didactiques des Mathématiques*, 17 (2), 167-209.

Pavlopoulou K. (1994): *Propédeutique de l'algèbre linéaire: la coordination des registres de représentation sémiotique*, Thèse de Doctorat, Université Louis Pasteur (Strasbourg 1).

Robert A., Robinet J. & Tenaud I. (1987): *De la géométrie à l'algèbre linéaire*, Brochure 72, I.R.E.M. Université Paris VII.

Tall D. (1994): A versatile theory of visualisation and symbolisation in mathematics. *Invited plenary lecture at the CIAEM Conference*, vol. 1, pp. 15-26.

SOME REMARKS ABOUT ARGUMENTATION AND MATHEMATICAL PROOF AND THEIR EDUCATIONAL IMPLICATIONS

Nadia Douek
I.U.F.M. de Creteil, Centre de Livry Gargan, 45 avenue J. Zay, 93190, Livry Gargan, France
ndouek@worldnet.fr

Abstract: *In spite of the undeniable epistemological and cognitive distance between argumentation and formal mathematical proof, argumentation and ordinary mathematical proof have many aspects in common, as processes and also as products. This paper aims at pointing out these aspects and sketching some educational implications of the fact of taking them into account.*
Keywords: *argumentation, formal proof, interaction*

1. Introduction

The general motivation for this study comes from the need to call into question the idea, widely shared among teachers and mathematics educators, that there are profound differences between mathematical thinking and thinking in other domains, and that these differences produce many difficulties in learning mathematics. In particular, some mathematics educators think that one of the main difficulties students face in approaching mathematical proof (one of the most characteristic and important mathematics subject) lies in their inability to grasp the differences between ordinary argumentation and mathematical proof. This position has been clearly presented by Duval (1991), and we will refer systematically to him in this article:

> "Deductive thinking does not work like argumentation. However these two kinds of reasoning use very similar linguistic forms and propositional connectives. This is one of the main reasons why most of the students do not understand the requirements of mathematical proof".

Duval's analysis (see 2.2.) offers a precise cognitive perspective for "formal proof" (i.e. proof reduced to a logical calculation). It suggests the following questions:

- What are the relationships between formal proof and proofs really performed in mathematics, in school mathematics as well as in the history of mathematics and the mathematics of modern-day mathematicians?
- What are the relationships between mathematical proof (as a written communication product), and the working mathematician's process of proving?
- In spite of the superficial analogies and profound differences between argumentation and formal proof, aren't there some deep connections between argumentation and mathematical proof (as products and as processes)?
- If those connections do exist, how can we take them into account, in order to manage the approach of students to mathematical proof?

In this paper I will try to explore only some aspects of these questions and show their relevance for the "culture of mathematical proof", which should be developed in teacher training and also for some direct educational implications. I will try to show how proving and arguing, as processes, have many common aspects from the cognitive and epistemological points of view, though significant differences exist between them as socially situated products. To provide evidence of the common aspects, it should be necessary:

- to compare the processes of producing an argumentation and producing a proof;
- to analyse the nature and limits of the reference corpus backing an argumentation and that backing specifically a mathematical proof;
- to differenciate the process of creating a proof and the product which is a written communicable object;
- to analyse the process of creation of proof and the process of writing a proof within social constraints;
- to analyse the structure of proof texts as particular argumentative texts.

This paper concerns mainly the first three points. I will focus only on "grounding connections" between argumentation and mathematical proof (see 3. and 4.) - although differences should be taken into account as well! I will also try to sketch some educational implications of my analysis (see 5.).

My approach will be phenomenological, i.e. I will consider how argumentation and mathematical proof "live" in different settings, today and in the past. My analysis will be mainly inspired by Thurston (1994) as concerns modern-day mathematicians' proofs. I will also refer to Lakatos (1985) as concerns definitions and proofs in the history of mathematics; Balacheff (1988), Bartolini et al. (1997) Arzarello et al. (1998); Simon (1996), Boero et al. (1996) and Harel & Sowder (1998) as concerns some epistemological, cognitive and educational aspects of proving; Lakoff and Nunez (1997) as concerns the idea of everyday experience as "grounding metaphor" for mathematics concepts; and Granger (1992) as concerns the relationships between formal proof and verification in mathematics.

2. About argumentation and proof

This section is intended to provide the reader with reference definitions and basic ideas for the following sections.

2.1 What argumentation are we talking about?

We cannot accept any discourse as an argumentation. In this paper, the word "argumentation" will indicate both the process which produces a logically connected (but not necessarily deductive) discourse about a given subject (from Webster Dictionary: *"1. The act of forming reasons, making inductions, drawing conclusions, and applying them to the case under discussion")* and the text produced by that process (Webster: *"3. Writing or speaking that argues"*). On each occasion, the linguistic context will allow the reader to select the appropriate meaning.

The word "argument" will be used as *"A reason or reasons offered for or against a proposition, opinion or measure"* (Webster), and may include verbal arguments, numerical data, drawings, etc. In brief, an "argumentation" consists of one or more logically connected "arguments". We may state that the discoursive nature of argumentation does not exclude the reference to non-discoursive (for instance, visual or gestural) arguments.

2.2 Formal proof

In this paper we will consider “formal proof” as proof reduced to a logical calculation. Duval (1991) offers a precise cognitive perspective for “formal proof”.

Duval performs *“a cognitive analysis of deductive organisation versus argumentative organisation of reasoning”*. I will quote some points here:

- as concerns “*inference steps*”: in argumentative reasoning, “*semantic content of propositions is crucial*”, while in deductive reasoning “*propositions do not intervene directly by their content, but by their operational status*” (defined as “*their role in the functioning of inference*”);
- as concerns “*enchaining steps*”: argumentative reasoning works “*by reinforcement or opposition of arguments*”. “*Propositions assumed as conclusions of preceding phases or as shared propositions are continuously reinterpreted*”. “*The transition from an argument to another is performed by extrinsic connection*”. On the contrary, in deductive reasoning “*the conclusion of a given step is the condition of application of the inference rule of the following step*”. The proposition obtained as the conclusion of a given step is “*recycled*” as the entrance proposition of the following step. Enchaining makes deductive reasoning similar to a chain of calculations.
- as concerns the “*epistemic value*” (defined as the “*degree of certainty or conviction attributed to a proposition*”): in argumentative reasoning “*true propositions have not the same epistemic value*”, while in mathematics “*true propositions have only one, specific epistemic value [...] - that is, certainty deriving from necessary conclusion”; and “proof modifies the epistemic value of the proved proposition: it becomes true and necessary*”. This modification constitutes the “productivity of proof”.

2.3 Mathematical proof

We could start by saying that mathematical proof is what in the past and today is recognized as such by people working in the mathematical field. This approach covers Euclid’s proof as well as the proofs published in high school mathematics textbooks, and current modern-day mathematicians' proofs, as communicated in specialized workshops or published in mathematical journals (for the differences between these two forms of communication, see Thurston, 1994). We could try to go further and recognize some common features, in particular: a common function, i.e. the validation of a statement; the reference to an established knowledge (see the definition of "theorem" as "*statement, proof and reference theory*" in Bartolini et al., 1997); and some common requirements, like the enchaining of propositions.

We must distinguish between the process of proof construction (i.e. "proving") and the result (as a socially acceptable mathematical text): for a discussion, see 4.

This distinction and preceding considerations point out the fact that mathematical proof can be considered as a particular case of argumentation (according to the preceding definition). However, in this paper "argumentation" will exclude "proof" when we compare them.

Concerning the relationships between formal proof and proofs currently produced by mathematicians, we may quote Thurston:

> "We should recognize that the humanly understandable and humanly checkable proofs that we actually do are what is most important to us, and that they are quite different from formal proof. For the present, formal proofs are out of reach and mostly irrelevant: we have good human processes for checking mathematical validity".

We may also consider some examples of theorems in mathematical analysis (e.g. Rolle's Theorem, Bolzano-Weierstrass' Theorem, etc.) whose usual proofs in current university textbooks are formally incomplete: completion would bring students far from understanding; for this reason semantic (and visual) arguments are frequently exploited in order to fill the gaps existing at the formal level.

2.4 Argumentation in mathematics

Argumentation can be performed in pure and applied mathematical situations, as in any other area. Argumentations are usually held informally between mathematicians to develop, discuss or communicate mathematical problems and results, but are not recognised socially in a research paper presenting new results: in that case proofs and "rigorous" constructions (or counter-examples) are needed.

As concerns "communication", Thurston writes:

> "Mathematical knowledge can be transmitted amazingly fast within a subfield of mathematics. When a significant theorem is proved, it often (but not always) happens that the solution can be communicated in a matter of minutes from one person to another within the subfield. The same proof would be communicated and generally understood in an hour's talk to members of the subfield. It would be the subject of a 15- or 20- page paper, which could be read and understood in a few hours or perhaps days by the members of the subfield. Why is there such a big expansion from the informal discussion to the talk, to the paper? One-to-one, people use wide channels of communication that go far beyond formal mathematical language. They use gestures, they draw pictures and diagrams, they make sound effects and use body language"

As concerns "rigour", it is considered here because it appears as a requirement of mathematical texts although it needs to be defined or rather to be questioned and historicised - see Lakatos (1985). The problem of rigour will be reconsidered later with the question of the epistemic value of statements.

2.5 Reference corpus

The expression "reference corpus" will include not only reference statements but also visual and, more generally, experimental evidence, physical constraints, etc. assumed to be unquestionable (i. e. "reference arguments", or, briefly, "references", in general). In Section 4.1. I will discuss the social determination of the fact that a "reference" is not questioned, as well as the necessary existence of references which are not made explicit.

2.6 Tools of analysis and comparison of argumentation and proof

Aren't there some criteria (even implicit ones) that enable us to accept or refuse an argumentation, as it happens for a proof? And are they not finally related to logical constraints and to the validity of the references, even if entangled with complex implicit knowledge? If we follow Duval's analysis, for argumentation it seems as if there is no recognised reference corpus for argumentation, whereas for proof it exists systematically. I do not think that this distinction is correct. The following criteria of comparison, inspired by Duval's analysis, will help me to argue this point in the next section: the existence of a "reference corpus" for developing reasoning; the means by which doubts about the "epistemic value" of a given statement can be dispelled; and the form of reasoning.

3. Analysis and comparison of argumentation and proof as products

3.1 About the reference corpus

No argumentation (individual or between two or more protagonists) would be possible in everyday life if there were no reference corpus to support the steps of reasoning. The reference corpus for everyday argumentation is socially and historically determined, and is largely implicit. Mathematical proof also needs a "reference corpus". We could think that this "reference corpus" is completely explicit and not socially determined, but we will see that this is not true.

A) Social and historical determination of the "reference corpus" for proof

In this subsection I will try to support the idea that the "reference corpus" for mathematical proof is socially and historically determined. In order to do so, I will exploit arguments of different nature (historical and epistemological considerations as well as reflections on ordinary school practices) that are not easy to separate.

The reference corpus used in mathematics depends strongly on the users and their listeners/readers. For example, in secondary school some detailed references can be expected to support a proof, but in communication between higher level mathematicians those may be considered evident and as such disregarded. As Yackel and Cobb (1998) pointed out, the existence of jumps related to "obvious" arguments in the presentation of a proof can be considered as a sign of familiarity with knowledge involved in that proof. On the other hand, some statements accepted as references in secondary school are questioned and problematised at higher levels; questions of "decidability" may surface. We may remark that today problems of "decidability" are dealt with by few mathematicians and seldom encountered in mathematics teaching (although in my opinion simple examples concerning euclidean geometry vs non euclidean geometries could be of great pedagogical value). Let us quote Thurston:

> "On the most fundamental level, the foundations of mathematics are much shakier than the mathematics that we do. Most mathematicians adhere to foundational principles that are known to be polite fictions".

Thus for almost all the users of mathematics in a given social context (high school, university, etc.) the problem of epistemic value does not exist (with the exception of the case: "true" after proving, or "not true" after counter-example) although it was and it still is an important question for mathematics as for any other field of knowledge. Mathematics concepts are the most stable, giving an experience of "truth" which should not be necessarily taken for truth.

Let us now consider other aspects of the social determination of the reference corpus which concern the nature of references. If we consider the "references" that can back an argumentation for validating a statement in primary school, we see that at this level of approach to mathematical work references can include experimental facts. And we cannot deny their "grounding" function for mathematics (see Lakoff and Nunez, 1997), both for the long term construction of mathematical concepts and for establishing some requirements of validation which prepare proving (e.g. making reference to acknowledged facts, deriving consequences from them, etc.). For instance, in primary school geometry we may consider the superposition of figures for validating the equality of segments or angles, and superposition by bending for validating the existence of an axial symmetry. Later on in secondary school, these references no longer have value in proving; they are replaced by definitions or theorems (see Balacheff, 1988). For instance, in order to prove that an axial symmetry exists, reference can be made to the definition of symmetry axis as the axis of segments joining corresponding points. For older students, similar examples can be found in the field of discrete mathematics, where a lot of familiar statements which are necessary to build a proof are not part of the elementary axiomatics.

In general, at a higher level it is a hypothesis or a partial result of the problem to solve that informs us of equalities, and not “experimental” validation (see Balacheff, 1988). At such a level the meaning of equality is not questioned as might (and should) happen at “lower” levels. We may note that, in the history of mathematics, visual evidence supports many steps of reasoning in Euclid’s “Elements”. This evidence was replaced by theoretical constructions (axioms, definitions and theorems) in later geometrical theories.

B) Implicit and explicit references

The reference corpus is generally larger than the set of explicit references. In mathematics, as in other areas, the knowledge used as reference is not always recognised explicitly (and thus appears in no statement): some references can be used and might be discovered, constructed, or reconstructed, and stated afterwards. The example of Euler’s theorem discussed by Lakatos (1985) provides evidence about this phenomenon in the history of mathematics. The same phenomenon also occurs for argumentation concerning areas other than mathematics. Let us consider the interpretations made by a psychoanalyst: we cannot fathom his ability unless we believe that he bases his work on chains of reasoning that refer to a great deal of shared knowledge about mankind and society, this knowledge being obviously impossible to reduce to explicit knowledge. And, in general, we could hold no exchange of ideas, whatever area we are interested in, without exploiting implicit shared knowledge. Implicit knowledge, which we are generally not conscious of, is a source of important “limit problems” (especially in non mathematical fields, but also in mathematics): in the “fuzzy” border of implicit knowledge we can meet the challenge of formulating more and more precise statements and evaluating their epistemic value. Lakatos (1985) provides us with interesting historical examples about this issue.

3.2 How to dispel doubts about a statement and the form of reasoning

Thurston writes:

> “Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do

> so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas".

And considering the example of Wiles's proof of Fermat's Last Theorem:

> "The experts quickly came to believe that his proof was basically correct on the basis of high-level ideas, long before details could be checked".

These quotations raise some interesting questions concerning the ways by which doubts about mathematics statements are dispelled. Formal proof "produces" (according to Duval's analysis) the reliability of a statement (attributing to it the epistemic value of "truth") . But what Thurston argues is that "*reliability does not primarily come from mathematicians formally checking formal arguments*". In Thurston's view, the requirements of formal proof represent only guidelines for writing a proof - once its validity has been checked according to "substantial" and not "formal" arguments. The preceding considerations directly concern the form of reasoning: the model of formal proof as described by Duval and based on the "operational status" of propositions rather than on their "semantic content" does not seem to fit the description of the activities performed by many working mathematicians when they check the validity of a statement or a proof. Only in some cases (for instance, proofs based on chains of transformations of algebraic expressions) does Duval's model neatly fit proof as a product.

Despite the distance between the ways of dispelling doubts (and the forms of reasoning) in mathematics and in other fields, the preceding analysis shows many points of contact - even between mathematical proof and argumentation in non-mathematical fields. Granger (1992) suggests the existence of deep analogies which might frame (from an epistemological point of view) these points of contact. Naturally, as concerns the form of reasoning visible in the final product, argumentation presents a wider range of possibilities than mathematical proof: not only deduction, but also analogy, metaphor, etc. Another significant difference lies in the fact that an argumentation can exploit arguments taken from different reference corpuses which may belong to different theories with no explicit, common frame ensuring coherence.

For instance, the argumentation developed in this paper derives its arguments from different disciplines (history of mathematics, epistemology, cognitive psychology); at present there is no mean to tackle the problem of coherence between reference theories belonging to these domains. On the contrary, mathematical proof refers to one or more reference theories explicitely related to a coherent system of axiomatics. But I would prefer to stress the importance of the points of contact (especially from an educational point of view: see 5.).

4. The processes of argumentation and construction of proof

In 2.3. I proposed distinguishing between the process of construction of proof ("proving") and the product ("proof"). Of what does the "proving" process consist? Experimental evidence has been provided about the hypothesis that "proving" a conjecture entails establishing a functional link with the argumentative activity needed to understand (or produce) the statement and recognizing its plausibility (see Bartolini et al., 1997). Proving itself needs an intensive argumentative activity, based on "transformations" of the situation represented by the statement. Experimental evidence about the importance of "transformational reasoning" in proving has been provided by various, recent studies (see Arzarello et al., 1998; Boero et al., 1996; Simon, 1996; Harel and Sowder, 1998). Simon defines "transformational reasoning" as follows:

> "the physical or mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated".

It is interesting to compare Simon's definition with Thurston (1994):

> "People have amazing facilities for sensing something without knowing where it comes from (intuition), for sensing that some phenomenon or situation or object is like something else (association); and for building and testing connections and comparisons, holding two things in mind at the same time (metaphor). These facilities are quite important for mathematics. Personally, I

> put a lot of effort into ”listening" to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quietening and focusing of my mind. Words, logic and detailed pictures rattling around can inhibit intuitions and associations". And then: “We have a facility for thinking about processes or sequences of actions that can often be used to good effect in mathematical reasoning”.

These quotations open some interesting research questions about metaphors:

- What are the relationships between metaphors and transformational reasoning in mathematical activities, especially in proving?
- What is the role of physical and body referents (and metaphors) in conjecturing and proving?

Metaphors can be considered as particular outcomes of transformational reasoning. For a metaphor we may consider two poles (a known object, an object to be known) and a link between them. In this case the “creativity” of transformational reasoning consists in the choice of the known object and the link - which allows us to know some aspects of the unknown object as suggested by the knowledge of the known object (“abduction”)(cf. Arzarello et al., 1998).

Coming to the second question, we may remark that mathematics “officially” concerns only mathematical objects. Metaphors where the known pole is not mathematical are not acknowledged. But in many cases the process of proving needs these metaphors, with physical or even bodily referents (sometimes their traces can be detected when a mathematician produces an informal description of the ideas his proof is based on: see 2.3., quotation from Thurston). In general, Lakoff and Nunez (1997) suggest that these metaphors have a crucial role in the historical and personal development of mathematical knowledge (“*grounding metaphors*”). The example of continuity is illuminating. Simon (1996) discusses the importance of a physical enactment in order to check the results of a transformation in transformational reasoning. In some situations the mathematical object is the known object and the other pole concerns a non-mathematical situation: the aim is to validate some statements concerning this situation, exploiting the properties of the mathematical model. In other cases it happens to relate two non mathematical objects (one known, the other not) by a mathematical, metaphoric link which sheds light on the unknown object and/or on its relationships with the known object. By these means, argumentative activities concerning non-mathematical situations rely upon mathematical creations (metaphors), an observation that should be taken into account in mathematics education (see 5.).

The example of metaphors shows the "semantic" complexity of the process of proving - and suggests the existence of other links with other mathematical activities. It also shows the importance of transformational reasoning as a free activity (in particular, free from usual boundaries of knowledge). However, metaphors represent only one side of the process of proving. Induction in general is relevant - and the need to produce a deductive chain guides the search for arguments to "enchain" when coming to the writing process (see Boero et al., 1996).

5. Some educational implications

Let us come back to the processes of argumentation and proof construction as opposed to the final static results. In my opinion, an important part of the difficulties of proof in school mathematics comes from the confusion of proof as a process and proof as a product, and results in an authoritarian approach to both activities. Frequently, mathematics teaching is based on the presentation (by the teacher, and then by the student when asked to repeat definitions and theorems) of mathematical knowledge as a more or less formalized theory based on rigorous proofs. In this case, authority is exercised through the form of the presentation (see Hanna, 1989); in this way school imposes the form of the presentation over the thought, leads to the identification between them and demands a thinking process modelled by the form of the presentation (eliminating every "dynamism"). This analysis may explain the strength of the model of proof, which gives value to the idea of the "linearity" of mathematical thought as a necessity and a characteristic aspect of mathematics.

If a student (or a teacher) assumes such "linearity" as the model of mathematicians' thinking without taking the complexity of conjecturing and proving processes into account, it is natural to see "proof" and "argumentation" as extremely different. But it is also important to consider the consequence of such a conception in other fields: it can reinforce a style of “thinking” for which no “sacred” assumption is challenged, only “deductions” are allowed (obviously, also school practice of argumentation may suffer from authoritarian models!). On the contrary, giving importance to “transformational” reasoning (and, in general, to non-deductive aspects of argumentation needed in constructive mathematical activities - including proving) can develop different potentials of thinking. On the possibility of educating manners of thinking other than deduction, Simon considers “transformational reasoning” and hypothesizes:

> “[...] transformational reasoning is a natural inclination of the human learner who seeks to understand and to validate mathematical ideas. The inclination [...] must be nurtured and developed.[...]school mathematics has failed to encourage or develop transformational reasoning, causing the inclination to reason transformationally to be expressed less universally.”

I am convinced that Simon’s assumption is a valid working hypothesis, needing further investigation not only regarding “*the role of transformational reasoning in classroom discourse aimed at validation of mathematical ideas*” but also its functioning and its connections with other “creative” behaviours (in mathematics and in other fields).

As concerns possible educational developments, the analyses perfomed in this paper suggest some immediate consequences:

- classroom work should include (before any “institutionalisation”) systematic activities of argumentation about work that has been done;
- validation, in mathematical work and in other fields, should be demanded whenever it can be meaningful;
- the fact that validation has not been done or was unsatisfactory or impossible should be openly recognized;
- and, finally, references as such should be explicitly recognized, be they statements, experiments or axioms (this does not mean that references are fixed as true once and for all, but rather that for at least a certain time we have to consider them as “references” for our reasoning).

The passage from argumentation to proof about the validity of a mathematical statement should openly be constructed on the basis of limitation of the reference corpus (see 3.2., last paragraph). It could be supported by exploiting different texts, such as historical scientific and mathematical texts, and different modern mathematical proofs (see Boero et al., 1997, for a possible methodology of exploitation).

6. References

Arzarello, F.; Micheletti, C.; Olivero, F. & Robutti, O. (1998): A model for analyzing the transition to formal proof in geometry. *Proceedings of PME-XXII*, Stellenbosch, vol. 2, pp. 24-31.

Balacheff, N. (1988): Une étude des processus de preuve en mathématiques, thèse d'état, Grenoble.

Bartolini Bussi, M.; Boero,P.; Ferri, F.; Garuti, R. & Mariotti, M.A. (1997): Approaching geometry theorems in contexts. *Proceedings of PME-XXI*, Lahti, vol.1, pp. 180-195.

Boero, P.; Garuti, R. & Mariotti, M.A. (1996): Some dynamic mental processes underlying producing and proving conjectures. *Proceedings of PME-XX*, Valencia, vol. 2, pp. 121-128.

Boero, P.; Pedemonte, B. & Robotti, E. (1997): Approaching Theoretical Knowledge through Voices and Echoes: a Vygotskian Perspective. *Proceedings of PME-XXI*, Lahti, vol. 2, pp. 81-88.

Duval, R. (1991): Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics*, 22, 233-261.

Granger, G. G. (1992): *La vérification*, Paris: Editions Odile Jacob.

Hanna, G. (1989): More than formal proof. *For the Learning of Mathematics*, 9, 20-23.

Harel, G. & Sowder, L. (1998): Students' Proof Schemes. In A. Schoenfeld et al (eds.), *Research on Collegiate Mathematics*, Vol. 3, M.A.A. and A.M.S. (in press).

Lakatos, I. (1985): *Preuves et réfutations*, Paris: Hermann.

Lakoff, G. & Nunez, R. (1997): The Metaphorical Structure of Mathematics. In L. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors and Images*, pp. 21-89, L.E.A., Hillsdale.

Simon, M. (1996): Beyond Inductive and Deductive Reasoning: The Search for a Sense of Knowing. *Educational Studies in Mathematics*, 30, 197-210.

Thurston, W.P. (1994): On Proof and Progress in Mathematics. *Bulletin of the A.M.S.*, 30, 161-177.

Yackel, E. (1998): A Study of Argumentation in a Second-Grade Mathematics Classroom. *Proceedings of PME-XXII*, vol. 4, pp. 209-216.

DISCRETE MATHEMATICS IN RELATION TO LEARNING AND TEACHING PROOF AND MODELLING

Denise Grenier, Charles Payan
Département de Mathématiques Discrètes, University Joseph Fourier, Laboratoire LEIBNIZ, 46 Avenue Félix Viallet, 38000 Grenoble, France
denise.grenier@imag.fr

Abstract: *The article presents the bases of a research aimed in the long run towards the construction of an « ambient environment » for discrete mathematics, a construction which would not only facilitate a first acquaintance to this mathematical field, but would also provide an alternative approach to some transversal concepts, such as proof and modelization. First, we give some results of our study about their status and role in pupils'curricula and manuals : we show that this mathematical field is in fact most often approached in a casual and unofficial manner, in combination with other mathematical knowledge and thus diverted from its real significance. We then develop our thesis on possibilities offered by discrete mathematics as a tool for learning proof and modelization. In order to support our thesis, we investigate some specificities of proof practice in discrete mathematics, especially from the viewpoint of truth checking and validation. These investigations are led from two points of view, a theoretical one and an experimental one.*
Keywords: *research situation, discrete mathematics, modelling, proof*

1. What are discrete mathematics?

Discrete Mathematics is not well known in France, even by mathematics teachers. However, this mathematical field has a different status in other countries, such as Hungary or United States. The subjects and the aims of Discrete Mathematics is to study configurations which can be described by some finite or countable relations (in a one-to-one relation with N). This mathematical field also develops its own specific objects and methods, for example, the decomposition/recomposition method, or structuration by coloration, or modelization by a graph. As an example, we mention the famous « four-colors theorem », which has been a conjecture for 200 years (Cayley, 1878), proved in 1976 by Appel and Haken (Appel & Haken, 1977).

We are particularly concerned with two characteristic features of this mathematical field : first, Discrete Mathematics is the mathematics of natural numbers, related to Peano axioms, and second, most classical problem statements in Discrete Mathematics are easy to understand, even by non specialists.

Didactical research about this mathematical field concerns almost only « counting » questions. For example, Batanero, Godino & Navarro-Pelayo studied the effects of the implicit combinatorial model on combinatorial reasoning in secondary school pupils (Batanero, Godino & Navarro-Pelayo, 1997).

2. Didactical transposition of discrete mathematics in France

Our analysis of the didactical transposition of Discrete Mathematics in France is given in Rolland (1995) and Grenier & Payan (1998). We describe here some results of this analysis. Discrete Mathematics is almost absent in French Curricula : a French student can become a mathematics teacher without having any knowledge in this mathematical field. Discrete Mathematics is taught only in rare courses, e.g. in optional courses, or in Mathematics for economics (Operational reserch, combinatorial optimization), or after the Fifth year in the University. The only exception is found in some chapters called *counting* or *combinatorics analysis* for 17 year pupils, where symbols such as C_n^p, A_n^p, $n!$, n^p are introduced. One can also find, in some parts of secondary schoolbooks, a few particular tools given to solve specific problems : graphs and trees. Unfortunately, these objects, which are complex concepts of Graph Theory (Berge, 1973), are usually given without any of their essential mathematical properties. As a consequence, not much can be done with them!

One can find, from time to time, discrete mathematics exercises, but it is shown that they are generally diverted from their real significance : the problematics of discrete mathematics is not recognized by the authors and the exercices are often combined with others mathematical knowledge. Here are two classical examples, found in some schoolbooks.

Example 1 : hijacking combinatorics.

The problem proposed at 15 years old pupils (in: Magnard, 3ème, 1989, rubrique *Casse-tête*) consists in partitioning the large square minus the small square, in five equal parts, of identical shape and area. The squares are « drawn » by wooden matches, and the partition has to be realized with « 10 matches », as follows.The authors indicated that « reflections are very useful to solve this problem ».

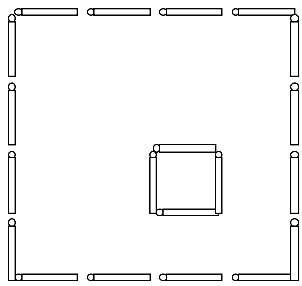


Fig. 1

A mathematical analysis shows in fact that reflections are not useful to find the solution (you can try !). The context under which the problem is posed is not very relevant : neither the number of matches nor their nature are useful. What knowledge, which notions are involved here ? What can pupils learn in such a situation ?

Yet, this problem can be formulated as a particular case of the following fundamental problem : « Tile a polymino with a basic polymino. » Here, the polymino to be tiled is a 4x4 square with a deleted square cell, and one has to tile it with identical triminos. This new formulation brings a real mathematical interest to the above problem. We analyse it in §4.1.

Example 2 : Euclid's rectangle.

We found the following problem in teachers'practices in the secondary school (13-15 year pupils). It has been studied from a didactical viewpoint by Arsac (Arsac et al., 1992, pp.89-104).

> *Draw a rectangle ABCD so that AB = 8 cm and BC = 5 cm.*
> *Take a point E on [AC] so that AE = 3 cm.*
> *Draw the line parallel to (AD) passing through E ; it intersects [AB] at a point N and [DC] at a point L. Draw the line parallel to (AB) passing through E ; it intersects [AD] at a point M and [BC] at a point K.*
> *Which of the two rectangles EMDL and ENBK has the larger area question question*

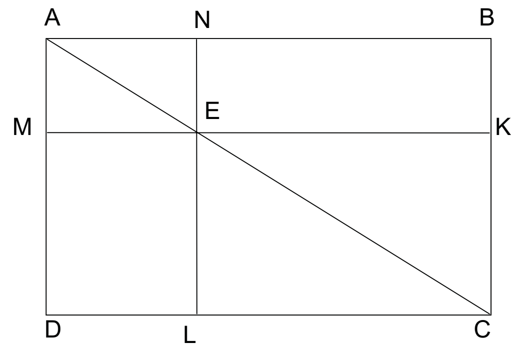


Fig. 2

To solve this problem using the given lengths, it is necessary to make an *exact* calculation, using Pythagoras or Thales theorems. But a more economical solution consists in analysing the figure in sub-figures, and comparing their areas : the latter solution is a very simple example of decomposition/recomposition method. It is usually not taken into consideration by secondary schools' teachers. Yet, the *combinatorial* point of view is pertinent for almost two reasons :

- it correspond to a very « economical » strategy for finding a solution, and there are real mathematical knowlegde behind it;

- it allows to reach minimal hypotheses in an easier way : in fact, the answer does not depend on measures and the condition « ABCD is a parallelogram » is sufficient in itself.

3. Global research questions

One of our research aims, in long term plans, is to induce changes in the teachers' and pupils' knowledge in discrete mathematics. Indeed, in a short-term plan, we think that discrete mathematics problems can play a specific role towards learning proof and modelization.

We study, in Grenier & Payan (1998, pp.75-81), the role and status of proof and modelization at school in France. We only present here our conclusions. Proof appears, in schoolbooks, essentially in geometry and from only two aspects : simple deductive reasoning and syntactical proof writing. Pupils have to learn how one can deduce given or evident conclusions from given and « true » hypotheses. There is no real place for questioning about truth, nor for raising conjectures. It is explicitly said in curricula that teachers have to use only those properties which are in the textbooks or given in the course, or given in an official program list. On the other hand, even though curricula specify that modelization is important, it has no place in teachers' practices and in classes. Many didactical researches have been conducted about proof problematics. For example, Arsac (1990) and Balacheff (1987) made propositions for new situations and problems to teach proof and demonstration in geometry.

We will now develop our thesis on possibilities offered by discrete mathematics as a tool for learning proof and modelization, especially from the viewpoint of truth checking and validation. These investigations are led from two points of view, a theoretical one and an experimental one. The problems which are studied here have been experimented, since several years, with teachers and students at different levels : First and Second year students at University, students at IUFM (Institutes for teachers training), and teachers of professional schools.

4. Proof and modelization in discrete mathematics.

We have selected problems presenting the following characteristics : the statement is easy to understand and there are various strategies to bengin to solve the problem, which are attainable by pupils, although a solution requires a « true mathematical activity » (proof and modelization).

For example, let us consider the following problem: *Is it possible to tile, with dominos, a given odd polymino with an arbitrary deleted square?*

If it is possible to find a tiling, the question is solved, but only for the given particular occurrence. However, in general cases, after several unsuccessful attempts, a proof is needed. This example has been experimented with different students. Various modelizations and proofs have been given, which attest that this kind of problems is wery relevant (Grenier & Payan, 1998).

4.1 An example of proof method: induction

In discrete mathematics, proofs by induction are very often constructed as follows, through a reasoning *by contradiction* (*reductio ab absurdum*): suppose there is a conter-example at some step n; take a *minimal* counter-example (step n_0); n_0 exists because of the Peano axiom : a non empty subset of N has a smallest element ; from this minimal configuration, the proof proceeds to producing another smaller one : we then have a contradiction. Often, the smaller one is obtained by splitting the minimal one. This scheme of proof gives, in the same time, an algorithm to construct the objects and the initialization of the induction.

Example 3

The problem given in example 1 is a particular case of the following problem: *Is it possible to tile, by L-shaped triminos, a 2^n-size polymino, with a deleted square?*

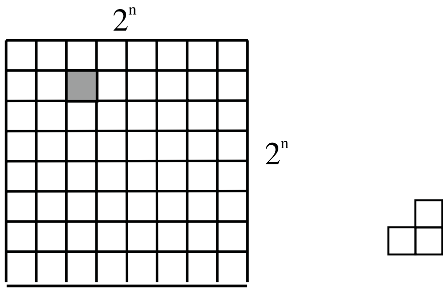


Fig. 3

An inductive proof does of course exist :

a) for n=0, the conclusion is obviously positive (empty configuration) ;

b) for n>0, splitting the square in four parts, and putting a trimino in the center of the square, as below, we obtain four 2^{n-1} size polyminos, each with a deleted square, which are, by hypothesis, tiled by L-shaped triminos.

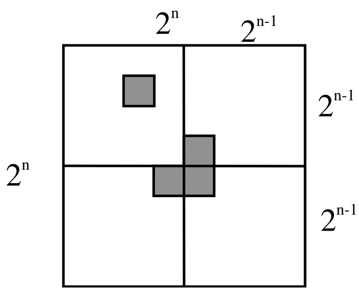


Fig. 4

This proof moreover gives a tiling algorithm, which is easy to use on any given example and, at the same time, which validates the proof.

This example has been experimented with student-teachers, in an attempt to look for the students' strategies of solutions and their conceptions about induction.

4.2 Two examples of modelization

Modelization is not a usual activity in French schools : most often, to a particular situation is associated only *one* mathematical model, and the role of the problem is only to get pupils learn how the appropriate model should work. In discrete mathematics, the large variety of basic models makes the modelization work necessary.

Example 4

Let us consider the following problem: *What is the number of all different configurations, when n identical balls are to be painted with at most k colours?*

We can, without modelization, find solutions for particular cases. However, it is difficult to find a solution for each n and each k.

A modelization approach allows a better investigation of the question : first, colour can be numbered e.g. 1, 2, etc... A representation of a configuration is then :

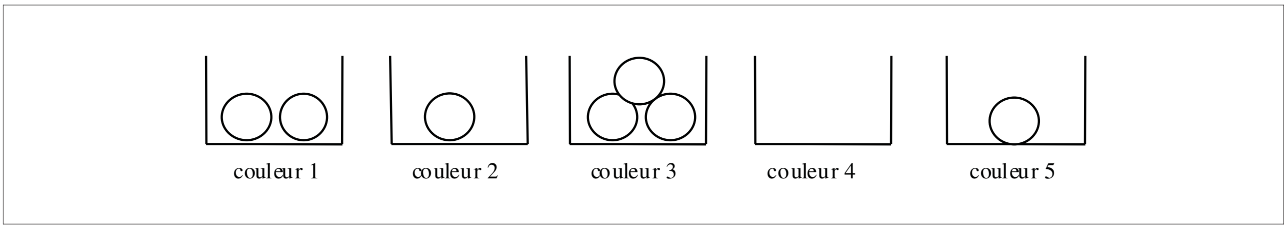


Fig. 5

An appropriate modelization consists in establishing a one-to-one relation between a configuration and a word made with n « 0 » and k-1 « 1 », such as :

O O / O / O O O / / O or 0 0 1 0 1 0 0 0 1 1 0.

The problem can then be reformulated as : « What is the number of different words with n « 0 » and k-1 « 1 » ? ». Now, it is easy to answer.

Example 5: Kœnigsberg bridges

Is it possible to a person to walk across each bridge, once and only once ?

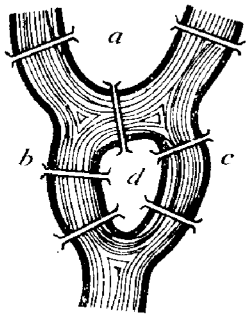


Fig. 6

We experimented this situation with students at different levels and with teachers in technical schools.

1. most people have an initial strategy which consists to try to explore all possible walks and then conjecture that it is impossible to walk across each bridge, once and only once. But a doubt remains and this implies the necessity of a proof.
2. in order to prove this conjecture, several attempts of modelization appear ; here are two examples, the first given by Fig.7, and the second by Fig.8:

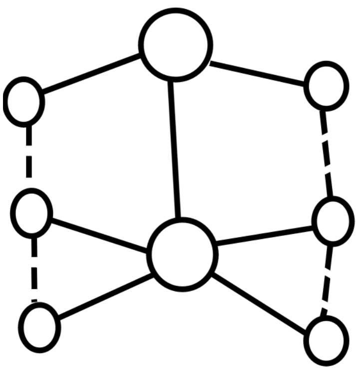


Fig. 7

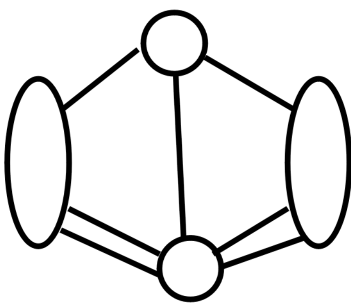


Fig. 8

The *model* represented in Fig. 7 is not relevant, because it does lead to the reasons for which there is no solution. On the other hand, the *model* represented in Fig. 8 is very close to the graph model which leads to the solution.

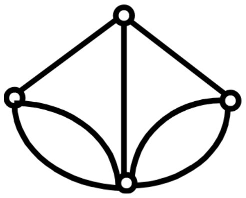


Fig. 8

The main asked questions are about the respective status of islands and banks, and their representation.

3. The reason why there is no solution appears with formulations like: *a region different from the beginning or the end of a walk must have as many entrances (coming in) than exits (coming out)*. This argument is found either directly on the original figure or on the elaborate/built/ model.

Although this model looks rather natural, especially since pupils happen to figure out similar representations, it may often raise a few difficulties: the isomorphism between a countryside stroll and a walk in a graph is not necessarily obvious.

A work on the model will help to understand the essential condition which decides for the existence or non-existence of a walk passing over each bridge once and only once, namely the parity of the number of edges reaching each vertex.

4.3 A particular tool: the pigeonholes principle

This is a very elementary principle which plays an important role in numerous reasonings in discrete mathematics. The so called "pigeonholes" principle simply asserts that when there are less holes than pigeons, there should be at least one hole where two pigeons have to settle.

Let us illustrate the strength of this "obvious" principle with a problem of combinatorial geometry : what is the smallest equilateral triangle in which k disks of diameter 1 can fit ? Let us mention that this problem is solved only for small values of k and for triangular numbers, namely whole integers of the form $k=q(q+1)/2$. This problem is a contemporary and actual research question (Payan, 1997) and it is well popularized (Steward, 1994).

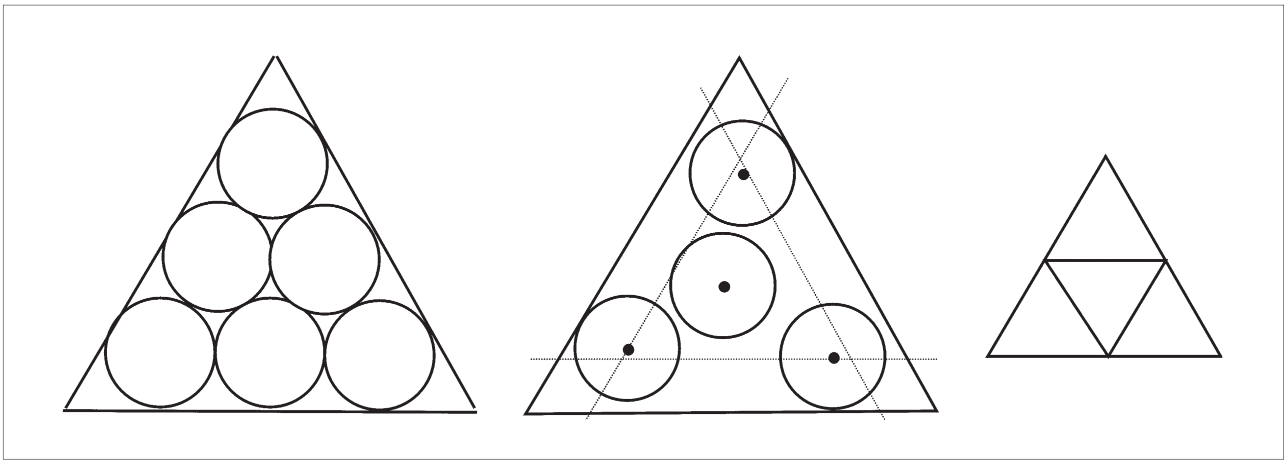


Fig. 9

On the above left figure, 6 disks of diameter 1 fit in an equilateral triangle of edge length $2+\sqrt{3}$. Can one put 5 disks (or maybe even 6) in a smaller equilateral triangle ? By a translation of 1/2 unit towards the interior of the triangle, in a perpendicular direction to the edges, one gets an equilateral triangle of length 2 which contains all circle centres. The question is thus reduced to the following one : find the smallest equilateral triangle containing 5 points with the property that any two of them are distant by at least 1 unit of length. Let us show that such a triangle cannot have an edge length less than 2. If it were the case, divide the triangle in four homothetic ones in the ratio 1/2, by joining the edge middle points. One of these triangles would contain at least two of the points. The distance between these two points would be less than the smaller triangles edge length, thus less than 1, which is a contradiction.

Experiments at various levels have shown that this problem is extremely interesting in view of the modelization activity. In particular, there are meaningfull questions pertaining to the equivalence between the problem posed in terms of disks and the problem posed in terms of points.

5. Conclusion

In this article, we have tried to illustrate the strength of some combinatorial models and tools which are likely to be in use in other taught domains of mathematics. Last but not least, these tools and models are easily apprehended and do not require sophisticated prerequisites. For example, the pigeonholes principle can be used to solve problems of a very different nature. In a more fundamental way, we have tried to present a few “simple” situations in which meaning and truth become central to the cognitive and teaching process. These situations are prototypal examples which may illustrate how discrete mathematics can be used for learning proof, modelization and some other transversal concepts in mathematics.

6. References

Appel K., Haken W. (1977): Every planar map is four colorable, Part I. Discharging. *Illinois J. Math.*, 21, 429-490.

Appel K., Haken W., Koch J. (1977): Every planar map is four colorable. Part II. Reducibility. *Illinois J. Math.*, 21, 491-567.

Arsac G. (1990): Les recherches actuelles sur l’apprentissage de la démonstration et les phénomènes de validation en France. *Recherches en Didactique des Mathématiques*, 9(3), 247-280.

Arsac G. et al. (1992): *Le raisonnement déductif au collège*. Presses Universitaires de Lyon.

Audin P. & Duchet P. (1992): La recherche à l’école : Math.en.Jeans. *Séminaire de Didactique des Mathématiques et de l’Informatique*, 121, Grenoble, 107-131.

Balacheff N. (1987): Processus de preuve et situations de validation. *Educational Studies in Mathematics*, 18(2), 147-176.

Batanero C., Godino J. D. & Navarro-Pelayo V. (1997): Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics*, 32, 181-199.

Berge C. (1973): *Graphes et Hypergraphes*. Bordas.

Grenier D. (1995): Savoirs en jeu dans des problèmes de combinatoire. In G. Arsac, J. Gréa, D. Grenier D. & A. Tiberghien (eds.), *Différents types de savoirs et leur articulation* pp. 235-251. Grenoble: La Pensée Sauvage

Grenier D. & Payan CH. (1998): Spécificités de la preuve et de la modélisation en mathématiques discrètes. *Recherches en Didactique des Mathématiques*, 18(1), 59-100.

Legrand M. (1996): La problématique des situations fondamentales. *Recherches en Didactique des Mathématiques*, 16(2), 221-280.

Payan Ch. (1995): La géométrie entre les lignes. *Cahier du séminaire DidaTech.*, Université Joseph Fourier, ed. IMAG, Grenoble.

Payan Ch. (1997): Empilement de cercles égaux dans un triangle équilatéral. A propos d'une conjecture d'Erdös-Oler. *Discrete Mathematics*, 165/166, 555-565.

Rolland J. (1995): *Le rôle ambigu des problèmes de combinatoire dans les manuels de troisième*. Memoire de DEA de Didactique des Mathématiques, IMAG, Université de Grenoble.

Stewart I. (1994): C'est du billard ! Compactage optimal de billes dans un triangle équilatéral. *Pour la Science*, 205 (nov. 1994), 96-100.

STUDENT'S PERFORMANCE IN PROVING: COMPETENCE OR CURRICULUM?

Lulu Healy, Celia Hoyles
Mathematical Sciences, Institute of Education, University of London, 20 Bedford Way, WC1H 0AL London, UK
l.healy@ioe.ac.uk
c.hoyles@ioe.ac.uk

Abstract: *In this paper, we describe some results from a nationwide survey of the proof conceptions of 14-15 year old students in England and Wales. To begin to examine the relationship between competence and curriculum in shaping students' performance on the proof questionnaire, we characterise the different responses of two students with differing views of proof and of mathematics.*
Keywords: *mathematical proof, curriculum, competence*

Students' difficulties in engaging with logical arguments in school mathematics are well documented (see Hoyles, 1997 for a recent summary). In 1989, the Mathematics curriculum of England and Wales prescribed a new approach to proving which follows a hierarchical sequence — where proofs involving logical argument are only encountered after extensive experience of inductive reasoning and of investigations where conjectures have to be explained.

1. Aims

The project, Justifying and Proving in School Mathematics[1], started in November 1995 with the aim to examine the impact of this new curriculum. It set out to:

- *describe the characteristics of mathematical justification and proof recognised by high-attaining students, aged 14-15 years;*
- *analyse how these students construct proofs;*
- *investigate the reasons behind students' judgements of proofs, their performance in proof construction and their methods of constructing proofs.*

2. Method

Two questionnaires were designed, a student proof questionnaire and a school questionnaire. The proof questionnaire comprised a question to ascertain a student's views on the role of proof, followed by items in two domains of mathematics — arithmetic/algebra and geometry — presented in open and multiple-choice formats. In the former format, students were asked to construct one familiar and one unfamiliar proof in each domain. In the latter format, students were required to choose from a range of arguments in support of or refuting a conjecture in accordance with two criteria: which argument would be nearest to their own approach if asked to prove the given statement, and which did they believe would receive the best mark. The school questionnaire was designed to obtain data about a school and about the mathematics teacher of the class selected to complete the proof questionnaire. These teachers also completed all the multiple-choice questions in the proof questionnaire, to obtain their choices of argument and to identify the proof they thought their students would believe would receive the best mark.

The survey was completed by 2,459 students from 94 classes in 90 schools. All the students were in top mathematics sets or chosen as high-attaining by the mathematics departments. Key Stage 3 test scores[2] provided a measure of students' relative levels of general mathematical attainment.

3. Results

We analysed students' views of proof, their scores on the four constructed proofs and the forms of argument used, their choices in the multiple-choice questions and their assessments of the correctness and generality of the arguments presented. We used descriptive statistics to describe patterns in response and multilevel modelling (using data from the school questionnaire) to identify factors associated with performance and how these varied between schools. In this paper, we focus on five of our findings (for a complete description of the results, see Healy and Hoyles, 1998).

High-attaining 14-15 year old students show a consistent pattern of poor performance in constructing proofs.

Student attempts to construct mathematical proofs were coded on a scale from 0 to 3. Table I describes the criteria for scoring and presents the distribution of scores on all four questions.

Constructed Proof Score	*Familiar algebra statement*	*Familiar geometry statement*	*Unfamiliar algebra statement*	*Unfamiliar geometry statement*
0 No basis for the construction of a correct proof	353 (14.4%)	556 (23.8%)	866 (35.2%)	1501 (62.0%)
1 No deductions but relevant information presented	1130 (46.0%)	1289 (52.4%)	1356 (55.1%)	690 (28.1%)
2 Partial proof, including all information needed but some reasoning omitted	438 (17.8%)	118 (4.8%)	154 (6.3%)	121 (4.9%)
3 Complete proof	537 (21.8%)	466 (19%)	83 (3.4%)	117 (4.8%)

Tab. I:

Distribution of students' scores for each constructed proof (% in brackets)

Table I shows that the majority of students did not use deductive reasoning when constructing their own proofs.

Students' performance is considerably better in algebra than in geometry in both constructing and evaluating proofs.

The picture of student proof construction as shown in Table 1 is slightly more positive for algebra than for geometry, particularly in terms of the ability to come up with at least a partial proof of a familiar statement. However, similar small proportions of students are able to construct complete proofs in comparable questions in both domains. Evidence from responses to other questionnaire items gave further support to this finding in that students were considerably better in algebra than in geometry at assessing whether each argument presented in the multiple-choice questions was correct and whether it held for all or only some cases within its domain of validity.

Most students appreciate the generality of a valid proof.

The majority of students were aware that, once a statement had been proved, no further work was necessary to check if it applied to a particular subset of cases within its domain of validity. For example, 62% of students recognised that if it had been proved that the sum of two even numbers was even, then the result did not have to be verified for particular cases. Similarly, 84% of students agreed that, if it had been proved that the sum of the angles of a triangle was 180o, then they need not check this statement for right-angled triangles.

Students are better at choosing a valid mathematical argument than constructing one, although their choices are influenced by factors other than correctness, such as whether they believe the argument to be general and explanatory and whether it is written in a formal way.

Significantly more students were able to select a proof that was correct from amongst various choices than to construct one. However, they were likely to make rather different selections depending on the two criteria for choice: own approach or best mark. For best mark, formal presentation was chosen frequently and empirical arguments infrequently – even when the latter would have provided a perfectly adequate refutation. Empirical choices were more common when students were selecting the argument closest to their own approach, although many more students favoured narrative or visual arguments which they believed to be general and which they found helpful in clarifying and explaining the mathematics in question. Formal presentation was not a popular choice for a student’s own approach.

Students’ views of proof and its purposes account for differences in their responses.

The students were asked to write everything they knew about proof in mathematics and what it is. Their responses were coded according to the categories listed in Table II which also shows the distribution of students’ answers[3].

View of proof and its purposes		n	%
Truth	References to verification, validity and providing evidence.	1234	50
Explanation	References to explanations, reasons, communicating to others.	895	35
Discovery	References to discovering new theories or ideas.	26	1
None/other	No response or one that indicated no understanding.	700	28

Tab. II: Distribution of students' views of proof

Table II indicates that students were most likely to describe proof as about establishing the truth of a mathematical statement, although a substantial minority ascribed it an explanatory function. Over a quarter of students, however, had little or no idea of the meaning of proof and what it was for.

Results from our statistical analysis indicated that students' views of proof and its purposes were consistently and significantly associated with performance. This interdependence of perception and response was evident in the following ways:

- *Students with little or no sense of proof were more likely to choose empirical arguments;*
- *Students who recognised the role of proof in establishing the truth of a statement were better at constructing proofs and evaluating arguments;*
- *In algebra, students who believed that a proof should be explanatory were less likely than others to try to construct formal proofs and more likely to present arguments in a narrative form.*

In order to tease out the meanings behind these statistical correlations, we moved to a second phase of research using a variety of qualitative techniques.

4. Some contrasting views of proof

On the basis of our statistical models of students' responses to the survey, we were able to pull out the variables that were influential on response and distinguish different profiles of students who were to a certain extent typical: for example, they expressed no view of proof, could not construct a proof, but could recognise a valid proof. We then selected a number of students for interview who corresponded to these different profiles or who exhibited surprising idiosyncratic responses. In the interviews, we aimed to find out more about the students' views of mathematics and of proof and to probe the reasons underlying particular responses they had made to the survey which we found interesting. We also interviewed their teachers to explore their views and to uncover any other contextualising data, particularly in the cases where whole schools exhibited similar profiles in their students' responses.

In the remainder of the paper, we report extracts from two student interviews to shed light on our contrasting profiles, and in particular, to illustrate how the students' views shaped their approaches to proving. The students' profiles were identical in terms of some of significant background variables in our model — they were girls and were of similar general mathematical attainment. However, the two girls had very different views of proof, made different choices of proof and made very different proof constructions. Sarah's responses were, in general, representative of a large number of students, except that her constructed proofs were rather better than most. Susie's responses stood out as different in almost every question.

Sarah — who likes to know why

Sarah performed rather well in the survey. The scores she obtained for her constructed proofs were higher than our statistical models predicted, with 3 out of 4 correct and complete. She was also good at evaluating arguments in terms of correctness and generality. In her interview, we wanted to probe why Sarah did better than the majority of the students surveyed. First, it soon became clear how much she liked mathematics and this positive attitude stemmed from a very early age:

Sarah: I had a very good maths teacher at junior school that taught me a lot of the things that I went on to do in secondary school and I had a greater understanding of them. I thought she was wonderful. She taught me all about Pythagoras and pi and everything when I was eleven years old.

Int: So you have always been quite interested in maths?

Sarah: Yes, but really it started with her then because she was so lively that it made it more interesting... she did things like, in 1991, she gave us 1 9 9 1 and said you use those numbers add, subtract and plus and divide and make all the numbers from one to hundred and we would sit there in groups and have great fun and it was in things like that she set up.

Sarah clearly enjoyed the open-ended investigations in the U.K. curriculum, which were also significantly, the very areas of mathematics where she had come across proof.

Int: So are there parts of maths that you do enjoy more than other parts?

Sarah: Yes, it is difficult to explain this. Some things that seem like a real slog to get through where it seems, not long winded but like you are slogging your way through instead of getting there step by step and going through it and understanding what you are doing......We are doing matrices at the moment and that seems boring to me...... you do the same thing time and time again, but there isn't a variation, there isn't a conclusion in many ways. You solve it and you draw it and great......I am more of an investigator.

\- - -

Int: So mostly you have come across proof in investigations. Is that right?

Sarah: Investigations and problem questions and things like that.....I think sometimes it can be very confusing, but once you get on the right path I think it is so great to be successful in it. If you see what I mean. To actually come to the conclusion, I think it is a good part of the lesson.

From her response to the questionnaire, Sarah's view of proof appeared rather unremarkable. She wrote only of its verification function, as shown in Figure 1:

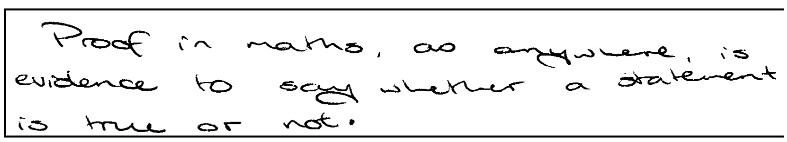

Proof in maths, as anywhere, is evidence to say whether a statement is true or not.

Fig. 1: Sarah's view of proof

She reiterated this view in here interview:

Int: What's it for? What's the point of proving?

Sarah: To prove that you were right or wrong.

However after examining the interview as a whole, it became apparent that she held a much more multi-faceted perspective of proof. As she discussed the empirical argument that had been offered as a proof that 'when you multiply 3 consecutive numbers, your answer is always a multiple of 6', (shown in Figure 2 below), it became clear that she thought examples *could* be 'proof', although not a 'conclusion' or 'explanation'.

Leon's answer

1 x 2 x 3 = 6	4 x 5 x 6 = 120
2 x 3 x 4 = 24	6 x 7 x 8 = 336

So Leon says it's true.

Fig. 2: A 'proof' that the product of three consecutive numbers is a multiple of 6

She explained:

Sarah: Well, it is enough for a proof but it is not a conclusion to me because it is not *why* … A proof that it is, not a proof plus conclusion, really - - - which is always what I prefer, because I like to find reasons rather than just examples.

Int: So a proof can be on the basis of just examples?

Sarah: Yes, you proved that the statement is wrong or right.

Int: So you have proved it's right?

Sarah: Yes, but it hasn't given the reason that I would find interesting....that's what I try and do because it has more reasoning than just an example.

So evidence, or a 'proof' was not enough to satisfy Sarah: she wanted an argument that also explained and it was this that motivated her. So she added to her view of proof as comprising empirical checks, the necessity for a convincing 'conclusion'. This interpretation is supported in the way she constructed proofs, which tended to be presented in a narrative form with an emphasis on explaining (see Figure 3) - type of proof construction that was very dominant in the survey.

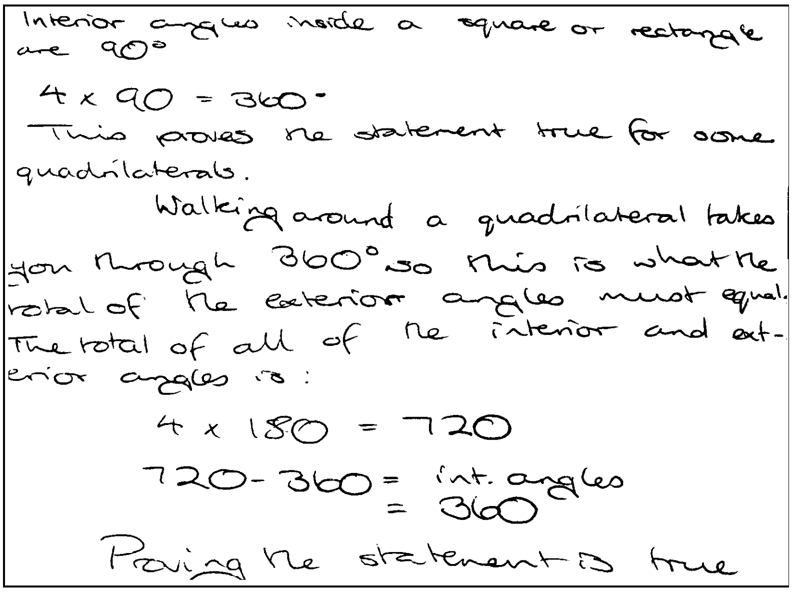

Interior angles inside a square or rectangle are 90°
4 x 90 = 360°
This proves the statement true for some quadrilaterals.
Walking around a quadrilateral takes you through 360°, so this is what the total of the exterior angles must equal. The total of all of the interior and ext-erior angles is:
4 x 180 = 720
720 - 360 = int. angles
= 360
Proving the statement is true

Fig. 3:
Sarah's proof that the sum of the interior angles of a quadrilateral is 360°

Also in line with survey findings was Sarah's conviction of the generality of a valid proof - - - it was simply obvious to her that a proof for all cases applied to any given subset:

Int: Right, it says here, suppose it has been proved when you add 2 even numbers the sum is even, do you need to do anything more about the sum of two even numbers that are square?

Sarah: When you add 2 even numbers that are square, your answer is always even. Its obvious, its already proved.

Int: You don't need to do any more? Because its obvious?

Sarah: Yes, as long as they are even, it doesn't matter if they are square, triangle. If they are even and you have proved it for all even numbers then that includes them.

Susie - - - proof is a formula which always needs examples

In contrast to Sarah, Susie was unable to write anything about proof on our questionnaire. In this respect she was similar to about one third of the students in the survey. But unlike most students in the survey, Susie was clearly confused about the generality of a mathematical argument. She tended to select empirical arguments as her own approach in all the multiple-choice questions, in geometry and in algebra, and saw them as both general and explanatory. Yet, she produced an almost perfect formal proof for the second geometry question – something only achieved by 4.8% of the students in the survey and which 62% of students did not even start. So why was Susie's profile of responses so strange?

First, compared to Sarah, Susie's view of mathematics was generally negative.

Int: Do you think you are good at maths?
Susie: No .... I think it's troublesome.
Int: How do you mean?
Susie: It's quite complicated and I hate doing it.

It was clear over the interview that Susie rarely engage with mathematics and only liked it when she could 'get it right'. Although Susie offered no description of proof or its purposes, it became clear in the interview that examples (*many* examples) took a central role. This contention is illustrated below when Susie described why she rejected Yvonne's visual argument in preference to Bonnie's empirical one, when evaluating proofs of the statement that 'when you add any 2 even numbers, your answer is always even' (see Figure 4).

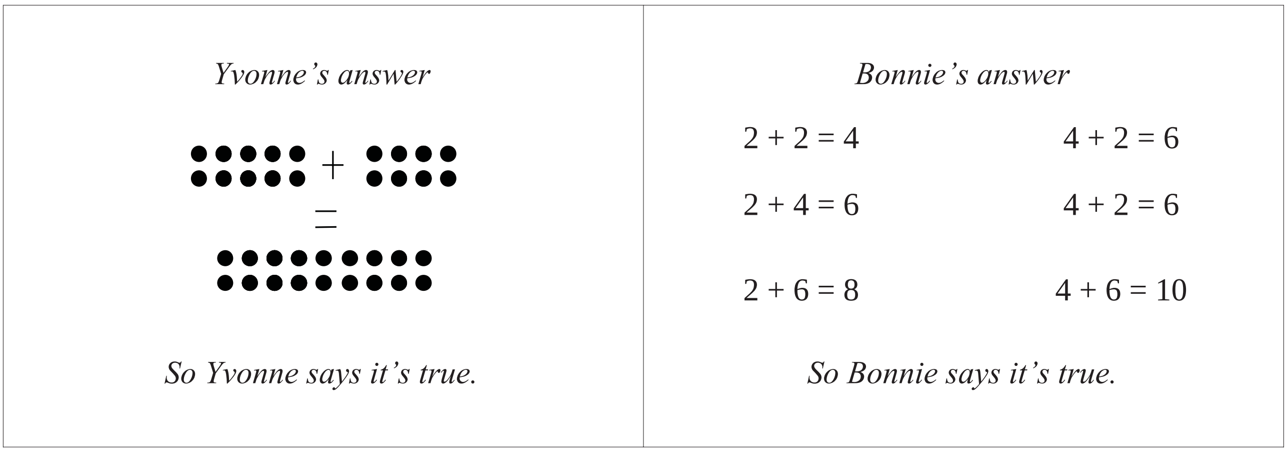


Fig. 4: A visual and empirical proof.

Int: What do you think of Yvonne by the way? You said it's okay, it doesn't have a mistake in it, but you don't think it's always true. So that you say is not always true - why did you say that - only sometimes true?

Susie: Because she just proves it one time. It is luck.

Int: Oh I see, this is just one time for the visual, so it's not enough.

Susie: Yes.

Int: Now, but it's interesting, Bonnie - which is the one you chose as nearest your approach. You said that it showed that the statement is always true - Bonnie's. Do you think that's the case?

Susie: Yes, from those examples, yes always true. They always have an even number result.

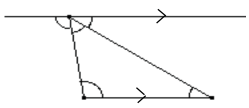
Int: Those examples. But does that show that it's always true?

Susie: Yes, because she has proved it many times.

As the interview progressed, however, it became clear that for Susie as well as examples, there was another important aspect of proof, namely a *rule or formula*. However its role was to obtain more marks rather than to confer any generality on the proof which she did not seek and did not appreciate. So, the examples were enough to prove a statement, but a rule was needed because that was what was expected of her. This interpretation is illustrated in Susie's choice of Dylan's 'proof' for her own approach and Cynthia's for the best mark when selecting an argument that proved that 'when you add the interior angles of any triangle, your answer is always 180° (see Figure 5).

Cynthia's answer

I drew a line parallel to the base of the triangle.



Statements	Reasons
$p = s$..................	Alternate angles between two parallel lines are equal
$q = t$....................	Alternate angles between two parallel lines are equal
$p + q + r = 180°$.....	Angles on a straight line
$\therefore\ s + t + r = 180°$	

So Cynthia says it's true.

Dylan's answer

I measured the angles of all sorts of triangles accurately and made a table.

a	*b*	*Int:*	*total*
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

They all added up to 180°.

So Dylan says it's true.

Fig. 5: Susie's choice of proof for best mark and own approach

She was asked to explain her choices.

Int: Now you said you would do Dylan but your teacher would give the best mark to Cynthia. Could you explain why Dylan and why you thought Cynthia would get the best mark?

Susie: .... Um… You can draw in triangle and measure each angle....You just need to put more examples. If you still get 180° it means it's always true....

Int: But you don't think Dylan would get the best mark?

Susie: No because he can't give… If the angles aren't those numbers it is… it can't, I mean he can't give out a rule for the proof. I think he needs to find out the rule of the question because for what I've done for the coursework the teachers proved the thing. You need to give a rule for the problem. A formula.

Int: And is that what Cynthia has done?

Susie: Yes

Int: You see you said for Dylan, you said it's fine - this is the one you chose - shows that the statement is always true. You did say that you agreed with that. It showed it's always true but you're now saying you're not so sure it wouldn't necessarily show it's true when the numbers are different.

Susie: Yes it will always be true but it will be better if there is a formula in it. If he can write a formula…

Her confusion over generality was evident when it became clear that she believed that even after producing a valid proof that the sum for the angles of a triangle is always 180°, more examples would be needed to be sure that the statement held for particular instances. She described the further work necessary to be sure that the 'formula' worked for right-angled triangles:

Susie: You can use the formula from… I don't know whether she has proved the formula… if she has she can add 90 to the formula.

Int: Okay if she's used the formula she can do a case with 90.

Susie: Yes.... she needs more examples as well.

When it came to construct proofs for herself, again Susie responded in a surprising way. Unlike the majority of students, Susie's proofs in geometry were far better than in algebra, and the proof she constructed for the second geometry question appeared to be much better than almost all other survey students. Her proof is presented in Figure 6.

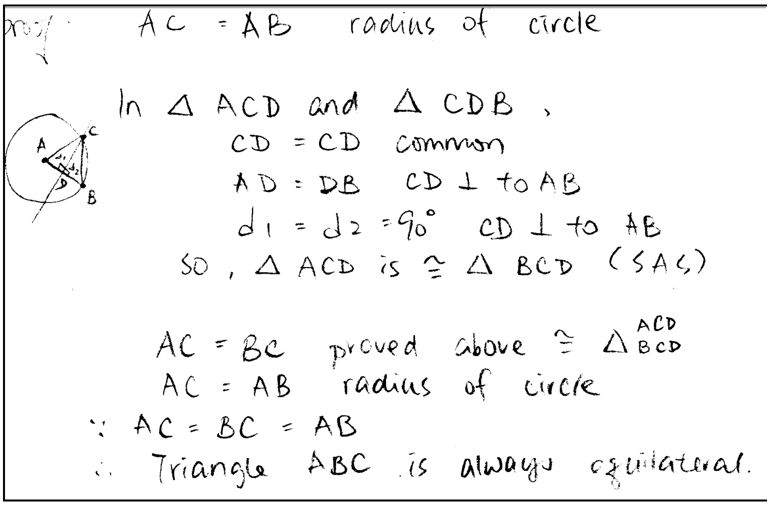


Fig. 6: Susie’s proof that Δ ABC is equilateral

Given the interview data above, it seemed that although Susie could write formal proofs, she did not see them as general and found them no more convincing than empirical evidence. Thus, her ‘perfect’ geometry proof appeared to be disconnected from anything about proving, let alone explaining or convincing.

5. Discussion

How can we interpret these different student responses? Are they a matter of differences in competence? We take a different starting point and suggest that to begin an explanation we have to consider the curriculum. Susie’s ‘odd’ profile can at least partly be explained by the fact that she had only studied mathematics in an English school for less than one year and had been educated up to then all here life in Hong Kong. Unlike most other students in the survey, she had been taught how to construct formal geometry proofs as well as to produce and manipulate algebraic formulae and she was clearly competent in both areas. Yet neither were about generality and moreover did not connect with argumentation or explanation.

Sarah, too, was also not a completely typical student. She had been exposed to proving at a young age and had identified an aspect, explanation, that she highly valued. So while the U.K. curriculum ‘delivered’ an empirical approach to proof in the context of investigations (collect examples and spot a pattern), Sarah wanted to do more than construct and check conjectures on the basis of evidence; she wanted to focus on the properties of the examples, and to construct from these, arguments that explained to her. Many students who follow the current U.K. curriculum are not like Sarah: they are likely to focus on measurement, calculation and the production of specific (usually numerical) results, with little appreciation of the mathematical structures and properties, the vocabulary to describe them, or the simple inferences that can be made from them. Yet those who are like Sarah have managed to develop a strong background on which to build more systematic approaches to proving — if they were exposed to them. Yet, the story of Susie cautions against replacement of a curriculum which de-emphasises proof with one in which students are simply 'trained' to write formal proofs. Curriculum changes must build on student strengths - in the U.K. case, on student confidence in conjecturing and arguing. Students like Sarah have responded positively in the teaching experiments that followed the survey, and risen to the challenge of attempting more rigorous proof alongside their informal argumentation. We have yet to analyse how this was achieved, yet we do know that Susie, on the other hand, made rather little progress, as she was starting from a perspective that we had not anticipated. We need to tease out different routes and devise different teaching approaches that ultimately lead to the same goal: a balance between the need to produce a coherent and logical argument and the need to provide one that explains, communicates and convinces. Clearly no universal solution is possible.

6. References

Hoyles, C. (1997): The Curricular Shaping of Students. Approaches to Proof. *For the Learning of Mathematics*, 17 (1), 7-15.

Healy, L and Hoyles, C. (1998): Technical Report on the Nationwide Survey. *Justifying and Proving in School Mathematics*, Institute of Education, University of London.

Notes

1. Funded by ESRC, Project Number R000236178
2. National tests administered to all students (aged 13-14 years) in England and Wales.
3. The total % is greater than 100, since some students incorporated several views in their descriptions, in which case their responses were coded into more than one category.

INTERACTION IN THE MATHEMATICS CLASSROOM: SOME EPISTEMOLOGICAL ASPECTS

Andreas Ikonomou[1], Maria Kaldrimidou[2],
Charalambos Sakonidis[3], Marianna Tzekaki[4]
[1]Training School of Teachers, Gambeta 93A, 54644 Thessaloniki, Greece
seletethe@hol.gr
[2]University of Ioannina, Ariba 37, 45332 Ioannina, Greece
mkaldrim@cc.uoi.gr
[3]Democritus University of Thrace, Dep.of Education, New Chili, 68100 Alexandroupolis, Greece
sakonid@edu.duth.gr
[4]Aristotle University of Thessaloniki, Dept. of Early Childhood Education, 54006 Thessaloniki, Greece
tzekaki@nured.auth.gr

Abstract: *This study relates the role of the teacher to the epistemological features of the classroom mathematics. Forty mathematics lessons given to 10-12 year old pupils were analysed in this respect. The results show that the teaching approaches used tended to treat the epistemological features of mathematics in a unified manner.*
Keywords: *mathematics, epistemology, teaching*

1. Introduction

There have been many theoretical approaches within research in mathematics education regarding the nature of mathematical knowledge and the processes of learning and teaching mathematics. Sierpinska and Lerman (1996), in an interesting review of the relevant literature, outline the basic elements and principles of the various approaches (constructivism, socio-cultural views, interactionism and theories based on the epistemology of meaning) and pose the question to whether, in relation to epistemology, all these approaches are competitive or complementary. Although the answer to this question cannot be conclusive, there are a few clear and interesting points that arise from the relevant debate:

- the classroom is a social, institutional creation,

- the study of learning and teaching involves phenomena of a heterogeneous nature,

despite the fact that epistemological views of mathematics do not apply directly to teaching, there are very strong links between these epistemologies and theories of mathematics education.

There is considerable evidence in the literature that the epistemological conceptions play an important role in the construction of mathematical knowledge. However, it is still unclear how these conceptions are shaped through the interaction within the mathematics classroom.

2. Theoretical framework

For pupils, mathematics acquires its meaning through school content as well as the relationships and interactions developed among the social and cognitive partners of the classroom (teacher - pupils). In addition, the organisation and inter-relations of the elements of the content; their relationship to problems, situations and representations; the evaluation and interpretation of these elements within the classroom, and the individual's functional role in and outside the classroom also play an important role in pupils' attempts to make sense of school mathematics (Sierpinska and Lerman 1996, Seeger 1991, Steinbring 1991).

The above consideration is based on the assumption that children learn directly, a procedure depending on the content of mathematics and their action on it, as well as indirectly through their participation in the cultural environment of the classroom. Thus, for example, pupils learn what is important in mathematics by observing what is emphasised and which types of solutions are marked by the teacher and the other pupils (Sierprinska and Lerman 1996, pp 850-853). It could be argued that children interpret the events by attributing to each of them a value proportional to its usefulness to the mathematics lesson. These interpretations concern the meaning of concepts and processes as well as the nature and value of concepts and processes of the system of knowledge.

In this context, the study of the nature and organisation of the mathematical content becomes of particular importance. This study requires an analysis of the ways in which epistemological elements of mathematics, such as the nature, meaning and definition of concepts, the validation procedures and the functionality of theorems appear in the classroom, especially in the process of (linguistic) interaction.

The role of the teacher in introducing the epistemological features in the mathematics classroom is more than important. S/he is considered responsible for the socio-mathematical norms of the classroom which, indirectly, determine what counts as mathematical thinking (Voigt 1995, p 197). S/he is seen as the agent who gives birth to culture (Bauersfeld 1995, p 274), the subject pupils address themselves to in order to attain their mathematical knowledge (Steinbring 1991, pp 67-73). Furthermore, the teacher, being the prevailing source concerning the content of mathematical knowledge, is considered to be the factor that determines the epistemological level of the development of the mathematical concepts in the classroom. As a result, as Steinbring suggests, many of the difficulties children have in understanding mathematics appear because teachers' knowledge and pupils' knowledge are situated on different levels.

The analysis that follows focuses on the study of the relationship between the role of the teacher on the one hand and epistemological features of the classroom mathematics on the other. In particular, an attempt is made to examine the way the teacher handles the nature, meaning and definition of the concepts, the validation procedures and the functionality of theorems in the classroom. In the following, these features are referred to as "epistemological features".

3. Methodology

The data collected for this study were 40 lessons in mathematics observed in various classes of the last three grades of the elementary school (10 - 12 years old) for over a week in the city of Ioannina (north-western part of Greece). The lessons chosen for the present study come from three different schools and nine teachers. While selecting the classes, two factors were taken into account: the type of pedagogical organisation and the interaction between teachers and pupils. More specifically:

- the pedagogical profile of the teacher (authoritarian (*A*), directive (*DR*), dialectic (*DL*));
- the forms of communication adopted in the classroom (teacher-centred, child-centred, pupil to pupil interaction);

Another factor taken into account was the degree to which the lesson depended on the curriculum and the textbook. In specific, we considered whether the teacher follows the textbook closely (*TC*), or in a more flexible way (*TF*) or s/he teaches based on his/her own ideas and ways of developing the topics taught and not on the textbook (*TN*)). It is worth noting that all Greek schools use the same textbooks, distributed free of charge by the Ministry of Education.

The above selection helps in examining the epistemological features of mathematics developed in a variety of forms of classroom interaction.

The lessons were taped, transcribed and analysed along the following two dimensions:

- the organisation and interrelationships of the various elements of the mathematical content (concepts, definitions of concepts, theorems and functionality of theorems);
- the organisation and selection of the elements of the mathematical activities (solving and proving processes, validation processes -checking and evaluating solving processes and solutions-).

4. Presentation of data and discussion

The distribution of the analysed lessons regarding the pedagogical organisation, the interaction within the classroom and the use of the textbook was as follows:

Pedagogical organisation: 2 authoritarian, 5 directive and 2 dialectic teachers.
Interaction within the classroom: teaching was teacher centred in all nine classes.
Use of textbook: 3 teachers followed the textbook closely, 3 in a flexible way and 3 teachers made little or no use of the textbook.

In the following, elements of epistemological features concerning the mathematical content and the activities found in the analysis of the transcribed lessons are presented. The focus was on the features concerning the mathematics "produced" in the classroom rather than teachers' or pupils' epistemology.

Episodes from the nine lessons are used in order to illustrate the findings.

4.1 Elements of mathematical content

In general, the way in which the nature, meaning and definitions of mathematical concepts are presented and dealt with in the classroom does not allow them to be distinguished from one another, regarding their meaning and their role in mathematics. Thus, these elements of the mathematical content lack epistemological characteristics.

Concepts

Concepts are often reduced to processes of manipulation, construction and recognition.

Example 1: In the episode that follows, the teacher (DL, TN) presents a decimal number explaining the process of recognition (10 years old):

T(eacher) *... decimal fraction units and decimal fractions are written in the form of a decimal number which is recognised by the comma or decimal point. The 1/10 is written as 0 integral units and 1 tenth ...*

Example 2: In this episode, the teacher (DR,TC) presents the area of a rectangle as a process of counting and through this he generalises (11 years old):

T. *... Count the (number of) boxes (square centimetres), how many boxes are there?*
P(upil). *There are 12 boxes.*

T. *So, the area of the rectangle is 12 square centimetres. Then, how can we find the area of a rectangle? What do we have to multiply?*

P. ......

T. *The area of the rectangle is: base by height and we measure it in square centimetres or square metres. …*

Example 3: In the following episodes, the concept of a triangle is reduced to the process of construction and manipulation of a "real" object (11 years old pupils / DR, TF teacher).

T. *… If I take three points and join them, what do I end up with?*

T. (In another occasion) *… The base is the side on which the triangle "sits". Could we use all three sides of the triangle as a base?*

P. *No!*

T. *Of course we can. But we use as base the side on which the triangle is better balanced. However, any side could be the base …*

The same happens in the following examples, where teachers develop a series of concepts:

T. *… fractional unit is one of the equal parts in which we divide the whole unit.* (construction)

T. *… decimal fractions are generated by the repetition of decimal fraction units.* (manipulation).

T. *… if I will draw a perpendicular from the angle* (from the vertex to the opposite side)*, this is the height* (construction and recognition).

Definitions of Concepts

Definitions, often confused with processes similar to those described in the previous section, do not define the concepts: they simply explain how to manipulate the elements to which they refer. The mathematical function of "definition", its differentiating and identifying role, does not appear anywhere. Pupils are often asked to learn definitions within the context of the lesson's demands, that are used nowhere else.

Example 1: Pupils measure the size of the angles of a triangle, in order to determine what an acute triangle is (12 years old pupils / DR, TF teacher).

T. *We measure and compare the angles.. What type of angles are they?*
P. *Acute angles*
T. *The triangle is an acute triangle.*

Example 2: Two pupils have measured the length of the three sides of a triangle in order to define the concept of perimeter (12 years old pupils / A, TN teacher).

T. *We add and how much do we find?*
P. *11 cm.*
T. *We call these 11 cm perimeter.*

Example 3: In the quotation that follows, the confusion between definitions and practical procedures becomes apparent (12 years old pupils / DL, TF teacher).

T. *Take the pair of scissors and cut this cardboard on which you have drawn the circle. How is this circular surface that has been created called?*
P. *Circular disk*
T. *And the circular line round the piece of paper you have cut out?*
P. *Circle*
T. *What is the difference between the circle and the circular disk, as you see it on the cardboard?*

Teachers ask and elicit definitions through construction or measuring processes. These “definitions” are absolutely oriented and restricted to the frame from which they come. They constitute the result of an action and not the setting of the limits of the concepts.

Theorems

Similar features are observed in the theorems/properties which are differentiated neither from the definitions nor from the processes.

Example 1: In the episode that follows, it is clear that the presentation of a theorem is not distinguished from definitions and processes (11 years old pupils / DR, TF teacher).

T. *If we construct an acute triangle and measure its angles, what will we see?*
P. *70°+50°+60°=180°*
T. *In the acute triangle the sum of the angles is 180°.*

The generalisation which will lead to the theorem has already been done.

In the consequent lesson, the classroom deals with the sides of a triangle.

T. *Let's construct an acute triangle and measure its sides.* (Children measure with the ruler and the teacher writes on the board)
T. *2,5+4+4,5=11 cm. Thus, the sum (of the length) of the three sides is 11cm. How do we call this?*
P. ....
T. *We call the sum (of the length) of the sides perimeter of the triangle.*

The generalisation that leads to the definition has already been made; the process is exactly the same as in the previous episode.

Example 2: In this episode, the way the teacher elicits the definition and the properties is absolutely muddled (11 years old, DR, TF teacher).

T. *... thus, what do we have in the isosceles triangle?*
P. *Two sides equal?*
T. *Yes, and what else?*
P. *And two angles equal*
T. (.. a little later) *So, what do we call equilateral triangle?*
P. *The one that has three equal angles and three equal sides.*

This negotiation refutes the view that one feature is enough to characterise an isosceles or an equilateral triangle and that the properties derive from the definition. The two features are placed on the same level through the teacher's questioning.

Furthermore, theorems and processes are often confused in teachers' discourses. Examples of this are:

T. *... when we compare decimals, we look at the integral part of the number and at the tenths* (decimal part)

T. *... when the denominator is 100, we have two digits after the decimal point.*

T. *... to write 3,5 in such a way so that it sounds like centimetre.*

4.2 Elements of mathematical activity

The activity developed in the mathematics classroom is often, almost entirely, deprived of the characteristics of mathematical processes which have to do with the pursuit of solving and proving procedures, as well as checking and confirming (validation).

Solving - Proving Procedures

The problem-solving methods which arise in the classroom do not constitute a process to negotiate or a subject matter of discussion, but a typical course towards the solution. The frequent processes of measuring used in the classroom to approach concepts and properties are a typical example of this.

Example 1: The teacher (DR, TF) provides the pupils with various triangles which they divide into categories according to the size of their angles, which they first measure. In the example that follows, the teacher suggests measuring as a process of proving (11 years old).

T. *Now, take your protractor and measure the angles and decided what kind of triangle it is.*

P. *In the first triangle there is a 90° angle.*

T. *How did you find it?*

P. *It looks like it, it seems to be a right angle.*

T. *In mathematics Billy, we can't claim something without being able to prove it. Take the protractor please, measure the angle and tell me whether it is in fact 90°.*

A week later, the same pupils are asked to solve the following problem “How much do the angles measure in the triangle below?” (isosceles triangle ABC (angle B= angle C), angle A=80°) (12 years old).

T. *The two angles are equal to each other. How many degrees are both together?*
P. *100°*
T. *So, each angle is ...?*
P. *50° each*
T. *Write it down.*
P. *We should measure them.*
T. *No, we shouldn't.*

A few days later, the same class uses again measuring as a proving procedure, but instead of the angles pupils worked with the sides of the triangle. In other words, there was a turning back with no explanation or negotiation.

In the above example, the teacher refuses to discuss the validity of the two methods of proof, although this issue is raised by the pupil. It is evident that the child believes that the measuring process provides more valid results compared to the application of the general theorem to a particular case. Furthermore, in using the measuring process a little later, the teacher reinforces the pupil's mistaken idea about the validity of one method (process) against another.

The following are typical examples of converting the solution - proof to a particular course of operations, which devalues the solving-proving process.

Example 2: The problem given is the same as in the last part of example 1. One pupil answers immediately (12 years old pupils / A, NT teacher):

P. *50°*
T. (He does not ask, for example, “how did you find it”, but) *A little better?*
P. (Changes her solution to the typical process) *I subtract 80°from 1800 and divide by two.*

Example 3: The problem asks for 3/8 of a kilo (10 years old pupils / DR, TF teacher).

T. *What will you do?*

P. *... Well ... I will divide the kilo by 8* (1000:8 = 125) *.... and then .... I will then multiply 125 by 3.*

Validation (Checking and Confirming) Procedures

Checking and confirming procedures fall into two categories in the transcribed lessons. In the first, the correction and confirmation are done directly by the teacher. In this first case, s/he uses expressions like “Yes”, “No”, “Well done”, “This is it”, “I would like better”, “Haven’t you forgotten something?”. Although it is often clear that s/he agrees with the answer, s/he asks for its confirmation.

Example 1: The teacher (DR, TF) provides three cases of triangles, where the sum of the size of the three angles is equal to 180° (12 years old).

T. *What do we notice here?*

P. *Sir, the sum of* (the size) *of* (all) *angles of a triangle is 180°.*

T. *Very well! That was my point.*

In the second case, in order to prove the correctness of an answer, the teacher addresses the class for consensus. One pupil corrects but the final confirmation is done by the teacher.

Example 2: (10 years old pupils / DR, TN teacher)

T. *0.5 and 0.35, which one is the larger?*

P. *The 0.35*

T. *I agree, who else agrees?*

P. (another pupil) *Nobody, it’s 0.5.*

T. *Correct.*

Example 3: In a problem of calculating an area, a pupil wrote that it is 16 metres (11 years old pupils / DR, TC teacher).

T. *Do you have anything to say?*

P. (another pupil) *16 s.m. and not 16 m*

T. *Well done.*

From the above examples, it could be argued that the assignment of the right to check, which means recognition of knowledge or ability to solve, is cancelled since it is accompanied by the final approval of the teacher. The existence of an external assessor creates difficulties in recognising the role of the proof in establishing the truth of a statement in mathematics.

5. Conclusions

As argued in the earlier, the teaching observed had different characteristics regarding the profile, the focus of communication and the degree to which the lesson depended on the curriculum and the textbook. However, a number of similarities in the presentation of epistemological features of mathematics have been located, despite the differentiation with respect to the type of interaction inside the mathematics classroom. These similarities are summarised below:

1. The presentation of the mathematical content shows a particular homogeneity inside the classroom.
   - the concepts and the definitions are reduced to procedures of manipulation, construction and recognition
   - the theorems are distinguished neither from the definitions nor from the processes.

2. The activity which is developed inside the mathematics classroom bears almost none of the epistemological features which characterise mathematical processes.
   - the methods of problem solving constitute a typical, non-negotiable route to the solution
   - the validation (checking and confirming) procedures are submitted to the teacher's final approval.

The above indicate that school mathematics, as a system, is differentiated from mathematics as a discipline as far as its basic features is concerned. It is highly probable that this differentiation shapes pupils' conceptions of mathematics and mathematical knowledge in such a way that it inhibits them later on from adapting the features of mathematics and mathematical activity (for example, see children’s difficulties with the concept of proof and the process of proving in the article by Hoyles and Healy in these proceedings).

3. The analysis of the role of the teacher in the preceeding episodes reinforces the views outlined in the beginning, that s/he is responsible and functions as an agent for the epistemological level of the development of mathematical knowledge in the classroom (Bauersfeld, 1995, Steinbring, 1991). This role of the teacher is manifested in the way in which s/he phrases the questions; in the types of the answers and processes s/he requires on a variety of occasions and in the way in which s/he assesses the answers or the solutions of the pupils.

The above findings lead to the following questions:

- Is the teachers’ behaviour solely a result of their role in the classroom and their understanding of this role or is it related to their views about mathematics and mathematics teaching?
- Given that the primary school teacher usually teaches almost all subjects, is it possible for the same person to change their role and framework in teaching one subject after the other?

A first attempt to answer these questions could be that teachers do not appreciate that there are epistemological differences between mathematics and the other subjects they teach and do not have clear or have incorrect ideas about the nature of mathematics.

Such an interpretation leads to a more general problem, this being the relationship between interaction and knowledge in instruction. Therefore, the improvement of mathematics education requires changes, as Seeger says, in “the very notion of content” (Seeger 1991), as it is related to teachers and teaching.

6. References

Bauersfeld. H. (1995): Language Games in the Mathematics Classroom: Their function and their Effects. In P. Cobb and H. Bauersfeld (eds.), *The Emergence of Mathematical Meaning. Interaction in Classroom Cultures*. Hillsdale, NJ: Lawrence Erlbaum Associates Publishers.

Seeger,F. (1991): Interaction and knowledge in mathematics education. *Recherches en Didactique des Mathématiques*, 11(2.3), 126-166.

Sierprinska, A. & Lerman, S. (1996): Epistemologies of Mathematics and of Mathematics Education. In A. J. Bishop (ed.), *International Handbook of Mathematics Education.* Dordrecht: Kluwer Academic Publishers.

Steinbring, H. (1991): Mathematics in teaching processes. The disparity between teacher and student knowledge, *Recherches en Didactique des Mathématiques*, 11(1), 65-108.

Voigt, J. (1995): Thematic Patterns of Interaction and Sociomathematical norms. In P. Cobb & H. Bauersfeld (eds.), *The Emergence of Mathematical Meaning. Interaction in Classroom Cultures*. Hillsdale, NJ.: Lawrence Erlbaum Associates Publishers

GROUP 2: TOOLS AND TECHNOLOGIES

COLETTE LABORDE (F)
ANGEL GUTIERREZ (E)
RICHARD NOSS (GB)
SERGEI RAKOV (UKR)

TOOLS AND TECHNOLOGIES

Angel Gutiérrez[1], Colette Laborde[2], Richard Noss[3], Sergei Rakov[4]
[1]Universitat de València, Didàctica de la Matemàtica, Apdo. 22.045 46071 Valencia, Spain
angel.gutierrez@uv.es
[2]Université Joseph Fourier - CNRS, Laboratoire IMAG-Leibniz, 46 avenue Felix Viallet, 38031 Grenoble Cedex1, France
colette.laborde@imag.fr
[3]University of London, Institute of Education, 20 Bedford Way, London WC1H 0AL, UK
rnoss@ioe.ac.uk
[4]Kharkov Pedagogical University, Regional Center for New Information Technologies, Bluchera street, 2, Kharkov, Ukraine, 310168
Rakov@usa.net

1. Tools and technologies in the didactics of mathematics

The thematic group discussed the role of tools and technologies in mathematics education on the basis of contributions of the nine attached presentations which cover various tools (including a range of programs and — interestingly enough — one non-computational tool consisting of semi-transparent mirrors). These spanned very different levels of schooling and topics: from primary school to university level, from numeration decimal system to calculus and geometry.

The case of the semi-transparent mirrors points to a general issue which emerged from the group: that focussing on the roles of computational tools, and how they mediate learning, is a special case of a focus on tools more generally. More important still, a primary outcome of this kind of focus is that it centres our attention on the ways in which tools mediate knowledge construction, and therefore, on quite general questions concerning mathematical learning — which are themselves independent of specific tools, whether computational or not.

It is useful to distinguish three embedded levels when analysing the use of tools in mathematics education:

- the level of the interactions between tool and knowledge

Tool <-------> Knowledge

- the level of interactions between knowledge, tool and the learner

Tool <------->
Knowledge

Learner

- the level of integration of a tool in a mathematics curriculum and in the classroom

Tool <------->
Knowledge

Learner

Teacher

2. Tools and knowledge

A key issue at the level of interactions between tool and knowledge concerns the question of how the tool mediates knowledge and how this process of mediation actually changes knowledge itself and its use. A paradigmatic example of this kind of change, which led to a lively discussion within the group, was provided by the following task:

The task began by asking for the enumeration of the various possibilities for the number of intersection points of four straight lines in the plane. It then continued:

1. *In how many points can the angle bisectors of a quadrilateral intersect question Use a dynamic geometry environment and choose a quadrilateral. Construct its angle bisectors.*
2. *By moving the vertices, can you obtain all possibilities which you listed in part 1?*
   Point out all the possibilities you found
   Which ones are missing ? Explain why.
   In a few sentences, write down your explanation.
3. *In how many points can the angle bisectors of a triangle intersect question?*
   Justify your answer.

One way of looking at this task (not the way of those of us who only read one part at a time!) is that it leads to a proof of the fact that the angle bisectors of a triangle intersect all in a point. But using a dynamic geometry environment affords a different (and more general) question concerning quadrilaterals. We do not propose to spoil this question for the reader by providing answers here, but the group was surprised to find that it led to a completely new way to think about an old question (and its answer) — put briefly, we came to see that the bisectors intersect in one point because they cannot intersect in three points!

Such a perspective simply would not arise in a paper and pencil environment, since in this case, there is no empirical evidence of the fact that the four angle bisectors of a quadrilateral cannot intersect in three points. The use of a tool in this example not only changes the way of exploring the question but even the meaning of the property: instead of appearing as a beautiful fact specific to a triangle, it becomes the by-product of a more general property. In one sense at least, it is not only the approach to the mathematical goal that changes, but the mathematical goal itself.

Thus some important questions for teaching of mathematics arise about a new epistemology of mathematics created by the use of technology. In particular it seems that modelling is more relevant in the computer era than it was before (cf. problems given in the paper of Belousova & Byelyavtseva). The nature of proof is also very much subject to change by the use of technology: mathematics might become the science of modelling rather than a fundamental science taught for its internal structure and specific ways of developing knowledge (we did not discuss the changes that are happening to mathematics itself — that is for another conference perhaps!).

A further category of changes through the mediation of the tool deals with the new behaviour of the objects due to the mediation. A very good example is given by the behaviour of points in a dynamic geometry environment. In a static environment, there is no reason to question the behaviour of objects when some basic elements are moved because there is no such possibility. But now it is natural to ask what could be or should be the trajectory of a point on a segment AB when one of its endpoints is dragged? There is no answer in Euclidean classical geometry because the question is meaningless. Thus the mediation of this geometry in a dynamic geometry environment actually creates objects of a new kind. The "danger" for the user is that the new nature of the object may be not visible. They may be transparent for the users who may believe that the objects are identical to those with which they are familiar.

The designers of tools specifically devoted to mathematics are thus faced with decisions about the behaviour (or properties) of the new objects they create. The group discussed the effect of these choices not only in geometry (cf. Jones and Dreyfus & al. papers) but also in calculus (cf. Gélis & Lenné paper). The changes made to knowledge by the use of the tool inevitably lead us to address the question of the meaning constructed by the user and in particular by the learner when using the tool.

3. Interactions between tool and learner

Tools are mostly used in mathematics teaching for their potential to foster learning. Integrating a spreadsheet into mathematics teaching, a CAS or a dynamic geometry environment is not primarily aimed at learning how to use them: it is essentially intended to improve the learning of mathematics by creating a context giving sense to mathematical activity. But there might be some distance between what the learner constructs from the use of the tool and the expectations of the teacher. Some papers of the group investigate the extent to which different environments may lead to different kinds of learning by observing strategies developed by students in both environments (cf. Price and Hedren papers about counting and calculating strategies).

Approaching the understandings constructed by the students when using tools and technologies (or in other words *emerging from the use of the tool*), and the evolution of these understandings, was seen by the group as a key issue of research. The group expressed the need for empirical research focusing on solution processes and the underlying constructed meanings. An example of such an investigation is proposed in Ainley & al. paper in the construction of a formula by 8-9 year-old children interacting with a spreadsheet. The tool and/or technology in this type of task is viewed as facilitating by its feedback the pupil awareness of some inconsistency in their data. It has to be noted that tools may be used as catalysts for making students aware of the erroneous character of their strategies or answers. By using very different tools as semi-transparent mirrors or computer software, it might happen that what the students observe as a result of their actions differs from what they expected. From a cognitive psychological point of view, we might consider this a source of a cognitive perturbation or 'internal conflict' which may lead to a cognitive progress.

4. Tools and technologies in the curriculum

One of the key issues for teachers is how to design tasks based on tools or technologies in which real questions for the learner emerge from the use of the tool, in which the tool is relevant and gives a new dimension to the task. Some participants of the group stressed that there is a danger in asking the students to solve simple tasks with a complex tool: a fascination may be created by the discovery of the tool in itself and the pleasure of using it. This led the group to distinguish two types of use of tools and technologies: as functional for their own sake or as used for a didactical purpose. Thus we were led to focus on a further distinction: between tools created for a specific teaching purpose or more “universal” tools like spreadsheets or interactive dynamic geometry environments. The latter do not involve a teaching agenda while the former ones may be based on a pedagogical strategy. The paper by Rakov & Gorokh gives some examples of use of such general tools for teaching at university level and how it is possible to use several tools for solving the same problem from different perspectives. Similarly semi-transparent mirrors are not designed for specific teaching interventions — an example of an extensive curriculum in geometry designed around their use is presented in the paper by Zuccheri.

5. The papers

You will find below the papers on which the work of the group was based. Their authors had a short time after the conference to modify their contributions on the basis of the group’s discussion. We hope that they will generate questions and reactions from readers. We are open to any kind of exchanges, especially by e-mail. The list of the e-mail addresses of the participants of the group is given below. We encourage readers to copy this list as it stands, and create (and add to!) an informal mailing list which can continue the work of the group, and discuss future possibilities for collaboration and communication.

E-mail addresses of the participants of the group

Ainley, Janet (University of Warwick, UK)	janet.ainley@warwick.ac.uk
Belousova, Ludmila (Kharkov State Pedagogical Univ., Ukraine)	rcnit@kharkov.ua
Byelyavtseva, tetyana (Kharkov State Pedagogical Univ., Ukraine)	rcnit@kharkov.ua
Dreyfus, Tommy (Center for Technological Education, Israel)	nttommy@weizmann.weizmann.ac.il
Gallopin, Paola (Università di Trieste, Italy)	zuccheri@univ.trieste.it
Gélis, Jean-Michel (INRP; France)	gelis@inrp.fr
Giorgolo, Bruno (Università di Trieste, Italy)	zuccheri@univ.trieste.it
Gutiérrez, Angel (Universidad de Valencia, Spain)	angel.gutierrez@uv.es
Hedrén, Rolf (Hoegskolan Dalarna, Sweden)	roh@du.se
Jones, Keith (University of Southampton, UK)	dkj@southampton.ac.uk
Laborde, Colette (Université de Grenoble, France)	colette.laborde@imag.fr
Noss, Richard (University of London, UK)	rnoss@ioe.ac.uk
Price, Peter (Queensland University of Technology, Australia)	pprice@storm.fed.qut.edu.au
Rakov, Sergey (Kharkov State Pedagogical University, Ukraine)	rcnit@kharkov.ua
Stevenson, Ian (University of London, UK)	i.stevenson@ioe.ac.uk
Yerushalmy, Michal (University of Haifa, Israel)	michalyr@construct.haifa.ac.il
Zuccheri, Luciana (Università di Trieste, Italy)	zuccheri@univ.trieste.it

CONSTRUCTING MEANING FOR FORMAL NOTATION IN ACTIVE GRAPHING

Janet Ainley[1], Elena Nardi[2], Dave Pratt
[1]Mathematics Education Research Centre, University of Warwick, CV4 7AL Coventry U.K.
janet.ainley@warwick.ac.uk
[2]University of Oxford, 15, Norham Gardens, OX2 6PY Oxford, U.K.
elena.nardi@linacre.oxford.ac.uk

Abstract: *Active Graphing has been proposed as a spreadsheet-based pedagogic approach to support young children's construction of meaning for graphs, particularly as a tool for interpreting experiments. This paper discusses aspects of a detailed study, illustrating how Active Graphing emerges as a facilitator of the children's passage from a vague realisation of relationships to the articulation of rules and finally the construction of formulae.*
Keywords: -

1. Background

This paper will present aspects of research on primary school children's use and interpretation of graphs, within the Primary Laptop Project (see Pratt & Ainley, 1997, for a detailed description of this project). We have discussed elsewhere the development of a computer-based pedagogic approach, termed *Active Graphing*, which may help children to develop such interpretative skills (Pratt 1994, 1995). Briefly,

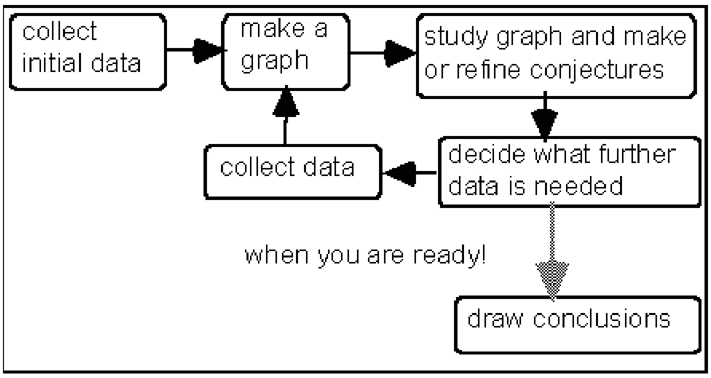


Fig. 1: A model of Active Graphing

children enter experimental data directly into a spreadsheet, and are able to graph this data immediately in order to look for trends and make decisions about what further data to collect. The physical experiment, the tabulated data and the graph are brought into close proximity. The significance of such proximity is strongly suggested by evidence from studies of data-logging projects (for example Brasell, 1987). The ability to produce graphs during the course of an experiment enables the graph to be used as an analytical tool for making decisions about future trials (Figure 1).

In the current study (funded by the Economic and Social Research Council) we are mapping out the ways in which 8 and 9 year-olds work through the Active Graphing process in activities in which they have to move between working with the experiment to collect data, tabulating the data on a spreadsheet, and producing and reading graphs. We are analysing the process of negotiation as the children draw on their pre-conceived expectations and their interpretations of the different modalities of *E*xperiment, *D*ata and *G*raph (the EDG triangle). We focus here on one aspect of this process which has emerged from the analysis: the ways in which children began to use, and to construct meaning for, the formal notation of the spreadsheet through interactions with the vertices of the EDG triangle (see Ainley, 1996, for preliminary work in this area).

2. Method

To explore the Active Graphing approach in more detail, we designed a sequence of four activities, re-using some from earlier phases of the research. The activities were designed to combine a range of features, one of which was the accessibility of the underlying mathematical structure. In this paper we use data from two activities which we shall refer to as *Display Area* (Act II) and *Sheep Pen* (Act IV).

In *Display Area*, the children are asked to make a rectangular frame from a 75 cm length of ribbon, into which they can fit as many miniature pictures as possible: i.e. they are asked to find the rectangle with maximum area for a perimeter of 75 cm. This activity arose as part of a project about Tudors: hence the interest in miniatures.

In *Sheep Pen*, the children are asked to design a rectangular sheep pen using 39 m of fencing, to be set against a wall, that would hold as many sheep as possible: i.e. they are asked to find the rectangle with maximum area when one length and two widths of this rectangle total 39 m. They modelled this using 39 cm straws.

Both activities produce similar graphs, in which the maximum value is found from a parabola. In each, the relationship between the length and the width of the rectangle is accessible to the children and amenable to algebraic modelling (e.g. $l = \frac{75 - 2w}{2}$ for *Display Area* , and l=39-2w for *Sheep Pen*).

The children recorded the results from each experiment on spreadsheets which (with help) they had set up to calculate the area of the rectangle, and made x-y scattergraphs of the width or length of the display area or sheep pen against its area. They were encouraged to make graphs frequently, and to discuss amongst themselves their ideas about the results so far, as well as to decide on what to do next in the experiment.

Each activity was used during one week with a class of 8 and 9 year olds, led by the class teacher, with a gap of around two weeks between activities. The children worked on the activity in small groups in sessions lasting up to two hours. For organisational reasons, the class was split into two halves, working on the activity on alternate days: thus each group worked for two sessions on each activity. The researcher (the second-named author) observed the work of four girl-boy pairs (two from each half of the class) using audio tape to record their conversations as they worked. She also recorded regular informal interviews reviewing their progress. There was a closing session at the end of each week's work, in which each group presented their work to the class: this session was also recorded. The data consist of the recorded sessions, the children's work and field notes.

Selected parts of the recordings were transcribed and these were combined with data from the field notes and examples of children's work to produce extended narrative accounts, describing the work of each pair on each activity. Significant learning incidents, varying in size and content, were then extracted and categorised. From this analysis emerged two broad themes: constructing meaning for trend, and constructing meaning for formal notation.

3. Analysis

We use the term *formalising* to encapsulate three categories of activity which we observed as contributing to the construction of meaning for formal notation:

- connecting a pattern based on the data with the experiment,
- connecting a pattern based on the data with a rule, and
- connecting a rule based on the data with a formula.

Our use of the word 'pattern' here is deliberately a little ambiguous, referring sometimes to an obvious numerical pattern, and sometimes to a fixed relationship between numbers. We will exemplify formalising by providing extracts from the data before discussing links with other themes in the research.

4. Connecting a pattern based on the data with the experiment

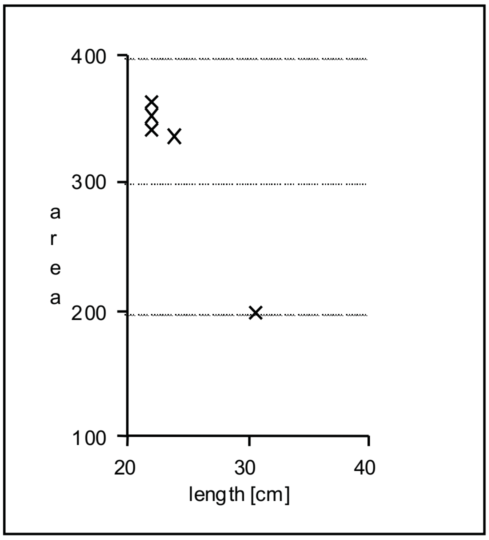
In both activities, the formula for the area was introduced in the beginning with the help of the teacher, so the notion of such a formula was familiar to the children. In *Display Area* the children began by working with a 75 cm length of ribbon, and pinning out rectangles on a display board. This naturally led to some inaccuracies as 'imperfect' rectangles were produced, although the children were not initially aware of this. Later the children went through a period of growing realisation that the length and the width are somehow interrelated. Consider this incident from work on *Display Area* . (Note: all boxed items are extracts from the data. The first person refers to the researcher. Numerical inaccuracies in these extracts are due to difficulties in making measurements, or to hasty calculations.)

> I ask Laura and Daniel whether their measurements have become more accurate. I observe how they are doing it and notice they are fixing the length. I ask them whether they can find out the width given the length. Daniel does the 23-length case: he doubles 23 and then the pair notice that their measurement of 16 cm for the width makes the perimeter 76 cm. They measure the 23 cm sides and the other comes out as 15.5 cm, so they record 15.5 cm as the width. (ACT II: session 1)

At this stage the children are deciding on (fixing) the length of the rectangle, but still measure the width. However they are not totally unaware of the interdependence of the two measures. When asked to predict the width if the length is given, Daniel starts by doubling the given length but then is distracted by the realisation that the previous measurement had been inaccurate. However, after several trials of the experiment, the children gradually saw the width-length relation more clearly and began to connect patterns based on the data with a rule.

5. Connecting patterns based on the data with a rule.

The following example from *Display Area*, begins with an attempt to correct or 'normalise' the appearance of the graph: we use the term *normalising* to describe this kind of behaviour, which emerged as a feature of the children's work on all four activities. The teacher (T in the transcript), who has experience in working on similar activities, recognises an opportunity to intervene to extend Chris' and Claire's thinking. The children's concern with normalising the graph leads to a realisation of how to calculate the width of a rectangle when given the length. Although Claire uses this calculation as a correcting procedure, applied to experimental data, she is still unaware of its potential as a data-generating tool.



width (cm)	length (cm)	area
14	24	336
6.5	30.5	198.25
16	22	352
15.5	22	341
16.5	22	363

Cl	ahhh...wicked...
T	What's happening there?
Cl	... it's doing a real pattern there
T	Why have you got three crosses all in a row - in a column rather?
[...]	
Cl	Maybe it's because we've got twenty two twenty two twenty two ...
T	Ahh!
[...]	
Ch	... it's the measurement...
Res	Do you think it's a problem?
T	Well, ... just have a look at that rectangle ... rectangle with a question mark!
Ch	Ah, I know ... it's ... it's not ... isn't it not right angles?
T	What do you think?
[...]	
Cl	I didn't understand why it was still twenty two twenty two twenty two so...
Ch	Sixteen, fifteen and a half and sixteen and a half
[...]	
Cl	Let's go back to these then ... shall we delete four, five and six?
Claire means rows 4, 5 and 6 on the spreadsheet.	
Ch	... can we measure this one again to see what that actual one is - it's sixteen and a half yes delete them two...
Res	How are you going to choose which one to delete?
[...]	
Res	Can you tell me how much this is going to be? If this is twenty two how much is the other? ... How would you find it?
Cl	Let's measure...
Res	Can you do it in your head, Chris? ... How would you find it out?
Cl	Twenty two add twenty two is ... that's forty four then you err then you have to try and make seventy five.
Res	OK, so how do you make seventy five from forty four ?
Ch	Forty four ... ohh I think I get it - what you do is ..
Cl	Twenty one
Res	Thirty-one
Ch	Oh yeah
[... *checking calculation*]	

Res	So then these two things would be thirty one what would each be?
Cl	...fifteen times two equals thirty, sixteen times two equals thirty two
Res	We have thirty one though, OK?
Cl	I don't know.
Res	It's very close what you are saying. If we have, this is twenty two and this twenty two and this is forty four so these two are thirty one both of them so how much each?
Ch	Divide it -
Cl	ahh so it's fifteen and a half
Res	Excellent so this is how to choose which one...
Cl	Now I get it! That one is right but these two aren't.

As this activity developed, the children demonstrated firmer knowledge of the width/length relationship, especially when asked to check the accuracy of their measurements. They added two widths and two lengths and if the result was 75 they accepted the measurement: if not, they didn't. However, the construction of an inverse process (given the length and the perimeter, find the width) is less straightforward. The following example is from session 2.

While trying to make the 19x19 square out of 75 cm, Laura realises that there doesn't seem to be enough string for the four angles of their quadrilateral to be right angles.

L	So we need to make it smaller! Instead of 19, make it 18!
D	Then the width has to be 20. (ACT II: session 2)

As the children's confidence in articulating their rules developed, the teacher again used this as a signal that she could intervene to move them towards the idea of using a formula to generate data.

WIDTH	LENGTH	AREA
15	23	345
7	30.5	213.5
14	23.5	329
16	21	336
6.5	31	201.5
11.5	26	299

Eli and Tarquin are busy using a rule to correct their measurements. The teacher joins in and asks what they have been doing. Eli explains the 26 case and also the previous 31 case. They explain how they get from width to two widths minus from the perimeter and then divided by two. After a discussion in which Tarquin confuses area with perimeter in his calculations, the teacher says she wonders whether they can find any other way of doing the calculation automatically. Tarquin mentions 'formula' but cannot say more. The teacher lets them go on. (ACT II: session 1)

Using their experience from *Display Area*, most of the children moved swiftly towards looking for a rule in *Sheep Pen*. The following example is from an early stage of Chris' work on the new activity.

I ask them to prepare a sheep pen for demonstration and save their file. Chris shows me why the width-29.5 sheep pen is impossible.

Ch You can't because you have to double the 29.5. (ACT IV: session 1)

Laura and Daniel both claim at this stage: "we don't have to make them" because "we've got a pattern". The following incident confirms the clarity with which Laura and Daniel 'see' the relationship between the length and the width of the sheep pen. When the teacher set up the activity initially, a length of 41m of fencing was chosen for the sheep pen, but this was later changed to 39m.

The children instantly suggest "taking 2 out of 41" because "If it's 41, if it's going to be 39 we are going to go 41, 40, 39 we are going to take two off everything so that makes that ...". (ACT IV: session 1)

Connecting their initial sense of a pattern with a rule they can articulate becomes gradually more overt.

It is the beginning of Activity IV for Laura and Daniel and they have chosen to start from width of 4 cm. Laura then suggests their choice of widths follows a pattern: ‘increase it by a half cm. So the next is 4.5 cm’.

Res	So, if you have 4.5 what’s gonna happen then?
L	It’s gonna be thirty the...in the 3s...
Res	So this is 4.5 and this is going to be?
D	That’s going to be 32!
Res	Clever, how did you do that?
D	The first one was 4 and then it’s 4.5 then both sides you take off a whole.

(ACT IV: session 1)

6. Connecting a rule based on the data with a formula

The process of constructing the width-length formula and putting it into use was an extended one for most pairs of children, and it is not possible to include extracts of such discussion here (detail of a similar incident can be found in Ainley (1996)). The teacher typically intervened with the suggestion that the children might ‘teach the spreadsheet’ their rule, once she felt they could articulate their rule clearly. At this point the teacher and/or the researcher were prepared to offer some help to the children in translating their rule into formal notation.

The example below shows the confidence with which Laura and Daniel were able to work systematically which such a formula to generate data in *Sheep Pen*. At this stage their spreadsheet contains three formulae. In the ‘Area’ column they have their original formula to multiply length by width. In the ‘Width’ column they have a formula to add .5 to the previous cell, so that they can increment the width by half a centimetre each time. They realised that there is a pattern in how the length changes when the width changes, and originally they used an iterative rule “increasing the width by a half cm decreases the length by 1 cm”, and entered the appropriate formula in the ‘Length’ column. After some discussion with other groups they decided to change their approach.

Later in the day, and once the children have introduced the spreadsheet formula, =39-2*F2, in order to connect the lengths in column E to the widths in column F, they decide to refine their measurements and generate data for an increment of a quarter for the width. Laura says that all they have to do is replace the 'sign for a half' with 'the sign for a quarter'. The discussion leads to deciding that .25 is the sign for a quarter and Laura says:

L So we have to put plus .25.

Res I think you have got the idea very well so can we see it: please, shall we start from 4?

D Can we do the wholes?

L We've done the wholes!

Res Because you start with halves: if you put half and half again you have a whole.

They insert the .25 formula. Daniel looks as if he is really clear about it but Laura does not. I ask her and she says:

L When the pattern comes up it will come to me.

D It's a different language, Laura, it's a different language!

They fill down the width column.

Res Can you tell me now, are you filling down? Good.

L 4.25, 4.5, 4.75 and then it does it again, it's repeating itself!

Res What do you mean repeating itself?

L 4.25, 4.5, 4.75, 5, 5.25, 5.5, 5.75 so it's like in a pattern.

(ACT IV: session 1)

Length	Width	Area
31	4	124
30.5	4.25	129.625
30	4.5	135
29.5	4.75	140.125
29	5	145
28.5	5.25	149.625
28	5.5	154
27.5	5.75	158.125
▪	▪	▪
▪	▪	▪
▪	▪	▪
1.5	18.75	28.125
1	19	19
0.5	19.25	9.625
0	19.5	0

7. Constructing meaning for trend and for formal notation

Despite the initial perception of constructing meanings for trend and for formal notation as two distinct analytical themes, it has gradually become possible to view them as interrelated. In the three observed categories of formalising activity mentioned above, the children began with a vague but increasingly persistent notion of the dependence of length on width. As their work progressed they proceeded to specify this dependence: twice the length plus twice the width should always make 75 cm, or estimating that for a width of 20 cm, length should be around 18 cm (*Display Area*). However these observations did not yet lead to the realisation that the length could actually be calculated from a given width. After numerous repetitions of the experiment and the evolution of their attempts to articulate a rule to express relationships (patterns) in the data, the idea of the formula arose as a natural step in which children understood both the structure of the formula, and its *utility* (see Pratt and Ainley 1997).

The children often reached this realisation in the context of checking whether the experimental data were accurate. As their sense of meaning for trend became more firmly established, they became more aware of points on the graph which seemed to be in the wrong place, that is, they did not fit with the trend. In a number of cases, the children recognised 'impossible' situations in a graph, or a set of data. An example of this was given earlier in this paper, when Claire and Chris realised that the three crosses in a vertical line on their graph could not be correct, as they represented three display areas with the same length but different widths. Perceived abnormalities in the graph such as these prompted children to check their data. In their efforts to normalise the graph or the data, they consolidated the notion of dependence between length and width. This consolidation accelerated their move towards formal notation. As the teacher became increasingly aware of normalising, she was able to recruit it as an intervention strategy, drawing children's attention towards discrepancies in the hope of moving them towards formalising.

Elsewhere we have described the data relating to constructing meaning for trend in terms of interactions with the EDG triangle (Ainley, Nardi & Pratt, 1998). In order to describe constructing meaning for formal notation, it is necessary to add the *F*ormula as a fourth modality. The richest episodes in our data exemplifying the construction of meaning for trend are characterised by intense interaction between the graph and the data and between the graph and the experiment. Similarly, the richest episodes in our data relating to formalising are characterised by the intensity of the data-formula interaction. In both cases the most powerful learning incidents seem to occur where there is more interaction among the different modalities.

We conclude that meanings do not emerge exclusively within any one of the four modalities, but are gradually established through constant interaction between them (see Nemirovsky & Rubin (1991)). In such interactions we commonly see children using patterns of language which draw on one modality, whilst working with another (see Ainley, Nardi & Pratt 1(998)). Nemirovsky (1998) terms such “talking and acting without distinguishing between symbols and referents” as *fusion*, which he sees as an expression of fluency in symbol use. In the activities we have described here, formal notation emerges as another, and particularly powerful way, of making sense of the phenomena. Children not only have opportunities to become familiar with the symbols of formal notation, but also experience their utility in describing the patterns they have observed within the experimental situation, and within the data, and in generating data to produce accurate graphs. As Active Graphing clearly encourages the interaction between different modalities, it emerges as an approach with high potential to encourage such fluency.

8. References

Ainley, J., Nardi, E. & Pratt, D. (1998): Graphing as a Computer Mediated Tool. In A. Olivier & K Newstead (eds.), *Proceedings of the 22nd Annual Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1 (243-258), Stellenbosch, South Africa.

Ainley, J. (1996): Purposeful Contexts for Formal Notation in a Spreadsheet Environment. *Journal of Mathematical Behavior*, 15(4)

Brasell, H. (1987): The Effect of Real-Time Laboratory Graphing on Learning Graphic Representations of Distance and Velocity. *Journal of Research in Science Teaching* 24

Nemirovsky, R. & Rubin, A. (1991): “It Makes Sense If You Think About How The Graphs Work. But In Reality ...”. In F. Furunghetti (ed.) *Proceedings of the 15th Annual Meeting, North American Chapter of the International Group for the Psychology of Mathematics Education*.

Nemirovsky, R. (1998): Symbol-Use, Fusion and Logical Necessity: on the Significance of Children’s Graphing. In A. Olivier & K Newstead (Eds.), *Proceedings of the 22nd Annual Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1 (259-263), Stellenbosch, South Africa

Pratt, D. (1994): Active Graphing in a Computer-Rich Environment. In J. P. da Ponte & J. F. Matos (eds.), *Proceedings of the 18th Annual Conference of the International Group for the Psychology of Mathematics Education*, University of Lisbon, Portugal

Pratt, D. (1995): Young Children’s Active And Passive Graphing. *Journal of Computer Assisted Learning*, 11

Pratt, D. & Ainley, J. (1997): The Design of Activities for the Abstraction of Geometric Construction. *International Journal of Computers for Mathematical Learning*, 1 (3)

TRAINING EXPLORATIONS ON NUMERICAL METHODS COURSE USING TECHNOLOGY

Ludmila Belousova, Tatyana Byelyavtseva
Department of Mathematics and Physics of Kharkov State Pedagogical University named after G.S. Skovoroda, Bluchera str., 2, Kharkov 310168, Ukraine
rcnit@kharkov.edu.ua

Abstract: *This article is devoted to describing general problems which arise while establishing study researches on Numerical Methods Course.*
Keywords: *-*

1. Introduction

Numerical methods courses play an important role in preparing future mathematics specialists, because students taking the courses have to master methods for solving practical mathematical problems which do not work with the strict methods of academic mathematics.

The numerical methods course can be considered as a "bridge" between the mathematical theories and objective reality. On the one hand, it is easy to discover that many numerical methods are the direct consequence of mathematical theorems projected onto practical problems. On the other hand, there are other methods which are so simple and obvious that one can work them out, not from the theoretical premises, but just relying upon the common sense or the geometrical interpretation of a problem. Nevertheless there are also some methods, which boggle imagination with their originality and distinctness of ideas, and which provide a non-standard approach towards solving problems.

For these reasons it is a pretty difficult task to construct a numerical methods course. In this paper we would like to outline some of the considerations which went into the design of a computer-based course about numerical methods, and describe some of the materials that have been produced to help students learn about this important mathematical topic.

2. Design issues

As we have suggested in the introduction, the theoretical part of a numerical methods course is fairly difficult for students to understand. The definition of numerical methods, on the one hand, requires a wide knowledge of academic mathematics in most of its aspects. On the other hand, the whole mathematical background of numerical methods rest upon assumptions which do not always seem very convincing. A student, moreover, has just to accept many of these assumptions, because their justification lies beyond the scope of their educational programme, and very often they are not even mentioned in school textbooks.

Everything that has been mentioned above is accentuated by the fact that, besides the theoretically established usage rules for one or another method, there are also practical rules which have no strict justification, other than they are easy and effective to use in practice. These rules define the area of applicability for given numerical methods, normally going beyond that which is described in theory, and provide criteria for the selecting the technical means to implement a chosen algorithm. They define simple procedures and equations used for controlling calculations and evaluating the problem solution to the required degree of accuracy. They are also important because these practical rules enable a given numerical method procedure to implemented using new technologies. However, the fact that the practical rules are not proved leaves some doubt as to their validity, and this can only be removed through the experience of using multiple numerical method in the same conditions that led to their creation.

It should also be observed that the numerical methods world is varied and idiosyncratic: each method has its own particularity and its own area for effective usage. The main task for a person selecting an appropriate method for solving a given problem efficiently, is to develop an ability for selecting shrewd combinations of different methods at various solution stages. Not only is theoretical knowledge in the field of numerical methods required, but also intuition, based on personal experience of using the methods in a practical context.

A numerical methods course must rest upon good laboratory routine due to its fundamentally practical nature, and its success depends, to a large extent, on the quality of the laboratory work. In standard presentations, however, the numerical methods laboratory element is reduced to finding and executing the calculations appropriate for solving the problem, in accordance with a chosen algorithm. The problem is that a student's activity can focus solely on reproducing the correct algorithm and doing the "spadework" on figures, but, in doing so, lose the problem's essence. From this point of view, using computers makes it possible to facilitate and automate this numerical work, so that the student can focus on the meaning of a problem and the methods used, rather than just their execution.

3. A course for numerical methods

We believe deeply that the significant changes needed to establish the numerical methods course and, as a result, the corresponding changes in students' mathematical training, can be reached only by transforming the laboratory routine into a cycle of what we call 'study researches'. They were devised with two things in mind. First, study researches should not be embedded into the routine as separate episodes, but they must constitute the essence of each laboratory work. Second, they should be computer-based, using the type of software that normally supports professional mathematical activity.

The first factor implied a restructuring of the whole course, giving lectures the specific role of providing a thematic overview of the various numerical methods used, becaming a necessary condition for the course's step-by-step development. Students' researches became more complex and deeper, with an emphasis on practical activity using the numerical methods described in lectures, and aimed at developing students' research skills and abilities through activity. One should note that the episodic usage of study researches in laboratory routine rules out as not reasonable, the principle of: "from time to time". Practice showed that without the study researches forming an integral part of the laboratory time, students do not comprehend the gist of given problems, and they do not gain sufficient levels of research skills. A result of implementing the student researches so that they did *not* form an integral part of the course was that instruction reverted to its customary style, giving the students' activities a certain element of randomness and scrapiness.

As for the second factor, the orientation of educational process in higher education establishments towards using the modern professional equipment, but not towards systems designed exclusively for learning, seems to be more expedient. Such orientation, on the one hand, enables students to develop the basic skills of using computer for professional purposes, and, on the other hand, demands a fairly high level of study research.

The kind of support packages which are used worldwide nowadays by professional mathematicians, are not designed for learning purposes. They provide tools for solving a wide range of standard mathematical problems, leaving the used solution methods hidden from the user. At the same time such packages have pretty powerful and comfortable built-in facilities, which allow expanding of package functions, including those which allow it to be adapted for the purpose of learning.

We have used the MathCAD package for establishing the study research works on our numerical methods course. We chose the MathCAD package, because it is widely used for solving practical mathematical problems, and it has a number of features which allows it to be used in teaching. In our case, MathCAD made it possible for us to create a dynamic screen page for which we have worked out a set of dynamic base outlines (DBO), that supported the implementation of all course themes. Coupled with a free cursor that scrolled all over the monitor and a fairly powerful built-in language, we were actually able to present students with a virtual lab for carrying out the calculation experiments.

Creating the DBO in a MathCAD environment consisted in developing a program which realized the appropriate numeric method algorithm, and building an interface that was convenient for entering data for a problem and presenting the results of the chosen algorithm on the screen. Mathematical features in the package were also used to check the quality of the results from each use of an algorithm. Two examples of DBO for piecewise interpolation are shown on Fig.1.

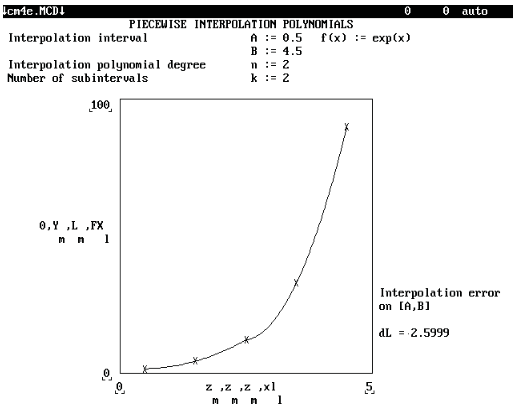


Fig. 1a:
DBO showing a Piecewise Interpolation

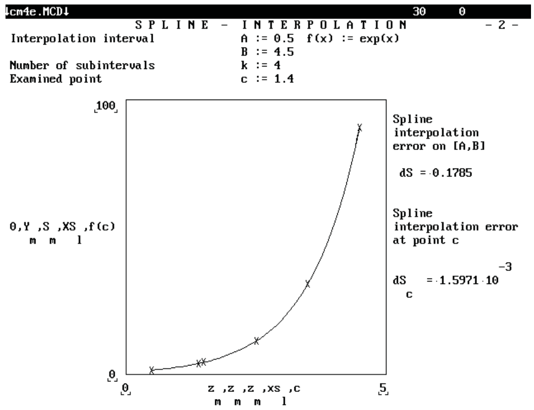


Fig. 1b:
DBO for a Spline Interpolation

Figure 1a. shows the result of interpolations using a piecewise interpolation, and Fig. 1b. show the same process but using a spline interpolation. Students were able to see the errors produced by each approximation, and, by varying the parameters of the interpolation algorithms, observe the conditions needed for a more accurate interpolation.

Each DBO was orientated for working with one of the numerical methods and gave an opportunity to make multiple tests of a method in various problems, displaying the results in numerical and graphic form. While carrying out this study research, a student had to fulfil a series of such tests. On the basis of the data, which was obtained due to observation of the displayed calculation process characteristics, and after comparing and analyzing the characteristics in question, a student was able to draw some conclusions about the quality of the algorithm being studied.

It should be noted that the problems that were to be solved by a student during the study research differed greatly from those ones which constitute the essence of traditional laboratory work. For example, while examining the numerical methods for solving equations, students have to determine which criteria should be selected to assess the proximity of an approximation to the desired value of an equation root. This involved making a decision about the degree of precision needed for either an approximation to a solution, or how a given approximate value iterates in relation to a previous one. The following problems were set for each of the methods examined to explore the difference between these:

- evaluate experimentally the order and speed of the method’s convergence
- find the main factors, which affect these characteristics
- determine the applicability of this method being investigated.

Students were offered the opportunity to study each of these situations for several equations roots in the examined segment, including the availability of complex roots.

At first sight, examining the interpolation formulae seemed clear; greater accuracy was obtained by using more interpolation nodes, and a larger polynomial degree. However, students had to make sure for themselves that the processes of interpolation do not always happens in this way. They soon discovered that to reach the required precision meant sometimes changing their tactics; for example, instead of building the nodes up, one might use fragmentation of the interpolation interval. Students were aiming to build up the best possible function approximation on an interval for a limited function value quantity. They had to ask themselves several questions such as: how do we go about finding the necessary information; which interpolation method gave the most reliable result? On the basis of these experiment, students set up a rule for selecting those values which should be taken into consideration when minimizing error, as they examined questions on the precision of function compensation using the interpolation formulae at intermediate point using a table.

4. Plan reports

To make student more conscious of their activities, and to aid them in reaching the predicted learning objectives, we have worked out a methodological support using what we call ‘plan-reports’ for each laboratory session. Plan-reports have the same structure consisting of two parts – the informative and the instructional. The informative part contains the themes and goals of the topic being studied, and includes any computer-based activity that may be required, specifying the characteristics of any numerical input and output or graphic data needed.

The instructional part contains completed runs with its key moments marked and stated. To orientate students towards the research implementation, they were given a list of special questions. Some of the questions were dedicated to developing students' intuitions on how the studied method worked in a particular situation, while other questions were dedicated to giving students a hint that such ideas might not be correct. Thinking over the given questions, students had a chance to get acquainted with a problem, understand it, and build up a working research hypothesis.

The remainder of student's work was devoted to checking and refining any hypothesis they had made to make it more specific. This work was implemented according to a suggested plan which defined separate research stages with specific problems to be solved at each stage, and experimental material that must be obtained. As the routine went on, instructions for students became less detailed, taking the form of suggestions rather than instructions. Some experiments were set as individual investigations, so that students had to think over the problem and implement solutions by themselves, without formal instructions. In laboratory work, the individual variants of problems packages were worked out, with each method being tested on these problem packages to gain experimental material corresponding to the learning objectives of the topic. If students wished, they could supplement these packages with the problems of their own choice.

The final stage of the research was to draw the necessary conclusions in the form of findings which were marked out in the plan-report with varying degrees of explicitness. Hints were given to students to re-inforce the results of the research, work out its structure, and draw their attention to the significant aspects of the research that formed the learning objectives of the activity.

Implementing the planned research gave students a fairly deep understanding of the features and specifity of the method being used, and this was reflected in a 'free topic composition'. This consisted of creating a practical problem, and solving it using the particular method being studied.

We should note that plan-reports could be produced for the students in both printed or electronic form. The latter was used along with DBO during the laboratory work runs, and was convenient for moving the experimental data from DBO to the prepared tables when preparing the record material.

5. Conclusion

The experience of introducing the described routine into the educational process at physico-mathematical school of Kharkov State Pedagogical University allows us to draw the following conclusions. The numerical methods course was found to be very important, with the number of methods examined being expanded to a great extent, and new types of problems being brought into the educational process. The study researches turned out to be an effective educational tool for developing different aspects of students' learning, given the proper programming and methodological software, as well as a bit of persistence on the teacher's part.

6. References

Belousova L., Byelyavtseva T., Kolgatin A. & Ponomaryova L., (1998): *Laboratory Works on Numerical Methods Course using MathCad Package.* Textbook, Kiev.

Rakov S., Nikolayevskaya M. & Oleinik T. (1993): *Learning Explorations on Mathematical Analysis using MathCad Package.* Textbook, Kharkov, "Osnova".

Kalitkin N. (1978): *Numerical Methods*. Textbook, Moscow, "Science"

Forsite J., Malcolm & M., Mouler C. (1980): Mathematical Calculations Machine Methods, Moscow, "Mir"

CABRI BASED LINEAR ALGEBRA: TRANSFORMATIONS

Tommy Dreyfus[1], Joel Hillel[2], Anna Sierpinska[2]
[1]Exact Sciences Department, Center for Technological Education, P. O. Box 305, Golomb street 52, 58102 Holon, Israel
nttommy@weizmann.weizmann.ac.il
[2]Department of Mathematics and Statistics, Concordia University, Montreal, Canada
jhillel@vax2.concordia.ca
sierp@vax2.concordia.ca

Abstract: *Many of the difficulties of beginning linear algebra students are due to confusions resulting from the indiscriminate use of different levels of description. Analysis of these difficulties suggests an entry into linear algebra starting from a coordinate-free geometric view of vectors, linear transformations, eigenvalues and eigenvectors, and the use of dynamic geometry software. A principled, epistemologically based design of such an entry to linear algebra is described, together with some experimental results on meanings students developed for the notion of transformation.*
Keywords: -

1. A geometric approach to beginning linear algebra

Only relatively recently have mathematics educators turned their attention to the teaching and learning of linear algebra (see e.g., Harel 1985, Robert & Robinet 1989, Rogalski 1990, Alves Dias & Artigue 1995; for a state of the art review, see Dorier 1997_2). Many of the students' difficulties noted by these researchers stem from the fact that elementary linear algebra uses three types of languages and levels of description (Hillel 1997), corresponding to three modes of thought (Sierpinska, Defence, Khatcherian & Saldanha 1997). They are:

(i) The geometric language of 2- and 3- space (directed line segments, points, lines, planes, and geometric transformations) corresponding to a synthetic-geometric mode of thought.

(ii) The arithmetic language of R^n (n-tuples, matrices, rank, solutions of systems of equations, etc.) corresponding to an analytic-arithmetic mode of thought.

(iii) The algebraic language of the general theory (vector spaces, subspaces, dimension, operators, kernels, etc.) corresponding to an analytic-structural mode of thought.

Furthermore, the geometric language is carried in a metaphoric way to the general theory (e.g. projections, orthogonality, hyperplanes, or the denotation of vectors using arrows). These three languages and modes of thought coexist, are sometimes interchangeable but are certainly not equivalent. Knowing when a particular language is used metaphorically, how the different modes of thought are related, and when one is more appropriate than the others is a major source of difficulty for students.

A very common approach in elementary linear algebra is to start with the arithmetic approach (in R^2 or R^3) with vectors as tuples and transformations as matrices, and then to make the link to geometry via analytic geometry, so that a vector (x,y) is represented as an arrow from the origin to the point P(x,y). Linear transformations are often introduced by a formal definition as transformations of vector spaces which preserve linear combinations of vectors. Very quickly, however, the example of multiplication by a matrix is given much prominence. Some multiplication-by-a-matrix transformations in R^2 such as reflection ($\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$), projections, rotations and shears are normally interpreted geometrically to help the students make the link between the new concept and their assumed high school knowledge.

This approach has several shortcomings and is a source of confusion for students. For example, it may prevent one from even talking or thinking about non-linear transformations. Also, tying a vector to a preferred system of coordinates, which is unavoidable in the arithmetic approach, leads to serious difficulties in distinguishing between the vector and its different representations relative to non-standard coordinate systems (Hillel & Sierpinska 1994). Furthermore, for many students the transition from the arithmetic to the structural view of linear algebra is a hurdle they never manage to take.

Why have we chosen a geometric entry into linear algebra? Some reasons for this can be found in the history of the domain. We refer to Dorier's (1997_1) analysis which stresses the importance of geometric sources of many algebraic concepts. For example, Grassmann openly admitted to a geometric inspiration when he introduced his notion of vector as a "displacement". In fact, the synthetic-geometric mode of thinking in linear algebra focuses on those properties that are independent of the choice of basis and thus brings one closer to the concept of the general vector space than the analytic-arithmetic mode does (Sierpinska 1996).

Contrary to the analytic-arithmetic approach to the teaching of linear algebra described above, a geometric approach easily allows for the consideration of examples of non-linear transformations. Since concept images are formed on the basis of examples and non-examples (Vinner 1983), this is essential for forming a solid concept image of linearity. A geometric approach also allows one to defer the introduction of coordinates until after vectors and transformations become familiar objects; moreover, coordinates can then immediately be introduced for a general basis rather than for the particular standard basis.

Such considerations suggest that a concrete, geometric but coordinate-free entry into linear algebra might help students to develop their analytic thinking about elementary linear algebra concepts. Geometry as a source for the development of intuitions related to linear algebra concepts has been strongly advocated both by mathematics educators and textbook authors (e. g., Banchoff and Wermer 1992). Presently, the latter idea is often realized with the support of computer graphics. The notions of linear transformation and eigenvector/eigenvalue have received special attention in this respect (e.g., Martin 1997). The computer constructions and visualizations of linear transformations and eigenvectors are sometimes quite ingenious, leading to interesting mathematical problems. They are certainly a joy for the mathematician, but it is not clear if and how they can be useful in the teaching and learning of linear algebra at the undergraduate level. While the mathematics underlying the computer visualizations and descriptions of some technicalities of the design of these visualizations have been studied, no accounts of teaching with their use and of students' reactions to such teaching are available. There seems to be a need, in mathematics education, for research in this direction.

Our research responds to this need: We investigated the option of using a geometric entry into linear algebra and, more specifically, to the notions of vector, transformation, linear transformation and eigenvector in two dimensions. We have designed and experimented in a controlled fashion a sequence of teaching/learning situations focused on these notions. The approach is conceptual rather than technical. The notion of transformation naturally leads to the question of the relation between a vector and its image, for example, whether the images of two (or more) vectors lying on a line, also lie on a line. In order to facilitate this approach and to afford students an exploratory environment, we chose to use a dynamic geometry software (Cabri II; Laborde & Bellemain 1994). Thus, a set of activities with Cabri is a central feature of our design allowing a geometric and exploratory introduction of notions such as linearity and eigenvectors.

Our study has a theoretical and an experimental component; the theoretical component is concerned with epistemological / content analysis as well as with the design of a sequence of learning activities. A first and brief version of six main stages of the design has been presented in Dreyfus, Hillel & Sierpinska (1997). Elements of an epistemological analysis of the design and its experimentation can also be found in Sierpinska (1997, in press).

The experimental component consists of a trial run with one pair of students, R and C, who took part in a sequence of six two-hour sessions with a tutor. The students were undergraduates majoring in a social or natural science domain who had taken a 'baby' linear algebra class during the preceding term, and had been classified as average to good students by their instructor. One aspect of the experimental component, namely the students' conceptions of the linearity of a transformation, has been reported in Hillel, Sierpinska & Dreyfus (1998). In this paper, we concentrate on another aspect, namely the students' conceptions of transformation, and in particular on how their conceptions may be linked to the design we have chosen and to the role Cabri activities play in this design.

2. The Cabri model for the 2-d vector space: The notion of vector

Several design decisions had to be taken with respect to the notion of vector. For example, in view of the historical roots of the concept it might have been natural to define a vector as a displacement; this would also have supported a structural notion of vector. However, we thought that the price to pay, in terms of conceptual complexity, would have been too high. Indeed, if R^2 is considered as a space of translations, then transformations of R^2 lose their geometric meaning as transformations of the plane. They become transformations of a space of transformations (translations) of the Euclidean plane, a notion which could be very difficult for the students to handle.

Vectors in our design are thus modeled by points in the Euclidean plane with a distinguished point called 'the origin' and labeled 'O'. The vectors are represented by arrows, emanating from O. We chose to represent vectors by arrows because we thought that the visual representations of transformations would be clearer on larger objects than on dots, and especially, that the use of arrows would make it easier to identify invariant lines.

We also decided to skip the process of abstraction of free vectors from the relation of equipollence, and work directly with one standard representative for each equipollence class. While this limits the geometrical figures one can work with, we believed that it would render the notion of equality of vectors trivial: Two vectors are equal if they are identical.

Operations on vectors are defined in our model by reference to geometric concepts: The sum of two vectors u and v is the vector w such that the figure Ouwv is a parallelogram. The vector kv is defined as a vector w such that w lies in the line Ov and is k times as far from O as v, with a convention linking the sign of k and the mutual orientation of v and w.

It is a crucial aspect of our model that an arrow representing a vector can be moved by dragging its endpoint, thus representing a variable vector. A detailed analysis of the notion of vector in our design is presented in Sierpinska, Hillel & Dreyfus (1998).

3. Transformations

In the second session, and before being introduced to the notion of linear transformation, the students were to be familiarized with the language and representation of transformations in general (not necessarily linear), in the Cabri model of the two-dimensional vector space. For this purpose, the tutor chose a (blue) vector v and a transformation (e. g., a rotation by 60 degrees around O, or a projection on a given line), and applied the transformation to the vector so as to obtain the image, i.e. another vector which is colored red and given a label of the type ‘T(v)’. The tutor dragged the endpoint of the arrow representing the vector v. He said to the students: ‘Look what happens to v and what happens to T(v)’.

The action of dragging and drawing the students’ attention to the relation between v and T(v) was intended to convey two ideas: The transformation is defined for any vector in the plane, and there is some constant relation between v and T(v).

In the third and fourth sessions, the students carried out a considerable number of activities concerning transformations; for example, after having been introduced to the notion of linear transformation, and presented with examples of linear as well as non-linear transformations, they were asked to check several transformations as to whether they were linear or not. They did this by choosing, say, a vector v and a number k and checking whether the vector T(kv) coincided with the vector kT(v) for the chosen v and k; even if they coincided, however, the students did not attempt to vary v by dragging them around the screen and check whether the relationship remains valid. A detailed analysis of the notion of transformation in our design is presented in Sierpinska, Dreyfus & Hillel (1998).

4. The students’ conception of transformations

The students seemed to understand the term ‘transformation’ as if it referred to a vector, namely, the vector labeled ‘T(v)’, where v is the variable vector of our Cabri model of the two-dimensional space. ‘Transformation’ did not, for them, seem to refer to a relation or dependence between v and T(v) but to an object T(v) whose position depends on the position of another object v.

The students were reading 'T(v)' as 'the transformation of v' and not as 'the image of v under the transformation T'. The tutor did not attempt, at first, to correct the students' language and even adopted the expression in his own discourse. While the expression 'the transformation of v' could function merely as a metonymy in the language, it can also be a symptom of a focus on images of particular vectors.

This conception came clearly to light in the fifth session when the students were asked to construct a projection in Cabri. On the Cabri screen, there were six vectors labeled v_1, v_2, w_1, w_2, v and T(v). The vector T(v) was constructed under the assumption that T is a linear transformation such that $w_k = T(v_k)$ for k=1,2. In other words, a macro was used which finds numbers a and b such that $v= av_1+bv_2$, and returns the vector aw_1+bw_2. The students did not have access to the macro, but could move the vectors v_1, v_2, w_1, w_2 freely and thus obtain different (linear) transformations. At this stage in the sessions they were supposed to have understood how the vector T(v) is determined by the mutual positions of the vectors v_1, v_2, w_1, w_2. They were expected to construct a projection onto a given line L by re-defining the vectors w_k as projections of the vectors v_k onto L, or at least move the vectors w_k to the positions of the projections of the v_k onto L.

The students, however, did not do this. They 'manually' searched for some position of the vectors v_k and w_k so that for the specific given vector v, T(v) coincided with and thus looked like the projection of v onto L. If v were dragged to another position, the relationship would have been destroyed.

R even wanted to check that the line between v and T(v) was perpendicular to L. He asked C (who was holding the mouse) to draw a perpendicular to L through v and then move v_1, v_2, w_1, w_2 so that T(v) would lie on L. C managed to 'position' (as she put it) the vectors v_k and w_k so that this happened, at least in appearance. R was satisfied, but C was still concerned about what would happen 'as soon as we move all these'. R dismissed her worries, saying that *that's what they asked for*.

This was his interpretation of the expectations of the designers: T(v) has to be the projection of v; for each position of v a special configuration of the vectors v_k and w_k can be found so that T(v) is the projection of v. He was clearly conscious of this situation:

R: Well, you see, if you move the vectors, T(v) won't be the projection anymore… 'Cause that's all they asked for. We don't have to do anything else. We have to find a way that T(v) is a projection. And there are many ways we can do that … It's very specific. It's only for this v that it works because of where v_1 and v_2 are.

We stress in particular R's use of T(v) in *We have to find a way that T(v) is a projection*. To him, the vector T(v) has to be a projection of v - nothing else.

5. Possible sources of the students' conceptions of transformation

When functions on real numbers are considered, it is common practice not to stress the distinction between f and f(x): we usually say 'the sine of x' not 'the image of x under the sine function'. This background seems to have influenced the students' conception of transformation more than the Cabri-environment which makes varying vectors natural and, in fact, difficult to avoid; and the tutor's use of language seems to have influenced them more than his regular and repeated demonstrations how to check properties "for all" vectors in the Cabri environment.

The intentions of the designers and the tutor when dragging the endpoint of an arrow representing a variable vector v around the screen was to convey the notion that v represents "any vector" and that the transformation T is defined for all vectors v. However, the variable vector v is often referred to as one single object: "the vector v". This could give students the idea a transformation refers to one single vector: If v represents one single vector and T(v) depends on it then T(v) also represents one single vector. Hence 'T(v)', read as 'the transformation of v' denotes a well defined object. The invariance of the relation inherent in the notion of transformation, as intended in the design, was replaced in the minds of the students, by the invariance of an object.

The deepest cause of the students' 'T=T(v)' conception of transformation was probably the fact that the design neglected the question of equality of transformations. The question is: When a vector v has been transformed into a vector v', and a vector w has been transformed into a vector w', how do we know whether they have been transformed by the same transformation? Can we decide it for any pair of vectors in a given vector space? The students were not given an opportunity to reflect on this question. But even if they were, it is not clear whether having v, T(v), u, S(u) would lead students to distinguish between a transformation and an image vector. For example, if u is dragged to overlap with v and the students notice that T(v) and S(u) coincide, they may still remain stuck with the 'T=T(v)' view of transformation. Cabri wouldn't let them move u and v simultaneously. Of course, the identity of two transformations S and T can be established by showing that they coincide on two linearly independent vectors u and v: If S(u)=T(u) and S(v)=T(v) then S=T. But in order to realize this, rather sophisticated knowledge is necessary; such knowledge is not easily available to students at the stage when the question of equality of transformations should be broached.

6. Conclusion

The students' inadequate conceptions appear to be due to several interacting and compounding factors. One of these is directly linked to the generic nature of objects in dynamic geometry. According to our design, a Cabri vector is generic, it represents 'any' vector. This considerably complicates the question when two vectors are equal as well as when two transformations are equal. If a vector is dragged to another place, is it still the same vector? If a linear transformation is defined by its images on two non-collinear vectors v_1 and v_2, is the transformation preserved when v_1 or v_2 are moved?

The objects - vectors and transformations - which our design made available to the students are different from those available in a paper and pencil environment. The students were provided with a representation of these concepts decided on by us, the designers, and implemented in Cabri. Students thus worked with and manipulated objects in Cabri which they could never work with in a paper and pencil environment.

During the design process, some crucial decisions had to be taken about the representation of the concepts. One of these was to decide whether the arrow representing a vector should have fixed length, fixed direction, and a fixed initial point.

As described above, we decided on a fixed initial point but variable length and direction. We thus created an object which might be called a ‘variable vector’, named v, which can take on ‘any’ length and ‘any’ direction. This is no different, in principle, from creating a variable number (scalar), named k, represented by a point on a number line which can take on ‘any’ real value. By the way, our environment also included such scalars; they were used when multiplying a vector by a scalar.

A variable vector has an unstable existence. Only while being dragged does it exist as such: a variable vector. When dragging stops, only a very partial record remains on the screen: An arrow with given, potentially variable length and direction. The variability remains only potential, in the eye (or mind) of the beholder. If the student looking at the screen is not aware of this potential variability, the variable vector has ceased to exist as such.

The specific notion of vector which we used has immediate implications for the notion of transformation: A transformation can only be described in terms of what is being transformed. And what is being transformed is a vector, in our case a variable vector. In fact, a single variable vector suffices to describe the entire transformation. But when this variable vector used to describe a transformation ceases to exist as above, due to the fact that it is not being dragged, and not even conceived of as being potentially variable, then the transformation disappears along with it. And this is precisely what happened in the projection activity reported above.

It is instructive to observe the extent to which representations of vectors and transformations are visible, and thus concrete. In a paper and pencil environment, typical representations of vectors are easily visible. In our Cabri environment, a static vector, as opposed to a variable one, would be similarly visible but is not usually an object of consideration. The more complex variable vector, however, is visible only to a limited extent, because of its fleeting appearance: At any instant in time, we see only a single one of its instantiations. In this sense, the variable Cabri vector is less visible than a paper and pencil vector. On the other hand, the Cabri environment gives far more visibility to transformations than a paper and pencil environment. In fact, a variable vector and its variable image under the transformation can be placed on the screen simultaneously. In this situation, the effect of a transformation is directly observable, thus indirectly lending some visibility to the transformation itself. Transformations are thus rendered far more concrete than in paper and pencil environments.

In our design, the central notions of vector and transformation are strongly dependent on the technological tool. It is therefore expected that students' argumentation and their notion of proof will also be influenced by the tool. For example, the tool can play a crucial role in the activity in which the students were asked to prove the linearity of a given transformation. In this activity, the relationship T(ku)=kT(u) must be verified for all vectors u and all scalars k. In words more appropriate to our Cabri representations, the relationship has to be valid for a variable vector u and a variable scalar k. While it is not possible to check the relationship for strictly all cases, it is definitely possible to check it for a variable vector v by implementing it for one (static) instance of the vector and then dragging the vector. Such checking, if carried out systematically, can be expected not only to provide students with a concrete geometrical meaning for the relationship T(ku)=kT(u), but also to make palpable the idea of checking the relationship for all vectors v and for all scalars k.

Similarly, the issue what constitutes a projection was expected to be dealt with for a variable vector. Above we have shown that this is not what happened. The students did not drag; in other words, they did not acquire the concept of a variable vector in the manner expected by the designers.

In summary, the notion of vector and the notion of transformation which we used in the design described in this paper are strongly tool dependent. As a consequence tools are expected to shape students' conceptualizations. The representations we used, and the sequence of activities which we asked the students to carry out were a first attempt at providing a dynamic representation of vector and transformation. It would have been unreasonable to expect that this first attempt is perfect, and clearly it was not. It did, however, provide the basis for a redesign which will be reported elsewhere.

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7. References

Alves Dias, M. & Artigue, M. (1995): Articulation problems between different systems of symbolic representations in linear algebra. In L. Meira (ed.), *Proceedings of the 19th International Conference on the Psychology of Mathematics Education*, Vol. II, 34-41. Recife, Brazil.

Banchoff, T. & Wermer, J. (1992): *Linear Algebra through Geometry*, New York: Springer-Verlag.

Dorier, J.-L. (1997_1): Une lecture épistémologique de la genèse de la théorie des espaces vectoriels. In J.-L. Dorier (ed.), *L'Enseignement de l'Algèbre Linéaire en Question, Panorama de la Recherche en Didactique sur ce Thème* (pp. 21-99). Grenoble, France: La Pensée Sauvage.

Dorier, J.-L. (ed.) (1997_2): L'Enseignement de l'Algèbre Linéaire en Question, Panorama de la Recherche en Didactique sur ce Thème. Grenoble, France: La Pensée Sauvage.

Dreyfus, T.; Hillel, J. & Sierpinska, A. (1997): Coordinate-free geometry as an entry to linear algebra. In M. Hejny & J. Novotna (eds.), *Proceedings of the European Research Conference on Mathematical Education*, pp. 116-119. Podebrady/Prague, Czech Republic.

Harel, G. (1985): *Teaching Linear Algebra in High School*. Unpublished doctoral dissertation, Ben-Gurion University of the Negev, Israel.

Hillel, J. (1997): Des niveaux de description et du problème de la représentation en algèbre linéaire. In J.-L. Dorier (ed.), *L'Enseignement de l'Algèbre Linéaire en Question, Panorama de la Recherche en Didactique sur ce Thème* (pp. 231-247). Grenoble, France: La Pensée Sauvage.

Hillel, J.; Sierpinska, A. & Dreyfus, T. (1998): Investigating linear transformations with Cabri. *Proceedings of the International Conference on the Teaching of Tertiary Mathematics*, Samos, Greece.

Hillel, J. & Sierpinska, A. (1994): On one persistent mistake in linear algebra. In J. P. da Ponte & J. F. Matos (eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education*, Vol. III, 65-72. Lisbon, Portugal: University of Lisbon.

Laborde J.-M. & Bellemain, F. (1994): *Cabri-Geometry II, software and manual*. Texas Instruments.

Martin, Y. (1997): *Groupe linéaire. Macros de base sur $GL_2(R)$*. AbraCadaBri, Novembre 1997. http://www-cabri.imag.fr/abracadabri

Robert, A. & Robinet, J. (1989): Quelques résultats sur l'apprentissage de l'algèbre linéaire en première année de DEUG. *Cahiers de Didactique des Mathématiques*, 52, IREM, Université de Paris 7.

Rogalski, M. (1990): Pourquoi un tel échec de l'enseignement de l'algèbre linéaire? Enseigner autrement les mathématiques en DEUG Première Année, Commission inter-IREM université. IREM de Lille, 279-291.

Sierpinska, A. (1996): Problems related to the design of the teaching and learning processes in linear algebra. Plenary talk at the *Research Conference in Collegiate Mathematics Education*, September 5-8, 1996, Central Michigan University, Mount Pleasant, Michigan (unpublished, available from the author).

Sierpinska, A. (in press): Recherche des situations fondamentales à propos d'une notion mathématique: le vecteur. *Actes des Premières Journées de Didactique des Mathématiques de Montréal*, sous la rédaction de J. Portugais, Montréal, 2-5 juin 1997.

Sierpinska, A. (1997): Vecteurs au Collège et à l'Université. Conférence Plénière au *Congrès de l'Association Mathématique du Québec, Trois-Rivières, Québec, 17-19 octobre 1997* (à paraître dans les actes).

Sierpinska, A., Defence, A., Khatcherian, T. & Saldanha, L. (1997): A propos de trois modes de raisonnement en algèbre linéaire. In J.-L. Dorier (ed.), *L'Enseignement de l'Algèbre Linéaire en Question, Panorama de la Recherche en Didactique sur ce Thème* (pp. 249-268). Grenoble, France: La Pensée Sauvage.

Sierpinska, A., Hillel, J. & Dreyfus, T. (1998): Evaluation of a teaching design in linear algebra: the case of vectors. Preprint, available from the authors.

Sierpinska, A., Dreyfus, T. & Hillel, J. (1998): Evaluation of a teaching design in linear algebra: the case of transformations. Preprint, available from the authors.

Vinner, S. (1983): Concept definition, concept image and the notion of function. *International Journal of Mathematics Education in Science and Technology* 14, 239-305.

INTEGRATION OF LEARNING CAPABILITIES INTO A CAS: THE SUITES ENVIRONMENT AS EXAMPLE

Jean-Michel Gélis, Dominique Lenne
Institut National de Recherche Pédagogique, 91 rue Gabriel Péri 92120 Montrouge France
gelis@inrp.fr
lenne@inrp.fr

Abstract: *This paper deals with the use of a Computer Algebra System (CAS) for the purpose of learning about sequences. Difficulties students have in working with a CAS in the field of sequences are pointed out. Then, SUITES, a learning environment based on a CAS, is described. This environment provides the students with suitable problem solving functions, didactic knowledge and help options based on a first level diagnosis of the students' activity. Finally, some evaluation results and future directions are given; in particular, we emphasise some design issues such as a better integration between learning and solving tools.*
Keywords: -

1. Introduction

A lot of research has been done to determine the potential of Computer Algebra Systems (CAS) in education and to specify the conditions of their integration into mathematics learning (Artigue et al 94, Lagrange 96). Significant difficulties, due to the fact that CAS have not been designed for learning, are often pointed out. However, few authors propose to enhance them with didactic capabilities. We think that this research direction, which has been explored in the eighties in artificial intelligence (Genesereth 82; Vivet 84) should now be revisited.

We have tried to add interactive learning environments to a CAS in some mathematical domains, in order to provide the students with suitable functions and appropriate help or advice. The goal was to achieve a closer integration between solving and learning environments.

In this paper, we describe the result thus obtained: SUITES (i.e. sequences), a learning environment in the domain of sequences, based on the widespread CAS "MAPLE V" ©. SUITES mainly aims at helping students with no experience in working with a CAS. The two steps of the SUITES design are: (1) A preliminary study including: (a) the choice of a field of knowledge and its related problems; (b) the identification of paper and pencil solution methods; (c) the identification of the difficulties to use these methods in a CAS; (2) The SUITES environment itself including its functions and help possibilities. Finally, we will give some evaluation results and introduce future possible lines of work.

2. Preliminary study

2.1 Domain and methods

We chose the topic of sequences because it is an important topic in the French curriculum at the end of the secondary studies. It is notoriously difficult; it involves a wide variety of mathematical objects and requires numerical, graphical and formal settings. In this paper, we will focus on a particular kind of problems we call relations problems. Figure 1 shows an example of such a problem.

To find out potential didactic difficulties, we carried out a paper and pencil pilot study with 25 secondary students of a French high school in April 1996. Students had to solve some relations problems. We observed that a lot of them had strategic difficulties. In the problem in Figure 1, for example, they mainly failed in establishing the recurrence relation satisfied by the v sequence, in order to prove that it was geometric. More precisely, they obtained different and useless relations involving the n and n+1 terms of the sequences u or v, without succeeding in establishing a recurrence relation of v. Our assumption was that the strategic steps they had to do were not clear. This experiment has given us information about the various methods used by the students to find out the recurrence relation. Figure 2 presents the most widely used method.

Let (u_n) be a sequence defined by the recurrence relation $u_{n+1} = \frac{-2u_n - 4}{u_n + 3}$. Find the limit of the sequence (u_n), when n approaches infinity. To this end, assuming that $v_n = \frac{u_n + 1}{u_n + 4}$, prove that the sequence (v_n) is a geometric one.

Fig. 1: An example of a relations problem

(1)	(2)	(3)	(4)	(5)
$v_{n+1} = \frac{u_{n+1} + 1}{u_{n+1} + 4}$	$v_{n+1} = \frac{\frac{-2u_n - 4}{u_n + 3} + 1}{\frac{-2u_n - 4}{u_n + 3} + 4}$	$v_{n+1} = \frac{-u_n - 1}{2u_n + 8}$	$v_{n+1} = -\frac{1}{2}\frac{u_n + 1}{u_n + 4}$	$v_{n+1} = -\frac{1}{2}v_n$

Fig. 2: An example of a student method to solve figure 1 problem

The fourth step of this method enables students to avoid computations. This way, the need does not arise to express u_n in function of v_n (by inverting the given relation), nor to substitute u_n nor to make hard calculations to obtain the final relation (step 5). Consequently, this method is well suited for paper and pencil work.

2.2 Identification of difficulties to use a CAS

We have observed that it is sometimes difficult for students to use MAPLE efficiently, especially if they are not accustomed to a CAS. The main reason is most probably because MAPLE has been designed for engineers and researchers. It has not been designed to teach mathematics.

For example, MAPLE does not include any specific function operating upon sequences, except the *rsolve* function, which establishes, in some particular cases, the explicit expression of a sequence defined by a recurrence relation. Consequently, usual operations on sequences (e.g. obtaining some graphical representations, deducing its recurrence relation or calculating some numerical values) cannot be performed with any MAPLE high level function. Some operations are difficult to make for inexperienced users of MAPLE, e.g. the spiral representation of a sequence defined by a recurrence relation (some computer program instructions are required).

We observed that when working with a CAS students often try to apply paper and pencil solution methods. These methods can sometimes be very hard to follow with a CAS. For example, the fourth step of figure 2 is very hard to simulate because MAPLE systematically simplifies any expression the user obtains and because high level CAS functions don't succeed in all cases in isolating the wanted expression. Furthermore, we have given some examples (Gélis, Lenne 97) (involving *logarithmic* and *exponential* functions), in which this method requires an acute knowledge of the CAS functions. Consequently, MAPLE is likely to add new difficulties for students trying to solve sequences problems.

In conclusion, we think that MAPLE is unsuitable for students who have to solve sequences problems, because of the lack of high level functions and of the potential difficulties in following paper and pencil methods. This is especially true for students not accustomed to working with a CAS.

3. The SUITES environment

The following paragraphs present the three parts of the SUITES environment: a functions set (allowing the students to easily solve their problems), some basic helps and a contextual help (providing the students with the more useful basic help).

3.1 Functions

The first part of the environment SUITES includes a set of suitable functions to provide the students with functions they lack. These functions have to respect the following principles: (1) a SUITES function has to represent a strategic step in a problem of the chosen field; (2) no function of the CAS is to be available to do what a SUITES function does; (3) a SUITES function must work according to a method students may understand and use; (4) the execution of a SUITES function must be explainable to students.

According to these principles, we conceived the following functions and we implemented them in the MAPLE programming language:

defsuite defines a sequence either by its explicit relation (as in *defsuite(a, a(n) = 2^n+3*n)*), or by its recurrence relation (as in *defsuite(u, u(0) = 2, u(n+1) = u(n)^2/(2*u(n)-1)*), or by a relation (as in *defsuite(v, v(n) = (u(n)-1)/u(n))*). Note that this function automatically creates a *sequence* object. Such objects do not exist in MAPLE. It also creates the calculating function associated with the sequence object, which is rather difficult to implement for students, particularly in the case of a sequence which is defined by a recurrence relation. The calculating function also stores the given relation and the initial suffix of the sequence.

recurrence allows one to establish a recurrence relation. For instance, *recurrence(v,u)* generates the recurrence relation of the sequence *v*, as soon as the sequence *u* is defined by a recurrence relation , and *v* is expressed as a function of *u*.

explicite computes the explicit expression of a sequence. For instance, *explicite(v, geom)* establishes the explicit expression of a sequence defined by a geometric recurrence relation. Moreover, *explicite(u,v)* generates the explicit expression of the sequence *u*, as soon as the explicit expression of *v* is known and *u* is expressed as a function of *v*.

To be able to solve a great range of problems involving sequences, we conceived other functions dealing for example with graphical possibilities and relations operations (such as inverting a relation).

Figure 3 shows how the relations problem of figure 1 can be solved with the SUITES functions. These functions make the calculations and thus enable the students to focus on strategic aspects.

```
> defsuite(u,u(0)=a,u(n+1)=(-2*u(n)-4)/(u(n)+3));
> defsuite(v,v(n)=(u(n)+1)/(u(n)+4));
> inverser_relation(u,v); # in order to prepare the next step
> recurrence(v,u); # in order to establish the recurrence relation of v
> explicite(v,geom); # the recurrence relation of v is a geometric one
> explicite(u,v); # the explicit expression of u is deduced from that of v
```

Fig. 3: Solving of the problem of figure 1
with SUITES functions

3.2 Basic help

The second part of the SUITES environment consists of help functions. Four basic types of help are available:

(1) *help on concepts*, such as the monotonicity of a sequence; (2) *help on functions*, including the descriptions of the suites functions as well as some MAPLE functions; (3) *help on basic methods*, explaining for example how to prove that a sequence is a geometric one; (4) *solving help*, providing the solution of a given relations problem, expressed in one of three ways (a) in a general way (without any reference to the given sequences); (b) in an applied way (giving for example the names of intermediate sequences and telling whether they are geometric or arithmetic); (c) as a sequence of SUITES functions to be used.

The first three kinds of help, allow the students to navigate in a hypertext mode and to use an index and a summary. Solving help relies on a small system solver finding solutions to relations problems. The students can be given access to these four help functions freely, partially or not at all, according to the teacher's decision.

3.3 Contextual help

When using the environment SUITES, students may have some difficulties in determining by themselves the most appropriate help. The third part of the SUITES environment takes this problem into account and provides the students with contextual help relying on help rules. Before designing these rules in 1997, we organised a pilot study with 10 secondary students of the same school as above. For two hours, students of various levels had to solve four relations problems with the learning environment SUITES; the questions were to establish explicit expressions of some sequences

The first goal of this experiment was to check that students could easily work with MAPLE and with the SUITES environment, including functions and basic help functions. The second goal was to identify the help that students need in addition. Three types of errors were encountered in students' protocols: (1) spelling errors, such as writing *recurence* instead of *recurrence*; (2) functional errors, such as asking for *inverser_relation(u, v)* when *u* is already expressed as a function of *v*; (3) strategic errors, such as trying to determine complicated and useless recurrence relations instead of picking out an arithmetic sequence and inferring the explicit expression wanted.

We have designed help rules that provide students with suitable advice. These help rules have to respect two principles:

(1) Whenever possible, help rules should not give a complete solution to the students. According to the particular student difficulties, help rules should recall general methods, appropriate concepts and correct syntax. We believe that when encountering obstacles, students should overcome the difficulties alone with the help of general advice; (2) if the students' attention is low, then the help rules have to provide him with more precise help.

Our help rule design relies on seven relevant indicators that were deduced from the above experiment. The most important indicators are: (1) *solving indicators* that allow to make a first level diagnosis of the students' work from a spelling, conceptual and strategic point of view; (2) *cognitive indicators* that evaluate the student's attention (for example by observing the number of helps the students asked for or by comparing the length of the student's solution to that of the system's solution).

These indicators are used by the help rules to achieve two tasks: (1) to determine the most appropriate basic help (including help on concepts, on functions, on basic methods and solving help); (2) to determine the form of this basic help (for example we can give general sentences without any reference to the given sequences or we can provide the students with the precise sequence of SUITES functions to apply to solve the problem). In figure 4 we show some examples of precise help rules and the kind of advice they provide the students with.

Rule that determines the relevant help		**Help given to the student**
IF AND THEN	the student has only strategic difficulties the student is attentive provide the student with the different solving methods.	*Here are some solving methods:* *method 1: check if the recurrence relation of the given sequence is an arithmetic or geometric one and draw conclusions;* *method 2: try to find a related sequence defined by an arithmetic or geometric recurrence relation and draw conclusions;* ....
IF AND THEN	the student has only strategic difficulties the student is not at all attentive provide the student with the applied solving help.	*Here is a general method to solve this problem:* *Find a related sequence defined by an arithmetic or geometric recurrence relation and draw conclusions;*
IF AND THEN	the student has only strategic difficulties the student is not at all attentive provide the student with the applied solving help.	*I suggest to use the following method to solve the problem:* *1. notice that the sequence a is an arithmetic one;* *2. infer the explicit expression of a;* *3. infer the explicit expression of c.*

Fig. 4: Examples of help rules and their application

So far, twenty help rules have been implemented. It is important to note that the rule set can be easily modified, and that the help rule design is closely based on the analysis of protocols and thus provides the students with help that teachers might have given in similar circumstances.

4. Evaluation and future work

In January 1998, fifteen students were observed while working with the SUITES environment to solve classical problems on sequences; these students could use the basic help but contextual help rules were not available at that time. As expected, students mainly used help on functions and methods because of their efficiency for solving problems. But more surprisingly, frequent calls to the conceptual help were also observed. New experiments are now planned to evaluate and improve the contextual help rules.

The design of learning environments (such as SUITES) based on CAS (such as MAPLE) brings up at least two important questions: (1) What level of granularity is desirable for the specific functions of the learning environment? (2) How closely should the learning environment and the CAS be integrated?

4.1 Granularity

The definition of new functions aiming at being more suitable to mathematics learning is often difficult. We distinguish four granularity levels:

Level 4: High level functions are available in the CAS or in the environment. They allow to solve a problem with a single command. Neither the environment SUITES nor MAPLE include such a general function for relations problems. Such functions would allow the students to verify results, rather than to learn strategic methods.

Level 3: Intermediate level functions are available in the environment. They allow the students to work comfortably at a strategic level and to obtain some help. That's the level we chose for the functions of the environment SUITES.

Level 2: Low-level functions are available in the CAS. For relations problems, such functions could have for example consisted in rewriting a relation between sequences at a given suffix or in recording intermediate relations that students establish for further utilisation. The **recurrence** function of the SUITES environment can be achieved by using several MAPLE and low level functions. We plan to design a low-level functions set and organize experiments using these functions.

Level 1: No specific function is available in the CAS to help the students. When encountering difficulties, students have to develop new strategies or try to know the CAS better.

4.2 Integration

Different degrees of integration between the learning environment and the CAS are possible. Three types of architecture can be designed (adapted from (Tatersall 92)): (1) divorced (CAS and learning environment are disconnected); (2) separated (CAS access is supervised by the learning environment); (3) integrated.

In the first type (divorced) the CAS and the learning environment are independent. The learning environment can contain some information on the CAS and can simulate some functions of the CAS in the specific field it is restricted to. On the one hand the main interest of this architecture is that the learning environment is not bound to the CAS, and can still be used if a new release of the CAS is issued. But on the other hand, no link exists between the learning environment and the CAS, and the students cannot use the potential of the CAS to solve a problem in the learning environment.

In the second type (separated), the students interact only with the learning environment, without having direct access to the CAS. The advantage is that the learning environment can offer a more natural interface, including direct manipulation (Laborde 98), possibly very different from the interface of the CAS. It can be more suitable for mathematics learning, but on the other hand, the students do not get accustomed to using the CAS.

With the third type (integrated) full access is given to the CAS. Functions of the learning environment can be composed of functions of the CAS. Therefore the interface is very close to the interface of the CAS and the learning environment has to be implemented with the use of the programming language of the CAS. We have used this last type for the design of suites.

5. Conclusion

In the future, CAS will probably be more and more used in the teaching and learning of Mathematics. But they need to be enhanced and improved to become useful instruments to solve problems as well as suitable for learning. The SUITES learning environment, that helps students to solve problems on sequences, is an attempt in this direction. It provides to the students suitable solution functions, didactic knowledge and help capabilities based on a first level diagnosis of students activity.

More generally speaking, we think that learning environments should be better integrated with solving environments. This approach brings up important questions on the granularity level of the functions and on the integration level of the environments.

6. References

Artigue, M., Drouhard, J.P. & Lagrange J.B. (1994): Acquisition de connaissances concernant l'impact de l'intégration de logiciels de calcul formel dans l'enseignement des mathématiques sur les représentations et pratiques mathématiques des élèves de l'enseignement secondaire, *Cahier de DIDIREM* N°24, IREM Paris VII, mai 1994.

Gélis, J.M. & Lenne, D. (1997): Conception de fonctions spécialisées dans un système de calcul formel : l'exemple des suites en Terminale S sous MAPLE. *Bulletin de l'EPI*, septembre 1997.

Genesereth M. (1982): The role of plans in intelligent teaching systems. In Sleeman & Brown (eds.), *Intelligent Tutoring Systems*. New York Academic Press.

Lagrange J.B. (1996): Analysing actual use of a computer algebra system in the teaching and learning of mathematics: an attitudinal survey of the use of DERIVE in French classrooms. *International DERIVE Journal*, Vol. 3, N°3.

Laborde J.M. (1998): Towards more natural interfaces. In Tinsley & Johnson (eds.), *Information and Communications Technologies in School Mathematics*. Chapman and Hall.

Pachet F., Giroux S. & Paquette G. (1994): Pluggable Advisors as Epiphyte Systems. *Proceedings of CALISCE'94*, Telecom, Paris.

Tattersall C. (1992): A New Architecture for Intelligent Help Systems. *Proceedings of ITS'92*, pp 302-316, Montréal.

Trouche L. (1996): *Etude des rapports entre processus de conceptualisation et processus d'instrumentation*. Thèse, Université de Montpellier.

Vivet M. (1984): *Expertise mathématique et informatique : CAMELIA, un logiciel pour raisonner et calculer*. Thèse d'état, Université Paris VI.

THE TEACHING OF TRADITIONAL STANDARD ALGORITHMS FOR THE FOUR ARITHMETIC OPERATIONS VERSUS THE USE OF PUPILS' OWN METHODS

Rolf Hedrén
Hoegskolan Dalarna, S-79188 FALUN, Sweden
roh@du.se

Abstract: *In this article I discuss some reasons why it might be advantageous to let pupils use their own methods for computation instead of teaching them the traditional standard algorithms for the four arithmetic operations. Research on this issue is described, especially a project following pupils from their second to their fifth school year. In this, the pupils were not taught the standard algorithms at all, they had to resort to inventing their own methods for all computations, and these methods were discussed in groups or in the whole class. The article ends with a discussion of pros and cons of the ideas that are put forward.*
Keywords: -

1. Introduction

A lot of calculation today is carried out by using electronic means of computation, with calculators and computers. Besides, the pedagogical disadvantages of the traditional written algorithms for the four arithmetic operations have been emphasised by researchers in mathematics education for a long time (see e. g. Plunkett, 1979). In my opinion these two facts have not been taken into consideration in the mathematics classrooms, at least not in my own country. It is high time to ponder about what kind of knowledge of mathematics is important in our present society and will be important in the society of tomorrow. It is likely that the drill of traditional algorithms for the four arithmetic operations, which is all too common in today’s elementary schools, should be dismissed or at least heavily restricted and replaced by the pupils’ invention of their own methods for computation.

I started a project in one class in their second school year, and the project went on until the pupils had finished their fifth year, i. e. in the spring term 1998. In this project, the pupils have not been taught the traditional algorithms for the four arithmetic operations. Instead they have always been encouraged to find their own methods. The pupils have used mental computation as far as possible and written down notes to help them when the computations have been too complicated for the pupils to keep the results in their heads. This latter kind of computation I will call written computation, although we did not make use of the standard algorithms.

2. Background

There were three reasons for starting this project:

1. The electronic devices for computation already mentioned;
2. An increasing demand for a citizen's number sense (numeracy);
3. Social constructivism as a philosophy of learning.

I will discuss these three points very briefly.

1. I think that when computation is carried out by calculators and computers, it is still important that we ourselves understand the meaning of the computation and are able to check that a result is correct. We must therefore possess understanding and knowledge of numbers and relationships between numbers and the meaning of the different arithmetic operations as well as skill in mental computation and estimation. It has been pointed out that the methods for pencil-and-paper computations, that the pupils invent themselves, are much more like effective methods for mental arithmetic and computational estimation than the standard algorithms are.
2. To be able to do mental computation and estimation, a person needs good comprehension and understanding of numbers and relationships between numbers (number sense). There are many aspects of number sense, but I will only mention a few here that in my opinion are most essential for the issue under discussion. (See e. g. Reys, 1991.) A pupil with good number sense:

- understands the meanings and magnitudes of numbers;
- understands that numbers can be represented in different ways;
- knows the divisibility of numbers;
- knows how to use the properties of the arithmetic operations.

3. Social constructivism has been discussed quite a lot lately, and I do not want to give another contribution to this discussion. (See e. g. Cobb, 1997; Ernest, 1991; Ernest 1994.) It is, however, very difficult for me to see that the teaching of ready-made mechanical rules for computation is in accordance with social constructivism. Research has also shown that the traditional drill of standard algorithms has not been very successful (Narode, Board & Davenport, 1993). In my opinion, letting pupils invent and discuss with each other and with their teacher their own methods for computation would better adhere to the ideas of social constructivism. A teacher should feel free to show methods that s/he has found effective (including the standard algorithms, when her/his pupils are ready to understand them), but s/he should never force a standard method on everybody.

3. Previous research

In the CAN-project (Calculator Aware Number) in Britain (Duffin, 1996) the children, beside using their own methods for written computation, always had a calculator available, which they could use whenever they liked. Exploration and investigation of "how numbers work" was always encouraged, and the importance of mental arithmetic stressed.

One of the reported advantages of the CAN-project was that the teachers' style became less interventionist. The teachers began "to see the need to listen to and observe children's behaviour in order to understand the ways in which they learn". (Shuard et al, 1991, p. 56.) The teachers also recognised that the calculator "was a resource for generating mathematics; it could be used to introduce and develop mathematical ideas and processes". (Ibid p. 57.)

Kamii (1985, 1989, 1994) worked together with the children's class teachers in grades 1 - 3 in a similar way in the U. S. She did not teach the traditional algorithms but encouraged the children to invent their own methods for the four arithmetic operations. She also devoted much time to different kinds of mathematical games.

According to Kamii et al "many of the children who use the algorithm unlearn place value …" (Kamii, Lewis & Livingston, 1993/94, p. 202). They give

$$\begin{array}{r} 987 \\ \underline{+\ 654} \end{array}$$

as an example and compare pupils, who use their own methods with those using the algorithm. The former start with the hundreds and say: " 'Nine hundred and six hundred is one thousand five hundred. Eighty and fifty is a hundred thirty; so that's one thousand six hundred thirty …' ". The latter "unlearn place value by saying, for example, 'Seven and four is eleven. Put one down and one up. One and eight and five is fourteen. Put four down and one up. …' ". They state that children, when working with algorithms, have a tendency to think about every column as ones, and therefore the algorithm rather weakens than reinforces their understanding of place value. (Ibid p. 202.)

It is also interesting to follow the research carried out by Murray, Olivier, and Human in South Africa (e. g. Murray, Olivier & Human, 1994; Vermeulen Olivier & Human, 1996). Like the researchers mentioned above, they had their pupils invent their own strategies for computation, and above all they discussed strategies used for multiplication and division. In a summary of the results of their problem centred learning they state among other things:

> … students operate at the levels at which *they* feel comfortable. When a student transforms the given task into other equivalent tasks, these equivalent tasks are chosen because the particular student finds these tasks more convenient to execute.
> (Murray, Olivier & Human, 1994, p. 405.)

Narode, Board and Davenport (1993) concentrated on the negative role of algorithms for the children's understanding of numbers. In their research with first, second and third graders the researchers found out that after the children had been taught the traditional algorithms for addition and subtraction, they discarded their own invented methods, which they had used quite successfully before the instruction. The children also tried to use traditional algorithms in mental arithmetic, they gave many examples of misconceptions concerning place value, and they were all too willing to accept unreasonable results achieved by the wrong application of the traditional algorithms.

4. My own research

In my own research I wanted to see what changes will occur in a class when the children are given the possibility to invent and develop their own methods for written computation. In particular, I wanted to try to get answers to the following questions:

1. How is the pupils' number sense affected?
2. How is the pupils' ability to do mental computation and estimation affected?
3. How is the pupils' motivation for mathematics affected?
4. Is there a difference between girls' and boys' number sense and ability in mental computation and estimation?
5. Is there a difference between girls' and boys' motivation for mathematics?

I followed an ordinary Swedish class from their second school year up to and including their fifth. The data collection was finished at the end of the spring term 1998. In short the following steps were taken in the experimental class:

1. The children were encouraged and trained to use *other paper-and-pencil methods* rather than the traditional algorithms to carry out computations that they could not do mentally. No special methods were taught or forced upon the children. The methods were discussed in groups and in the whole class. The children's parents were also encouraged to help their children to use alternative computational methods and not to teach them the standard algorithms.

2. *Mental arithmetic and estimation* were encouraged and practised. The children were encouraged to invent their own methods, which were discussed in class.
3. The children had *calculators* in their desks. They were used for number experiments, for more complicated computations, and for checking computations made in other ways.
4. With the exceptions mentioned in points 1 - 3, the children followed a *traditional course*. The ordinary teacher had full responsibility for the mathematics periods. My own task was to design the project, to encourage and give advice to the teacher, and to evaluate the project.

Although the calculator could be said to be one of the reasons for the realisation of the experiment, it was not itself a major issue in it. However, I chose to let the pupils use calculators on some occasions, as it would have been illogical to pretend that they do not exist or that they are a resource that should only be used outside the classroom.

For the evaluation I used mainly qualitative methods:

- Clinical interviews,
- Observations,
- Copies of pupils' writing on the observed occasions,
- Interviews with pupils,
- Interviews with the teacher and weekly phone calls to him.

As the results stated below are only taken from observations, I will concentrate on them here. I undertook the observations when the pupils were working in small groups. In this way it was possible for me to follow the interaction in the groups. I used a tape-recorder during these sessions and made copies of the pupils' written work. Every pupil of the class was observed in this way at least twice per school year.

5. Some results

In this section I will give a few examples of the pupils' methods of computation. I will state if a computation was done only mentally or if some parts of the computation was written. However, I think there was very little difference between a computation that a pupil made in her/his mind and one where s/he made some notes to help her/him remember some intermediate results. I always interviewed the pupils about their solutions. I have thus been able to note the pupils' way of reasoning, even when the exercises were solved mentally. However, in the following text the mental computation, too, is written in mathematical symbols. I see no reason to translate the pupils' words into English, as part of the their thoughts and intentions would get lost anyhow.

A typical solution to an addition exercise with two three-digit numbers, 238 + 177, was: 200 + 100 = 300; 30 + 70 = 100; 7 + 8 = 15; 238 + 177 = 415. (This solution was written.)

A more special solution in addition looked like this: 157 + 66 = 160 + 63; 60 + 60 = 120; 100 + 120 + 3 = 223. (On the paper the boy only wrote 160 + 63 = 223. Thus, the solution was mainly done mentally.)

Several different methods were used in subtraction. This is one example for 147 - 58: 40 - 50 = - 10; 100 - 10 = 90; 7 - 8 = - 1; 90 - 1 = 89. (The solution was written.)

Another pupil solved the same exercise in the following way: 140 - 50 = 90; (90 + 7 = 97); 97 - 8 = 89. (Mental arithmetic.) The boy explained that he could not do the sum 7 - 8 but he managed 97 - 8.

I give a third example in subtraction, where a pupil uses even hundreds and tens. A boy computed 514 - 237 in this way: 500 - 200 = 300; 300 - 30 = 270; 270 - 7 = 263; 263 + 14 = 277. (Mental arithmetic.)

However, many pupils made mistakes in subtraction, because they mixed up numbers from the first and from the second term. E. g., two girls computed 514 - 237 as

500 - 200 = 300; 10 - 30 = 20; 4 - 7 = 3, and the answer was 323. (The solutions was written.)

A similar misunderstanding: 53 - 27: 50 - 20 = 30; 3 - 7 = 0. The answer was 30. (Mental arithmetic.)

I will give two examples in multiplication. In the first, the distributive property is used, in the second repeated addition.

7 x 320: 7 x 3 = 21; 2100; 7 x 2 = 14; 140. (The boy explained why he added one and two zeros respectively.); 2100 + 140 = 2240. (He only wrote the product 2100, the rest was done mentally.)

6 x 27: 27 + 27 = 54; 54 + 54 = 108; 108 + 50 = 158; 158 + 4 = 162. (The boy only wrote the number 54 and the final answer, the rest war done mentally.)

Finally, I turn to division. Again, I will give two examples, one, where the pupil partitioned the numerator in a sum of two terms and tried to divide one term at a time, and one, where the pupil guessed the quotient more or less intelligently and then tested its correctness with multiplication or addition.

236 ÷ 4: The girl first wrote 200 ÷ 4 = 50; 30 ÷ 4 + 6 ÷ 4. She then altered her writing to 36 ÷ 4 but made a mistake and got 19. After trying first 8 and then 9 she got the correct answer.

236 ÷ 4: The girl proceeded by trial and error. She found the number 50 pretty soon but had some trouble with the units. After trying 7 and 8, she found the number 9. She wrote:

	4	236	
9	9	9	9
50	50	50	50

and told me the answer 59.

As the girl in the first example of division, many pupils tried to partition the numerator into hundreds, tens and units, a method which they had used successfully in the three other arithmetic operations.

6. Discussion

6.1 Introduction

From these examples we can see that the pupils of the experimental class understood place value and were able to use it by partitioning in hundreds, tens and units. They could also use other ways to partition numbers, when these were more convenient. In the exercises they clearly showed their mastery of the following aspect of number sense: "Understands that numbers can be represented in different ways".

The pupils also gave many examples of their mastery of the properties of the four arithmetic operations. They sometimes used a kind of compensation in addition and also subtraction to simplify the computations. In addition, they used the distributive property for multiplication and division over addition.

Many pupils solved the exercises mentally with few or no intermediate results written. Even the pupils who wrote very detailed notes used strategies that were very similar to those that are used in mental arithmetic.

However, we have to consider the gains and losses with an instruction, where the pupils are allowed to use their own methods for computation and the traditional algorithms are not taught. I will therefore give my own opinion of the advantages of pupils' own methods and traditional algorithms respectively.

6.2 Advantages of the pupils' own methods of computation

- When the pupils get the chance to develop their methods themselves, these will in some sense be "the pupils' property".

- The methods are more like methods for mental arithmetic and computational estimation.
- As in mental arithmetic, the pupils almost always start computing from the left. By considering this position first, the pupils might get a sense of the magnitude of the result. We can also compare with computational estimation, where it might be enough to calculate with the numbers formed by the far left positions with a sidelong glance at the other digits.
- It is more natural to start reading from the left.
- The pupils practise their number sense when they are working in this way. They can clearly see what happens to the hundreds, the tens etc.
- It is easier to understand what happens in the computation. Thereby, the risk that the pupils will misunderstand the method and make systematic errors or forget what to do will be reduced.
- This way of working is in accordance with social constructivism.

However, in connection with the fourth statement I want to point out that there are also algorithms for all the four arithmetic operations, where one starts from the left, although they do not seem to be very common, except in division.

6.3 Advantages of the traditional standard algorithms

- They have been invented and refined through centuries.Today, they are therefore very effective methods of computation.
- They can be used in about the same way, no matter how complicated the numbers involved are. If the computations are very laborious - e. g. multiplication of two three digit numbers or division by a two digit denominator - they are probably the only way, if one can only use paper and pencil.
- They are a part of the history of mathematics and are thus a cultural treasure that we should be careful with.

7. A final word

To be fair, I have to add that a non-standard method of computation can also be an algorithm in a negative sense - a plan that the pupil follows without being aware of what s/he is doing. However, as long as the pupil has invented her/his method herself/himself, there is no risk, but if s/he has taken over the method from a class mate or from the teacher without really understanding what s/he is doing, the risk is present.

As I see it, every pupil should start his learning of computation by inventing and using his own methods. We must look at computation as a process, where the pupil has to be creative and inventive, and from which the pupil can learn something.

However, the question whether we should teach the algorithms at all and, if so, when it should be done, remains. One extreme is not to teach them, because they are not needed in the society of today. When the computations are so complicated that we cannot use non-standard methods, we can turn to calculators or computers. In the other extreme, we introduce the standard algorithms pretty soon after the pupils have started developing their own methods. After that the pupils might be allowed to choose their methods at will.

Personally, I doubt if it is necessary to teach the standard algorithms at all. If teachers and pupils (or pupils' parents) insist, the teaching of them should be postponed to perhaps the sixth or seventh school year. By then, the pupils have, hopefully, already acquired good number sense, and therefore the teaching of the algorithms will not do any harm.

8. References

Cobb et al. (1997): Reflective Discourse and Collective Reflection. *Journal for Research in Mathematics Education* Vol. 28 No. 3, 258 - 277.

Duffin, Janet (1996): *Calculators in the Classroom*. Liverpool: Manutius Press.

Ernest, Paul (1991): *The Philosophy of Mathematics Education*. London: The Falmer Press.

Ernest, Paul (1994a): Varieties of Constructivism: Their Metaphors, Epistemologies and Pedagogical Implications. *Hiroshima Journal of Mathematics Education* Vol. 2, 1-14.

Kamii, Constance (1985): *Young Children Reinvent Arithmetic. Implications of Piaget's Theory.* New York : Teachers College Press.

Kamii, Constance (1989): *Young Children Continue to Reinvent Arithmetic. 2nd Grade. Implications of Piaget's Theory.* New York: Teachers College Press.

Kamii, Constance (1994): *Young Children Continue to Reinvent Arithmetic. 3rd Grade. Implications of Piaget's Theory.* New York: Teachers College Press.

Kamii, Constance; Lewis, Barbara A. & Livingston, Sally Jones (1993/94): Primary Arithmetic: Children Inventing Their Own Procedures. *Arithmetic Teacher* Vol. 41 No. 4, 200 - 203.

Murray, Hanlie; Olivier, Alwyn & Human, Piet (1994): Fifth Graders' Multi-Digit Multiplication and Division Strategies after Five Years' Problem-Centered Learning. *Proceedings of the Eighteenth Interntional Conference for the Psychology of Mathematics Education*. Vol. 3, pp. 399 - 406. Lisbon: University of Lisbon.

Narode, Ronald; Board, Jill & Davenport, Linda (1993): Algorithms Supplant Understanding: Case Studies of Primary Students' Strategies for Double-Digit Addition and Subtraction. In Becker, Joanne R. & Pence Barbara J. (eds.). *Proceedings of the Fifteenth Annual Meeting, North American Chapter of the International Group for the Psychology of Mathematics Education* Vol I. Pacific Groves, Ca (USA).

Plunkett, Stuart (1979): *Decomposition and All that Rot. Matthematics in School* (8)2, 2-5.

Reys, Barbara J. (1991): *Developing Number Sense in the Middle Grades.* Curriculum and Evaluation Standards for School Mathematics. Addenda Series Grades 5 - 8. Reston, Va (USA): National Council of Teachers of Mathematics.

Shuard, Hilary et al. (1991): *Prime. Calculators, Children and Mathematics.* London: Simon & Schuster.

Vermeulen, Nelis; Olivier, Alwyn & Human Piet (1996): Students' Awareness of the Distributive Property. *Proceedings of the Eighteenth Interntional Conference for the Psychology of Mathematics Education.* Vol. 4, pp. 379 - 386. Valencia: University of Valencia.

STUDENT INTERPRETATIONS OF A DYNAMIC GEOMETRY ENVIRONMENT

Keith Jones
Research and Graduate School of Education, University of Southampton, Southampton SO17 1BJ, United Kingdom.
dkj@southampton.ac.uk

Abstract: *It seems that aspects of student interpretations of computer-based learning environments may result from the idiosyncrasies of the software design rather than the characteristics of the mathematics. Yet, somewhat paradoxically, it is because the software demands an approach which is novel that its use can throw light on student interpretations. The analysis presented in this paper is offered as a contribution to understanding the relationship between the specific tool being used, in this case the dynamic geometry environment Cabri-Géomètre, and the kind of thinking that may develop as a result of interactions with the tool. Through this analysis a number of effects of the mediational role of this particular computer environment are suggested.*
Keywords: -

1. Introduction

There is considerable evidence that learners develop their own interpretations of the images they see and the words they hear. This evidence also suggests that, although individuals form their own meanings of a new phenomenon or idea, the process of creating these meanings is embedded within the setting or context and is mediated by the forms of interaction and by the tools being used. Such considerations have recently been turned to examining student learning within dynamic geometry environments (DGEs), as such tools have become more widely available (for example, Laborde and Capponi 1994, Hölzl 1996, Jones 1997).

An important issue in mathematical didactics, particularly given the abstract nature of mathematical ideas, is that student interpretations may not coincide with the intentions of the teacher. Such differences are sometimes referred to as “errors” (on behalf of the students) or “misconceptions”, although this is not the only possible interpretation (see, for example, Smith et al 1993). A key perspective on these differences in interpretation, and the theme of this paper, is highlighted by Brousseau (1997 p82), “errors ... are not erratic or unexpected ... As much in the teacher’s functioning as in that of the student, the error is a component of the meaning of the acquired piece of knowledge” (emphasis added). This indicates that we, as teachers, should expect students to form their own interpretations of the mathematical ideas they meet and that their ideas are a function of aspects of the learning environment in which they are working. Within a dynamic geometry environment, Ballachef and Kaput (1996 p 485) suggest, student errors could be a mixture of true geometric errors and errors related to the student’s understanding of the behaviours of the learning environment itself (based on an examination of work by Bellemain and Capponi 1992 and Hoyles 1995).

The focus for this paper is the interpretations students make when working with a dynamic geometry environment (DGE), in this case Cabri-Géomètre, particularly their understanding of the behaviours of the learning environment itself. One of the distinguishing features of a dynamic geometry package such as Cabri is the ability to construct geometrical objects and specify relationships between them. Within the computer environment, geometrical objects created on the screen can be manipulated by means of the mouse (a facility generally referred to as ‘dragging’). What is particular to DGEs is that when elements of a construction are dragged, all the geometric properties employed in constructing the figure are preserved. This encapsulates a central notion in geometry, the idea of invariance, as invariance under drag.

This paper reports on some data from a longitudinal study designed to examine how using the dynamic geometry package Cabri-Géomètre mediates the learning of certain geometrical concepts, specifically the geometrical properties of the ‘family’ of quadrilaterals. In what follows I illustrate how the interpretation of the DGE by students is a function of aspects of the computer environment. I begin by outlining the theoretical basis for this view of DGE use as tool mediation.

2. Theoretical framework

The concept of tool mediation is central to the Vygotskian perspective on the analyses of cognitive development (Wertsch 1991). The approach suggested below begins with the assumption that tools and artifacts are instruments of access to the knowledge, activities, and practices of a given social group (an example of such an approach is given by Lave and Wenger 1991). Such analyses indicate that the types of tools and forms of access existent within a practice are intricately interrelated with the understandings that the participants of the practice can construct.

This suggests that learning within a DGE involves what Brousseau refers to as a dialectical interaction, as students submit their previous knowings to revision, modification, completion or rejection, in forming new conceptions. The work of Meira (1998) on using gears to instantiate ratios, for example, challenges the artifact-as-bridge metaphor, in which material displays are considered a link between students' intuitive knowledge and their mathematical knowledge (taken as abstract). Meira notes that the sense-making process takes time and that even very familiar artifacts (such as money) are neither necessarily nor quickly well-integrated in the students' activities within school. Cobb (1997 p170) confirms that tool use is central to the process by which students mathematize their activity, concluding that "anticipating how students might act with particular tools, and what they might learn as they do so, is central to our attempts to support their mathematical development".

This theoretical framework takes the position that tools do not serve simply to facilitate mental processes that would otherwise exist, rather they fundamentally shape and transform them. Tools mediate the user's action - they exist between the user and the world and transform the user's activity upon the world. As a result, action can not be reduced or mechanistically determined by such tools, rather, such action always involves an inherent tension between the mediational means (in this case the tool DGE) and the individual or individuals using them in unique, concrete instances. Such theoretical work suggest some elements of tool mediation which can be summarised as follows:

1. Tools are instruments of access to the knowledge, activities and practices of a community.

2. The types of tools existent within a practice are interrelated in intricate ways with the understandings that participants in the practice can construct.
3. Tools do not serve simply to facilitate mental processes that would otherwise exist, rather they fundamentally shape and transform them.
4. Tools mediate the user's action - they exist between the user and the world and transform the user's activity upon the world.
5. Action can not be reduced or mechanistically determined by such tools, rather such action always involves an inherent tension between the mediational means and the individual or individuals using them in unique, concrete instances.

Examples of mathematics education research which make use of the notion of tool mediation include Cobb's study of the 100 board (Cobb 1995), Säljö's work on the rule of 3 for calculating ratios (Säljö 1991), and Meira's examination of using gears to instantiate ratios (Meira 1998).

Applying such notions to learning geometry within a DGE suggests that learning geometrical ideas using a DGE may not involve a fully 'direct' action on the geometrical theorems as inferred by the notion of 'direct manipulation', but an indirect action mediated by aspects of the computer environment. This is because the DGE has itself been shaped both by prior human practice and by aspects of computer architecture. This means that the learning taking place using the tool, while benefiting from the mental work that produced the particular form of software, is shaped by the tool in particular ways.

3. Empirical study

The empirical work on which the observations below are based is a longitudinal study examining how using the dynamic geometry package Cabri-géomètre mediates the learning of geometrical concepts. The focus for the study is how "instructional devices *are actually used and transformed by students in activity*" (Meira 1998, emphasis added) rather than simply asking whether the students learn particular aspects of geometry "better" by using a tool such as Cabri.

The data is in the form of case studies of five pairs of 12 years old pupils working through a sequence of specially designed tasks requiring the construction of various quadrilaterals using Cabri-géomètre in their regular classroom over a nine month period. Students were initially assessed at van Hiele level 1 (able to informally analyse figures) and the tasks designed to develop van Hiele level 2 thinking (able to logically interrelate properties of geometrical figures), see Fuys et al (1988). The version of Cabri in use was Cabri I for the PC. Sessions were video and audio recorded and then transcribed. Analysis of this data is proceeding in two phases. The first phases identified examples of student interpretation as a function of tool mediation, a number of which are illustrated below. The second phase, currently in progress, is designed to track the genesis of such tool mediation of learning.

4. Examples of student interpretations

Below are four examples of extracts from classroom transcripts which reveal student interpretations of the dynamic geometry environment.

4.1 Example 1

Student pair Ru and Ha are checking, part way through a construction, that the figure is invariant when any basic point is dragged.

Ru	Just see if they all stay together first.
Ha	OK.
Ru	Pick up by one of the edge points. [H drags a point]
Ha & Ru (together)	Yeah, it stays together!

In this example the students use the phrase "all stay together" to refer to invariance and the term "edge point", rather than either radius point (or rad pt as the drop-down menu calls it) or circle point (as the help file calls that form of point), to refer to a point on the circumference of a circle.

It is worth reflecting that in the implementation of Cabri I the designers found it necessary to utilise a number of different forms of point: basic point, point on object, (point of) intersection - not to mention midpoint, symmetrical point, and locus of points, plus centre of a circle and also rad pt (radius point) and circle point (a term used in the onscreen help). In addition, there are several forms of line: basic line, line segment, line by two pts (points) - not to mention parallel line, perpendicular line, plus perpendicular bisector, and (angle) bisector, and two different forms of circle: basic circle, circle by centre & rad pt. With such a multitude of terminology, it may not be totally unexpected that students invent their own terms.

4.2 Example 2

Pair Ho and Cl are in the process of constructing a rhombus which they need to ensure is invariant when any basic point used in its construction is dragged. As they go about constructing a number of points of intersection, one of the students comments:

Ho A bit like glue really. It's just glued them together.

This spontaneous use of the term "glue" to refer to points of intersection has been observed by other researchers (see Ainley and Pratt 1995) and is all the more striking given the fact that earlier on in the lesson the students had confidently referred to such points as points of intersection. Hoyles (1995 pp210-211) also provides evidence of the difficulty students have with interpreting points of intersection.

4.3 Example 3

Pair Ru and Ha are about to begin constructing a square using a diagram presented on paper as a starting point (see Appendix B). The pair argue about how to begin:

Ha If ...I .. erm ..
I reckon we should do that circle first [pointing to the diagram on paper].
Ru Do the line first.
Ha No, the circle. Then we can put a line from that centre point of the circle [pointing to the diagram on paper].

Ru Yeah, all right then.

Ha You can see one .. circle there, another there and another small one in the middle [pointing to various components of the diagram on paper].

The student pair had, in previous sessions, successfully constructed various figures that were invariant under drag including a rhombus and, prior to that, a number of arrangements of interlocking circles (see Appendix A). In particular they had successfully constructed a rhombus by starting with constructing two interlocking circles. Following the above interchange they followed a very similar procedure. The inference from the above extract of dialogue is that previous successful construction with the software package influences the way learners construct new figures.

An influence here might well be the sequential organisation of actions in producing a geometrical figure when using Cabri. This sequential organisation implies the introduction of explicit order of operation in a geometrical construction where, for most users, order is not normally expected or does not even matter. For example, Cabri-géomètre induces an orientation on the objects: a segment AB can seem orientated because A is created before B. This influences which points can be dragged and effectively produces a hierarchy of dependencies in a complex figure (something that has commented on by Balacheff 1996, Goldenberg and Cuoco 1998 and by Noss 1997, amongst others).

4.4 Example 4

Students Ru and Ha have constructed a square that is invariant under drag and are in the process of trying to formulate an argument as to why the figure is a square (and remains a square when dragged). I intervene to ask them what they can say about the diagonals of the shape (in the transcript Int. refers to me).

Ru They are all diagonals.

Int No, in geometry, diagonals are the lines that go from a vertex, from a corner, to another vertex.

Ru Yeah, but so's that, from there to there [indicating a side of the square that, because of orientation, was oblique].

Int That’s a side.

Ru Yeah, but if we were to pick it up like that .... like that. Then they’re diagonals [indicating an orientation of 45 degrees to the bottom of the computer screen].

Student Ru is confounding diagonal with oblique, not an uncommon incident in lower secondary school mathematics (at least in the UK). What is more, the definition provided by me at the time does not help Ru to distinguish a diagonal from a side, while the drag facility allows Ru to orientate any side of the square so as to appear to be oblique (which in Ru’s terms means that it is ‘diagonal’). Of course, such oblique orientation is not invariant under drag, whereas a diagonal of a square is always a diagonal whatever the orientation. This example illustrates that, in terms of the specialised language of mathematics, the software can not hope to provide the range of terms required to argue why the figure is a square, nor could it be expected to do so. Such exchanges call for sensitive judgement by the teacher in terms of how such terminology is introduced, together with judicious use of the drag facility.

5. Some observations on the examples

The examples given above are representative of occurrences within the case studies arising from this research project. A number of comments can be made on these extracts which illustrate how student interpretations of the computer environment is shaped by the nature of the mediating tool. As Hoyles (1995 p211) explains, it is something of a paradoxical situation that student interpretations can be traced to the idiosyncrasies of the software design rather than the characteristics of the mathematics, yet it is just because the software demands an approach which is novel that its use can throw light on student interpretations.

First, it appears that learners find the need to invent terms. In example 1 above, the student pair employ the phrase “all stay together” to refer to invariance and coin the term “edge point” to refer to a point on the circumference of a circle. To some extent this parallels the need of the software designers to provide descriptors for the various different forms of point they are forced to use. Yet research on pupil learning with Logo suggests that learners use a hybrid of Logo and natural language when talking through problem solving strategies (for example, Hoyles 1996). This, I would argue, is one effect of tool mediation by the software environment. The software designers found it necessary to use hybrid terms. As a consequence, so may the students. Further analysis of the data from this study may shed some light on how this hybrid language may foster the construction of meaning for the student and to what extent it could become an obstacle for constructing an appropriate mathematical meaning.

A second instance of the mediation of learning is when children appear to understand a particular aspect of the computer environment but in fact they have entirely their own perspective. In example 2 above it is the notion of points of intersection, In this example, one student thinks of points of intersection as ‘glue’ which will bind together geometrical objects such as lines and circles. This, I would suggest, is an example of Wertsch’s (1991) ‘ventriloquating’, a term developed from the ideas of Bakhtin, where students employ a term such as intersection but, in the process, inhabit them with their own ideas. In other words, it can appear that when students are using the appropriate terms in appropriate ways, they understand such terms in the way the teacher expects. The evidence illustrated by example 2 suggests that students may just be borrowing the term for their own use.

A third illustration of the mediation of learning is how earlier experiences of successfully constructing figures can tend to structure later constructions. In example 3 above, the pair had successfully used intersecting circles to construct figures that are invariant under drag and would keep returning to this approach despite there being a number of different, though equally valid, alternatives.

Following from this last point, a further mediation effect can be that the DGE might encourage a procedural effect with children focusing on the sequence of construction rather than on analysing the geometrical structure of the problem. Thus pair Ru and Ha, rather than focusing on geometry might be focusing rather more on the procedure of construction. This may also be a consequence of the sequential organisation of actions implicit in a construction in Cabri-Géomètre.

A fifth illustration of the mediation of learning within the DGE is that even if the drag mode allows a focus on invariance, students may not necessarily appreciate the significance of this. Thus hoping points of intersection will ‘glue’ a figure together, or that constructing a figure in a particular order will ensure it is invariant under drag, does not necessarily imply a particularly sophisticated notion of invariance.

From the examples given above, a sixth illustration of the mediation of learning is provided by an analysis of the interactions with the teacher (in this case the researcher). The challenge for the teacher/researcher is to provide input that serves the learners' communicative needs. As Jones (1997 p127) remarks "the explanation of why the shape is a square is not simply and freely available within the computer environment". It needs to be sought out and, as such, it is mediated by aspects of the computer environment and by the approach adopted by the teacher.

6. Concluding remarks

In this paper I have suggested some outcomes of the mediational role of the DGE *Cabri-Géomètre*. While such outcomes refer to only one form of computer-based mathematics learning environment, these outcomes are similar to those emerging from research into pupils' learning with Logo (adapted from Hoyles 1996 pp103-107):

1. Children working with computers can become centrated on the screen product at the expense of reflection upon its construction
2. Students do not necessarily mobilise geometric understandings in the computer context
3. Students may modify the figure "to make it look right" rather than debug the construction process
4. Students do not necessarily appreciate how the computer tools they use constrain their behaviour
5. After making inductive generalisations, students frequently fail to apply them to a new situation
6. Students can have difficulty distinguishing their own conceptual problems from problems arising from the way the software happens to work
7. Manipulation of drawings on the screen does not necessarily mean that the conceptual properties of the geometrical figure are appreciated

As Hoyles remarks, such indications are intended to capture some of the general in the specific and thereby generate issues for further research.

None of the above is necessarily a criticism of Cabri. In the implementation of such software, decisions have to be made. Goldenberg and Cuoco (1998), for example, quote Jackiw, a principle designer of the DGE Geometer's Sketchpad as saying that "at its heart 'dynamic geometry' is not a well-formulated mathematical model of change, but rather a set of heuristic solutions provided by software developers and human-interface designers to the question 'how would people like geometry to behave in a dynamic universe?'" The point is that the decisions that are made mediate the learning and influence student interpretations. As Hoyles (1995 p210) writes: "the fact that the software constrains children's actions in novel ways can have rather positive consequences for constructivist teaching. The visibility of the software affords a window on to the way students build conceptions of subject matter". The finding from this study of the dynamic geometry package Cabri-Géomètre may well prove useful both to teachers using, or thinking about using, this form of software and to designers of such learning environments, as well as contribute to the further development of theoretical explanations of mathematics learning.

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7. References

Ainley, J. and Pratt, D. (1995): Mechanical Glue? Children's perceptions of geometrical construction. *Proceedings of the Third British Congress on Mathematical Education. Manchester*. Part I, p 97-104.

Balacheff, N. (1996): Advanced Educational technology: knowledge revisited. In T. T. Liao (ed.), *Advanced Educational Technology: research issues and future potential*. Berlin: Springer.

Balacheff, N. & Kaput, J. (1996): Computer-based Learning Environments in Mathematics. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (eds.), *International Handbook on Mathematics Education*. Dordrecht: Kluwer. Part 1, Chapter 13.

Bellemain F. & Capponi, B. (1992): Spécificité de l'organisation d'une séquence d'enseignement lors de l'utilisation de l'ordinateur. *Educational Studies in Mathematics* 23(1), 59-97.

Brousseau, G. (1997): *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer.

Cobb, P. (1995): Cultural Tools and Mathematical Learning: a case study. *Journal for Research in Mathematics Education* 26(4), 362-385.

Cobb, P. (1997): Learning from Distributed Theories of Intelligence In E. Pehkonen (ed.) *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*. Helsinki, Finland. Volume 2, 169-176.

Fuys, D., Geddes, D. & Tischer, R. (1988): *The Van Hiele Model of Thinking in Geometry Among Adolescents*. Reston, Va. National Council of teachers of Mathematics.

Goldenberg, E. P. & Cuoco, A. (1998): What is Dynamic Geometry? In R. Lehrer & D. Chazan (eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space*. Hilldale, NJ: LEA.

Hölzl, R. (1996): How does 'Dragging' affect the Learning of Geometry. *International Journal of Computers for Mathematical Learning*. 1(2) 169-187.

Hoyles, C. (1995): Exploratory Software, Exploratory Cultures? In A. A. DiSessa, C. Hoyles & R. Noss with L D Edwards (eds.), *Computers and Exploratory Learning*. Berlin: Springer.

Hoyles, C. (1996): Modelling Geometrical Knowledge: the case of the student In J-M. Laborde (ed.), *Intelligent Learning Environments: the case of geometry*. Berlin: Springer-Verlag.

Jones, K. (1997): Children Learning to Specify Geometrical Relationships Using a Dynamic Geometry Package. In E. Pehkonen (ed.) *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*. Helsinki, Finland, Volume 3, 121-128.

Laborde C. & Capponi B. (1994): Cabri-géomètre Constituant d'un Milieu pour L'Apprentissage de la Notion de Figure Géométrique. *Recherches en didactique des mathématiques* 14(1), 165-210.

Lave, J. & Wenger, E. C. (1991): *Situated Learning: legitimate peripheral participation.* New York: Cambridge University Press.

Meira, L. (1998): Making Sense of Instructional Devices: the emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education*. 29(2), 121-142.

Noss, R. (1997): Meaning Mathematically with Computers. In T. Nunes & P. Bryant (eds.), *Learning and Teaching Mathematics: an international perspective*. Hove: Psychology Press.

Säljö, R (1991): Learning and Mediation: fitting reality into a table. *Learning and Instruction* 1(3), 261-272.

Smith, J., diSessa, A. & Roschelle, J. (1993): Misconceptions Revisited: a constructivist analysis of knowledge in transition. *Journal of the Learning Sciences*. 3(2), 115-164.

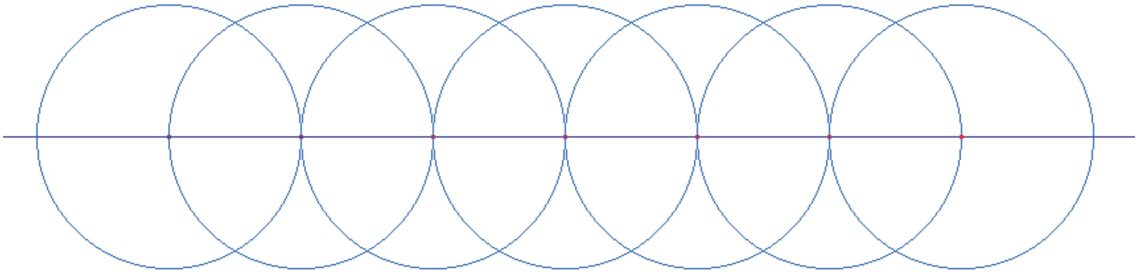
Wertsch, J. V. (1991): *Voices of the Mind: a sociocultural approach to mediated action*. London: Harvester.

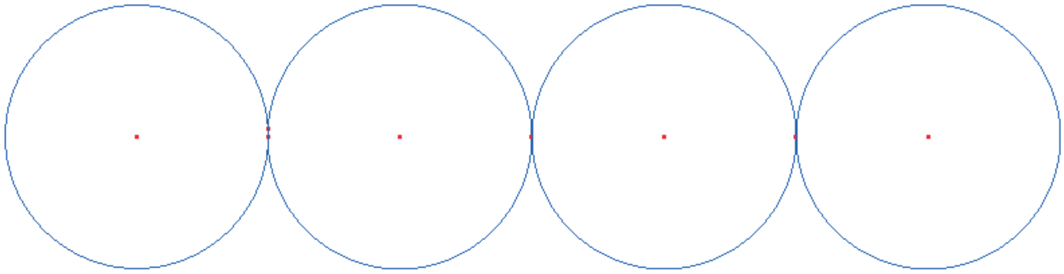
Note

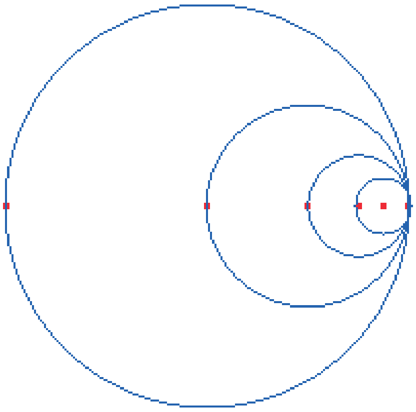
In the appendices that follow, the use of the phrase 'cannot be "messed up"' rather than 'invariant under drag' is based on the suggestion of Healy, L, Hoelzl, R, Hoyles, C, & Noss, R (1994). Messing Up. *Micromath*, 10(1), 14-16.

Appendix A: a task undertaken by pupils during their introduction to *Cabri-Géomètre*

Lines and Circles
Construct these patterns so that they cannot be “messed up”.
In each case, write down how you constructed the pattern.





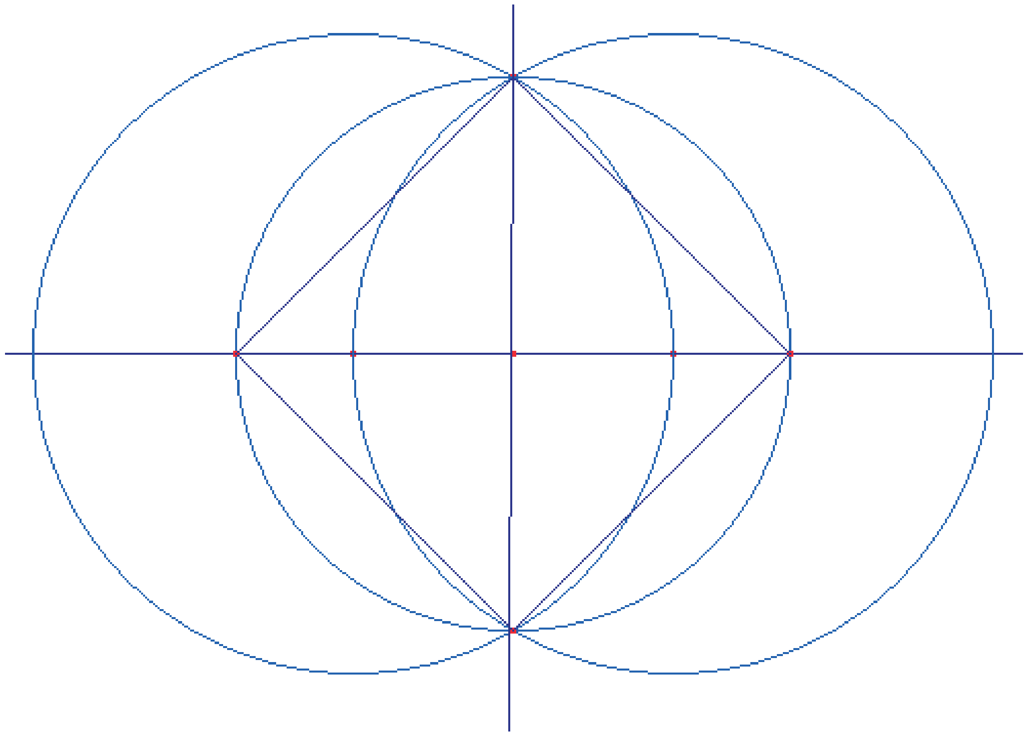


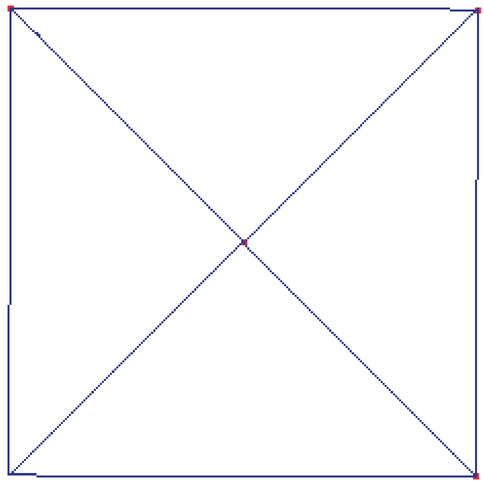
Now construct some patterns of your own using lines and circles.
Make sure you write down how you constructed them.

Appendix B: a task asking pupils to construct a square that is invariant under drag.

The Square

Construct these figures so that they cannot be “messed up”.





What do you know about this shape from the way in which you constructed it?

Think about ..... sides
diagonals

Explain why the shape is a square.

USING PLACE-VALUE BLOCKS OR A COMPUTER TO TEACH PLACE-VALUE CONCEPTS

Peter Price

Centre for Mathematics and Science Education, Queensland University of Technology, Brisbane, Australia

pprice@storm.fed.qut.edu.au

Abstract: *Since place-value blocks were introduced in the 1960s they have become the predominant material of choice for teaching place-value concepts to young school students. Despite their apparent advantages for modelling multidigit numbers, some students who use them still develop faulty conceptions for numbers. A study currently in progress in Australia is comparing the use of computers and of place-value blocks by Year 3 students as they learned place-value concepts. Data analysed to date reveal important differences in the behaviour of students in the two groups. Students using the software tended to focus on the quantities being modelled, whereas students using blocks spent large amounts of time counting and re-counting the blocks.*

Keywords: -

1. Introduction

Place-value concepts are a key foundation for many areas of the mathematics curriculum in schools. Understanding of the base-ten numeration system is a necessary prerequisite for work in computation and measurement in particular, and underlies the use of multidigit numbers in any application. Resnick (1983, p. 126) pointed out the importance of this understanding and the difficulty teachers have in teaching it with her comment that “The initial introduction of the decimal system and the positional notation system based on it is, by common agreement of educators, the most difficult and important instructional task in mathematics in the early school years.”

It is clearly important that teachers effectively assist their students to develop place-value concepts, with a minimum of faulty or limited conceptions. Sadly, however, research evidence shows that frequently students *do* develop conceptions of numbers that are not accurate or complete, and this appears to affect their competence with computation and problem-solving. This paper is a report of a research study presently in progress. The study involves Year 3 students at an Australian school, who used either conventional place-value blocks or a computer software package as they answered questions involving two- and three-digit numbers.

2. Physical materials as models for numbers

Because of the abstract nature of numbers, it is necessary to use physical models of some sort to enable discussions with young children about number properties and relationships. A wide variety of models have been used over the past few decades, including varieties of commercial or teacher-made counting material, abacuses and play money. Advantages and disadvantages of these various materials have been found, which make some materials more effective than others.

The material that is probably the most frequently used in developing place-value concepts (English & Halford 1995, p. 105) is place-value blocks (known also as Dienes blocks). The reasons for the prevalence of use of place-value blocks relate to the systematic structure of the blocks as a system, and the parallels between the "blocks system" and the base-ten numeration system. The sizes of the blocks are proportional to the numbers represented, so that they form a system of proportional analogues of numbers. Actions on the blocks, such as trading, can be mapped onto actions on numbers, such as regrouping, which are reflected in various computational algorithms.

The generally positive belief in place-value blocks as effective models of numbers has to be tempered by other comments, however. A number of authors have pointed out drawbacks for their use, including misunderstandings of what the blocks actually do in terms of assisting children to picture numbers. First, it has been pointed out (Hunting & Lamon 1995) that the mathematical structure of multidigit numbers is not contained in any material; the structure of the numbers themselves has to be constructed in the mind of the individual student. Thus there is no guarantee that use of physical materials will lead to better understanding of numbers. Baroody (1989, p. 5) added support to this point with his statement that learning can only take place when the "[learning] experience is meaningful to pupils and . . . they are actively engaged in thinking about it”.

Research into the effects of instructional use of place-value blocks has produced equivocal results. Hunting and Lamon (1995) suggested that there exist many variables that may affect results from use of materials, including the type of material, length of time used and teacher training. Thompson (1994) suggested that research studies needed to give attention to the broad picture of the teaching environment to discern reasons for the success or failure of instructional use of concrete materials.

The study reported on here used a descriptive approach to data gathering, relying on videotapes of groupwork sessions to reveal important aspects of the learning environment and its effects on students’ learning.

3. Research design

3.1 Research questions

Two broad questions are addressed in this research:

- Are there differences in children’s development of conceptual structures for multidigit numbers when using two different representational formats (place value blocks and computer software)?
- What differences emerge as children of different ability groups learn place value concepts using two different representational forms and associated processes?

3.2 Method

Four groups of four students were involved in a series of ten groupwork sessions in which they answered questions designed to develop concepts of two- and three-digit numeration. The students had previously learned about two-digit numbers, but not three-digit numbers. The sessions were conducted with a teacher-researcher present, who directed the students to complete tasks on cards, helping and correcting them where necessary. The tasks were representative of tasks reported in the place-value literature, and were of five types: number representation, regrouping, comparing and ordering numbers, counting on and back, and addition and subtraction. The tasks were set at two levels, involving first two-digit, and later three-digit, numbers. Two groups used conventional place-value blocks, and two used a software application, called *Hi-Flyer Maths* (Price, 1997), on two computers. All sessions were audio- and video-taped, and a researcher's journal, students' workbooks and audit files produced by the software were collected as supporting sources of data.

Prior to and after the ten sessions, each student was interviewed individually to ascertain his or her understanding of two- and three-digit numeration concepts. Each interview consisted of 27 questions, divided into eight question types. The questions required participants to demonstrate a number of skills, including representing numbers with place-value blocks, counting forward and backward by 1 or 10, comparing the values represented by pairs of written symbols and solving problems involving novel ten-grouping situations.

The raw data from the groupwork sessions, consisting mostly of audio and video tapes, have been transcribed, and data analysis has commenced. The analysis approach adopted is to pursue a close analysis of all relevant aspects of the learning environments and their relationships with the evident learning by the participants.

3.3 Participants

Participants in the study reported here were selected from the population of Year 3 students (aged 7-8 years) at a primary school in a small Queensland rural town. Participants were selected at random from two pools: students of either high or low mathematical achievement, based on the previous year's *Year Two Diagnostic Net*, a state-wide test used in Queensland to identify students at risk in the areas of literacy and numeracy. Students were assigned to groups of four, matched for ability, on the assumptions that maximum learning would be possible if children in each group were of similar mathematical ability. Each group comprised two girls and two boys. Four groups were used, with one high- and one low-achievement level group using each of the blocks or the computer.

3.4 The software

The software application used in the study has been designed to model multidigit numbers from 1 to 999 using pictures of place-value blocks on screen (Figure 1). The screen blocks can be placed on a place-value chart, labelled “hundreds”, “tens” and “ones”, and counters keep track of the number of blocks put out. The number represented by the blocks can also be shown as a written symbol, and as a numeral expander, that contains labels for the hundreds, tens and ones places that can be individually shown or hidden. The verbal name of the number can also be accessed as an audio recording played through the computer’s soundcard.

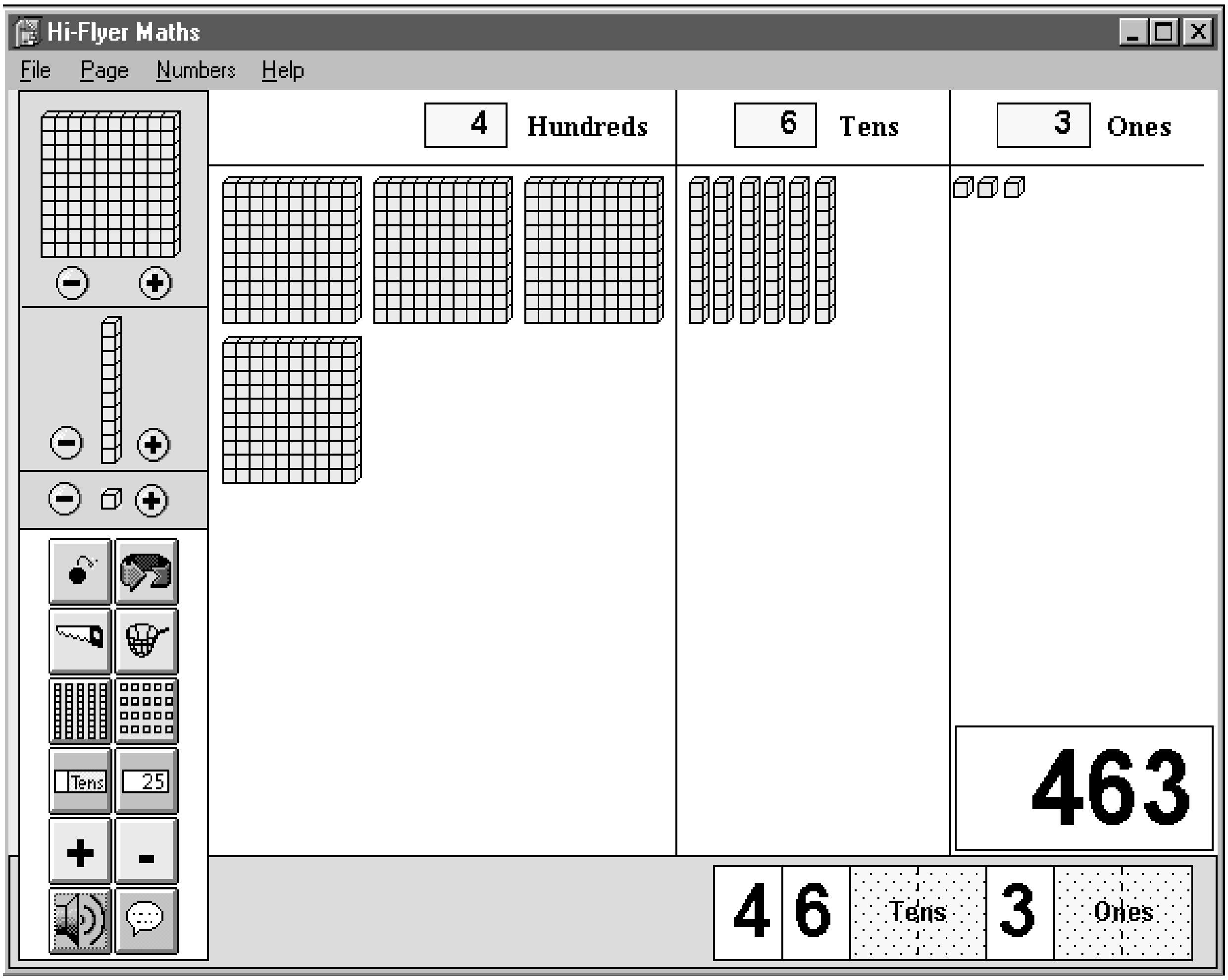


Fig. 1: “Hi-flyer Maths” screen showing block, written symbol and numeral expander representations of the number 463

As well as the representation of numbers, the software enables manipulations of the blocks, that are designed to model actions taken on the represented quantities. A “saw” tool allows a hundred or ten block to be dynamically re-formed into a collection of ten ten blocks or ten one blocks, respectively, which are then moved into the relevant column. In reverse, a “net” tool is used to “catch” ten blocks to be re-formed into one of the next place to the left. By modelling regrouping actions dynamically on screen, it is hoped that the software will assist children to gain better understanding of regrouping processes applied to numbers.

3.5 Results

Transcripts of the two interviews of each participant conducted before and after the groupwork sessions showed that understanding of place-value concepts improved for the majority of participants (see Figure 2). For each of the 27 interview questions a response was coded with a nominal score of 2 if it was completely correct on the first attempt, 1 if correct on the second attempt or if there was a simple miscount, and 0 for all other responses. Thus each participant was awarded two scores with a maximum possible of 54. Figure 2 shows that increases in this measure of understanding of place-value concepts for most students using either blocks or the computer. Of the 16 participants, 11 improved their score from interview 1 to interview 2, 1 had no change, and 4 achieved a lower score on the second interview. The greatest increase was a gain of 12 points, and the biggest decrease was a loss of 5 points. Comparing the two treatments, the median improvement for blocks participants was 7.4%, and for computer participants was 5.6%.

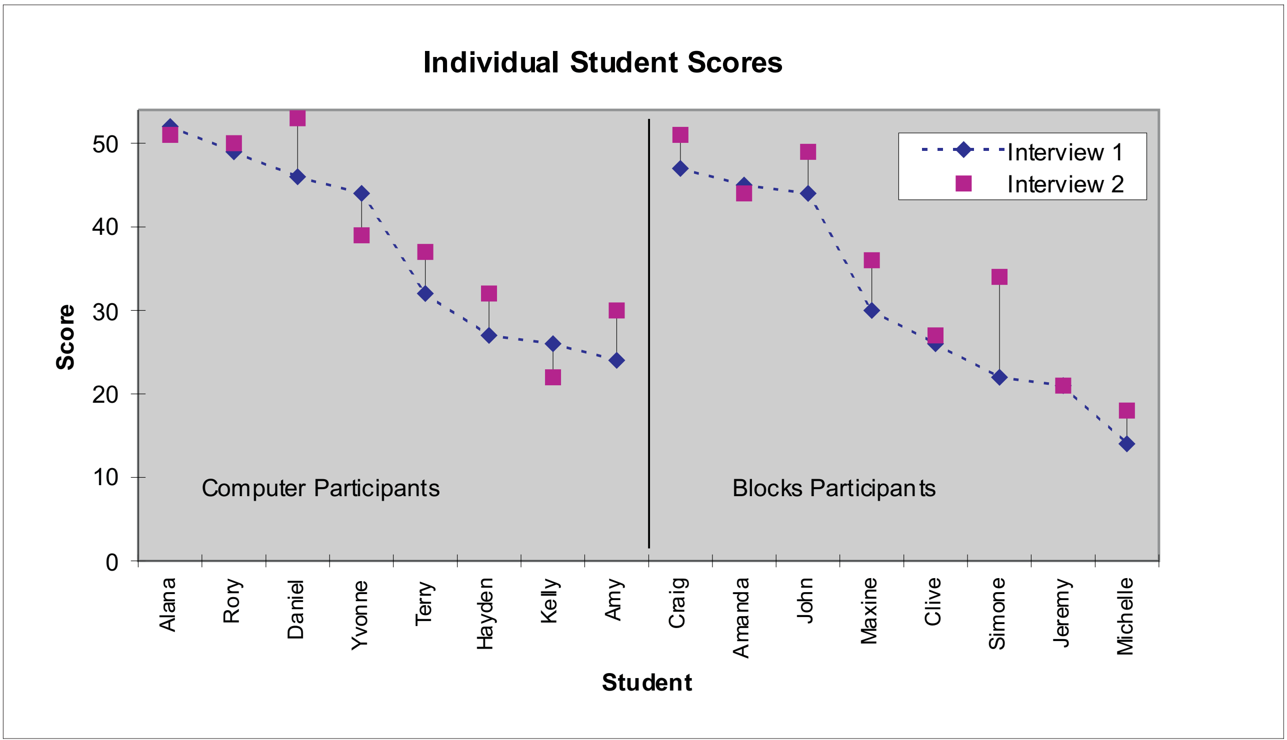


Fig. 2: Summary of First and Second Interview Scores for Individual Participants

4. Discussion

Though the above numerical scores indicate that both computers and place-value blocks were effective for developing understanding of place-value concepts, these scores alone do not indicate important differences that emerged in the use of the two representational formats. Analysis of transcripts has revealed a number of trends in the data that are the subject of further on-going investigation and verification. The most evident trend identified so far, discussed below, is the frequency and nature of counting activity by participants using the two materials. In brief, students using the computer counted much less often than students using blocks.

A simple count was made of the occurrences of the word "count" (including words such as "counting" and "counts") in the first three transcripts for each of the four groups. This showed that in computer group transcripts, the word "count" occurred 16 times in 6 transcripts, whereas in block group transcripts it occurred 244 times. A closer look at these transcripts showed that students using the computers occasionally counted a number sequence aloud, or counted on fingers to work out a simple sum, and on one occasion a student started to count the blocks displayed on screen, until his friend pointed out that the computer would do it for him. In comparison, students in block groups counted blocks as they were first placed in a representation, re-counted subsets of groups of blocks, mis-counted blocks, counted to find the answer to questions relating to the numbers represented, and so on.

The statement that students using blocks were frequently observed to count the blocks comes as no surprise. In the absence of any other means for determining the numbers of blocks selected to represent a number, counting is a necessary adjunct to the use of place-value blocks. Thus a student asked to “show 257 with the blocks” will typically count out 2 hundreds, 5 tens and 7 ones blocks, and place them in a group in front of himself or herself. In comparison, a student using the software needs to click with the computer mouse until the same blocks are visible on the place-value chart on the screen. However, counting is not needed for this task, as the software displays a counter above each column, which shows how many blocks are in each place at any time. For example, it was observed many times that a student using the computer clicked the relevant button too many times, resulting in too many blocks on screen. However, the screen counters enabled the student to notice and correct this quickly, without having to count the blocks.

One aspect of counting activity in students using blocks that is of particular interest is the prevalence of *re*-counting. Students using blocks frequently counted groups of blocks more than once, for a number of apparent reasons. First, students sometimes had to re-start counting because they were distracted or lost count while counting a group of blocks. Second, students frequently re-counted a group of blocks simply to confirm that the result of their previous count was correct. Third, it appeared that students sometimes needed to re-count blocks because they simply forgot how many they had counted previously. It is hypothesised that this may be the result of too great a demand on the students’ cognitive capacity to complete the task at hand.

A worrying aspect of the block group students’ counting behaviour is the fact that often students did not predict the numbers of blocks that ought to have been in place at a given time. Generally it seemed clear that the students did not have a good enough understanding of numeration to make links between block representations and manipulations made on the numbers so represented. For example, if a student had 4 tens and 7 ones, and traded a ten block for ten ones, it was rare for the student to realise that there should be 17 ones. Thus, rather than being able to say how many blocks there should have been after a particular transaction, they needed to count blocks to reveal how many blocks there were. This was particularly so for the low-achievement students, but was also observed in students of high achievement. This difficulty was compounded sometimes when the act of counting caused the student to forget what the task was, as demonstrated in the following vignette. Clive and Jeremy (low achievement students using place-value blocks) were asked to show 58 with blocks, trade a ten block for ten ones, and record how many blocks resulted.

C(live): [Counts ten blocks] 2, 4, 5. [Puts tens down] 8! 58, 58. [Counts out ones two at a time] 2, 4, 6, 8. [Writes in workbook] 58 equals 5 tens and 8 ones.

I(nterviewer): OK, this is 58 now, Jeremy. And Clive's just doing the swap.

C: [Swaps ten for ones, counts them in his hand] 2, 4, 6, whoops, 8, 10. [Puts them on table] Now that means ... [Counts ones] 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58. 58, again. [Pauses, smiles, pauses again.] I need some help.

I: You've done 5 tens and 8 ones, which you've got to write down. How many tens and ones do you have now, Clive?

C: Ah, ooh. That's what I missed. [Starts to count one blocks]

I: Write the tens down first. You know how many tens there are.

C: [Writes in book] 58 equals 4 tens and ... [counts ones] 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17! 17 ones.

Following this transaction Clive argued with the girls in the group about whether there should be 17 or 18 ones. Clearly Clive did not have the understanding to predict that it ought to be 18.

Analysis of the data is still at an early stage. However, analysis is showing that students in the computer groups paid more attention to the quantities involved, and the blocks and written symbols representing them, than their counterparts using blocks. For example, in the following short vignette, Hayden and Terry realised that they did not need to count the on-screen blocks, and "discovered" that after a ten block was regrouped into ten ones, the number represented [77] had not changed:

H(ayden): [Pointing to screen as he counts] 10, 20, 30, 40, 50, 60, ... [Puts up fingers on his left hand as he continues] 61, 62, 63, [Goes back to pointing to screen] 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77.

T(erry): [As Hayden gets to ‘70’] Hey, no! Why didn’t you ask ... [To interviewer] There’s an easier way to do it.

H: [Laughs] Oh, yeah. [Starts to use mouse]

COMPUTER: [audio recording] 77.

H: [To Terry, with surprised look] 77!

T: Oh! We’ve still got ... Oh, cool, that’s easy! [Writes in workbook] Seventy ... 77! [To interviewer] How does it do that? It’s still got 77. [Interviewer looks at him, but does not respond] Oh yeah!

H: [points to screen] It’s still ... You cut it up, and it’s still 77! [Looks at Terry]

It is conjectured that since they did not need to count the blocks, these boys were freed to concentrate on other mathematical aspects of their activities, rather than on the block representations alone.

Though it may appear that the lack of counting carried out by students in the computer groups is of only superficial interest, on the contrary, it appears that this had a critical impact on the students’ conceptualising. The lesser amount of time spent counting apparently had two advantageous effects on the learning environment experienced by these students: increased efficiency and a lowering of the cognitive demand imposed by the tasks. Firstly, by spending less time counting, many tasks were completed more quickly than by students in the blocks groups. Even high-achievement students using blocks spent long periods of time counting, because of mistakes and failing to remember the number of blocks they had counted. By comparison, students using the computer were often able to complete tasks with little difficulty, simply by placing blocks on the screen and referring to the screen display to see the numbers of blocks.

The second advantage for students in the computer groups was that the cognitive demands placed on them were less than for students using the blocks. This is related to, but conceptually different from, the first advantage. By spending less time counting the students were freed to concentrate on the tasks at hand. A short description of students carrying out a task may help to make this point clearer. Suppose that the task required them to "regroup a ten out of 56 and write the numbers that result". Typically, a student using blocks might read the task, count out 5 tens and 6 ones, check the task, remove a ten and swap for ones, re-count the blocks, re-read the task, then record their answer. A student using the computer would typically read the task, use the software to show 5 tens and 6 ones, check the task, click the "saw" cursor on a ten block, observe the result, and record their answer. Tellingly, students using the computer were observed on several occasions to notice that the total number represented was unchanged, and to try to make sense of that fact. On the other hand, students using the blocks frequently made mistakes at some stage, and by the time they wrote their answer, apparently had little idea of what it meant. The actions of (a) counting several quantities and holding them in their minds, and (b) carrying out tasks on the blocks, seemed to cause a number of students to forget what they were doing during the process, and thus to find it harder to make sense of it all. Analysis is continuing; results emerging in this study suggest strongly that appropriate software has the potential to assist students to develop number concepts in ways not possible with conventional physical materials.

5. References

Baroody, A. J. (1989): Manipulatives don't come with guarantees. *Arithmetic Teacher*, 37(2), 4-5.

English, L. D. & Halford, G. S. (1995): *Mathematics education: Models and processes*. Mahwah, NJ: Erlbaum.

Hunting, R. P. & Lamon, S. J. (1995): A re-examination of the rôle of instructional materials in mathematics education. *Nordisk matematikkdidaktikk*, 3.

Price, P. S. (1997): Hi-Flyer Maths (Version 1.5) [Computer software]. Brisbane, Australia: Author.

Resnick, L. B. (1983): A developmental theory of number understanding. In H. P. Ginsburg (ed.), *The development of mathematical thinking* (pp. 109-151). New York: Academic Press.

Thompson, P. W. (1994): Concrete materials and teaching for mathematical understanding. *Arithmetic Teacher*, 41, 556-558.

COURSEWARE IN GEOMETRY (ELEMENTARY, ANALYTIC, DIFFERENTIAL)

Sergei Rakov, V. Gorokh
Kharkov Pedagogical University, Regional Center for New Information Technologies, Bluchera street, 2, Kharkov, Ukraine, 310168
Rakov@usa.net

Abstract: *The paper deals with computer support of teaching and learning in a geometry course for universities and a special course for high school. Four computer packages and activities in their environments are discussed. All the discussions are closely connected with the context of the modern textbooks by Academic A.A. Borissenko of the National Academy of Ukraine.*
Keywords: *-*

1. Introduction

Information technologies have changed all kinds of human activities. Mathematics is no exception - it has become technologically dependent. The most important changes have taken place in the process of doing mathematics - discovering new facts and their proof. Mathematical packages offer the user a suitable environment for undertaking computer experiments to find mathematical regularities as the first step of exploration and then support the process of proof with the powerful opportunities of computer algebra. No doubt that the future of mathematics is in a symbiosis of a human and computer "thinking". Using mathematical packages has become the inalienable component of mathematical culture.

In mathematical education innovative trends lie in the framework of a constructivist approach - involving students in the process of constructing their own mathematical system which consists of mathematical knowledge and beliefs. One of the most effective ways of realizing a constructive approach is in explorations in which students explore open-ended problems on their own. Solving open-ended problems can be regarded as a model of the professional mathematical work. It is therefore natural to use information technologies in mathematical education just in the same manner: computer experiments as the source of powerful ideas, and computer algebra as a tool of the deductive method. Using information technologies to arrange learning explorations and carrying out proofs can not only do this work more effectively but it can acquaint students with modern technologies in mathematics.

We have design a guidebook “Information Technologies in Analytic Geometry Course” to provide computer support for a popular Ukraine textbook by Academic A.Borissenko which is based on several years experience in curricula at the Physics and Mathematics department of the Kharkov State Pedagogical University. It is oriented toward use of the packages Derive, Cabri-Geometry, Geometry-A, Tragecal (the last two are developed by the programmers M.Nicolayevskaya and A.Garmash in the framework of their Ph.D. thesis under the Dr. S.Rakov supervision and are specialized for Analytic and Plane geometry respectively). The kernel of the guidebook is a collections of problems on all the topics of the course. The spectrum of computer use is rather wide: computer algebra for analytic solutions (through the definition of the corresponding hierarchy of functions giving the general solution of the problem), demonstrations (in the main visualization), computer experiments for conceptualization and insight, proofs with the help of symbolic transformations etc. The solution of model problems in each paragraph are given as well as hints, answers and short descriptions of the packages.

We describe below the matter of computer activities for explorations of some geometric problems in the context of some model problems.

2. Examples

Problem 1. Check that four planes given by their general equations:

$P_i : A_i x+B_i y+C_i z+D=0, i=1,2,3,4$

define a tetrahedron.
Find the equation of the bisector line of the trihedral angle formed by the three planes given by i=1,2,3.

Hints for solving Problem 1 in Derive environment:

- Declare the matrix of coefficients of the general equations of the given planes P_i.
- Declare the function TetrTest which checks if the given four planes define a tetrahedron.
- Declare the function TetrVert which returns the set of vertices of the tetrahedron defined by four planes (given by matrix M) in the form of the vector of vectors - $[[x_1,y_1,z_1],..,[x_4,y_4,z_4]]$.
- Declare the function Bisect3D(M,i) which returns the equation of the bisector line of the trihedral angle formed by the three planes given by i=1,2,3.

We give below the protocol of the solution of this problem in the Derive environment with appropriate comments (Derive lines are indented).

Solution of Problem 1 in the Derive environment

Let us take the tetrahedron with vertices: [0,0,0], [1,0,0], [0,1,0,], [0,0,1] as a test. In this case the equations of the planes which define the tetrahedron are: x=0, y=0, z=0, x+y+z=1. The correspondent matrix M of the coefficients of these equations is given in the line #1:

$$\#1: \quad M := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

The function TetrTest in line #2 checks if the given four planes define a tetrahedron:

```
#2:    TetrTest(M):= if(Product(det(minor(m,i,4),i,1,4)*det(M)=0,
                        «Tetrahedron isn't defined», «Tetrahedron is
                        defined»)
```

The meaning of the condition in the function *if* is: each of the four triples of planes uniquely defines the vertex of a tetrahedron and not all the planes belong to the same bundle of the planes.

Let us test the function TetrTest(M) with matrix M(regular case) and singular matrix given directly (lines #3 - #6):

```
#3:    TetrTest(M), <Simplify>
#4:    «Tetrahedron is defined»
#5:    TetrTest([[1,0,0,0], [0,1,0,0], [0,0,1,0], [1,1,1,0]]) , <Simplify>
#6:    «Tetrahedron isn't defined»
```

The auxiliary function System(i) returns the system obtained from the original system avoiding i^{th} equation. The function TetrVertices returns the vector of vertices of the tetrahedron defined by the four planes (given by matrix M) in the form $[[x=x_1,y=y_1, z=z_1],..,[x=x_4,y=y_4,z=z_4]]$.

```
#7:    System(i):=vector((element(M,mod(i+k,4)+1).[x,y,z,1]=0,k,0,2)
#8:    TetrVertices(M):=vector(solve(system((i),[x,y,z],i,1,4)
       Let us test the function TetrVertices(M):
#9:    TetrVertices(M), <Simplify>
#10:   [[x=1,y=0,z=0], [x=0,y=1,z=0], [x=0,y=0,z=1], [x=0,y=0,z=0]]
```

Remark that the way in which the vertices are represented in line #10 not always suitable (for example if we want to use them in further calculations with Derive).

The function TetrVert below returns the set of vertices of the tetrahedron in the form of the vector of vectors: $[[x_1,y_1,z_1],..,[x_4,y_4,z_4]]$. Functions Drop(i,j) and RP(i) are auxiliary. Function Drop(i,j) as a function of j returns the sequence of integers in which the number i is dropped. Function RP(i) returns the vector of the right parts of the system of equations given by the matrix M in which i-th element is dropped.

```
#11:   Drop(i,j):=if(j<i,j,j+1)
#12:   RP(i):=vector(element(element(M,Drop(i,j),4),j,1,3)
#13:   P(i):=minor(m,i,4)-1.(-RP(i))
#14:   TetrVert(M):=vector(P(i),i,1,4)

Let us test the function TetrVert(M):
#15:   TetrVert(M),
#16:   [[1,0,0], [0,1,0], [0,0,1], [0,0,0]]
```

The function Bisect3D(M) returns the equation of the bisector line of the trihedral angle formed by the three planes given by the three first rows of the matrix M. The auxiliary function

Reduce(v,k) avoid k last elements of vector v, function ModV(v) returns the module of vector v. Function D(i) returns expression of the distance between the point with coordinates [x,y,z] and plane P_i given by the i-th row of the matrix M. Function Bisect3D(M) returns the system of equations which defines the bisector line.

```
#17:   Reduce(v,k):=vector(element(v,i),i,1,dimension(v)-k)
#18:   ModV(v):=sqrt(v.v)
#19:   D(i):=sign((element(M,i).Append(P(i),[1]))(element(m,i).[x,y,z,1])/
                ModV(Reduce(element(M,i),1))
#20:   Bisect3D(M):=[D(1)=D(2), D(1)=D(3)]

Let us find the bisector line for our test csase:
#20:   Bisect3D(M), <Simplify>
#21:   [x=y, x=z].
```

Problem 2. Explore the composition of two geometric transformations at the plane. (Tragecal)

The package Tragecal offers the user the tools for discovering the properties of the group of geometric transformations by a series of computer experiments with geometric figures and transformations of them defined in interaction.

The user can work with package TRAGECAL in two modes:

- **command mode**, which offers interactive possibilities in drawing geometric figures and performing their transformations with the help menu;
- **automatic (program) mode**, which offers the possibility of writing algorithms for drawing geometric figures and performing their transformations on a specialized geometrically oriented language OXYGEN.

It ought to be emphasized that from the geometry point of view the possibilities of both modes coincide, and it is one of the cornerstones of the package. Both modes are completely independent and self-sufficient. At the same time the user can performs computer experiments in the command mode at first and then can to program its algorithm without any correction and then perform it automatically. Examples described below and copies of the screens have to clarify the matter of both modes as well as the matter of the program language OXYGEN.

In a framework of the geometry course we propose to students perform learning explorations under the special guide. We give below an example of the exploration “Composition of two symmetries” in Command mode for insight the equivalence of such composition to the translation on the vector with the beginning at the center of the first symmetry and the end at the center of the second symmetry).

Extracts from the students’ guide to exploration “Composition of two symmetries” in the TRAGECAL environment

Fig. 1: Screenshot 1

- Draw an arbitrary figure F_1.
- Copy the figure F_1 to figure F_2 and paint it in another color.
- Mark two arbitrary points as the centers of symmetries and label them as O_1 and O_2.
- Make the figure F_2 active and draw its symmetric image F_2’ of it first with the center O_1 at first and then with the center O_2.
- Explore the possibilities of transforming figure F_1 into figure F_2’.
- Find the simplest way of transforming the figure F_1 into figure F_2’.
- Prove the theorem (or find a counterexample to it).

In Fig. 1 the screen shows the result of the performing the two symmetries.

Similar experiments can be performed with the program Sym2PInt.oxy for checking this theorem and can be used as a demonstration program as well. In this manner students can write the OXYGEN programs for discovering the various geometric conjectures with interactive input of data or with random data.

The set of geometric transformations which can be used in the TRAGECAL is not restricted only to moves (translations, rotations, reflections and symmetries). It includes the dilatation[1] (as a consequence the orthogonal group can be explored), and 'oblateness'[2] (as a consequence the group of Athena transformations can be explored). Moreover there are possibilities for exploring the properties of inversion and any other transformation which can be expressed by an analytic formula.

Problem 3. A second-order surface is given by its equation. Evaluate the type, parameters and canonical equation of the surface, natural parameterization and the matrix of the transformation to the canonical coordinate system.(Geometry-A).

For these purposes the package Geometry-A is supplied with special tools for visualizing and manipulating with geometric objects. Geometry-A offers the user the opportunity to work in two modes: expert (for explorations of the surfaces given by the user) and control (for presenting the theory and testing students' performance in the form of dialog about second-order surfaces, the equation of which are generated in random way).

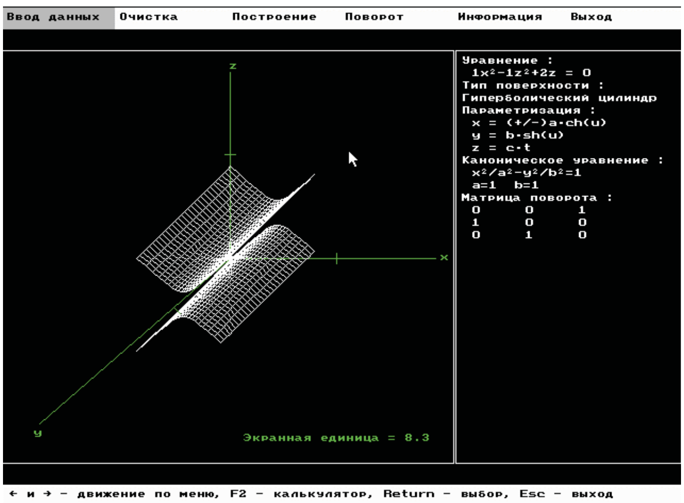


Fig. 2: Screenshot 2

In Fig. 2 the screen of work in expert mode with Geometry-A is shown. The geometric image can be rotated with respect to the axis or directly rotated with the mouse. Information about the surface is given in the right part of the screen. Unfortunately the package has Russian interface. Let us give some comments to the information presented at the screen. The main menu (at the top of the screen) has such devices: Input Data, Screen, Drawing, Rotation, Information, Quit. Information in the right part of the screen includes: Equation, Type of the Surface, Parameterization, Canonical Equation, Matrix of Transformation (from the current to the canonical coordinate system). In the bottom part of the screen the auxiliary available commands are given.

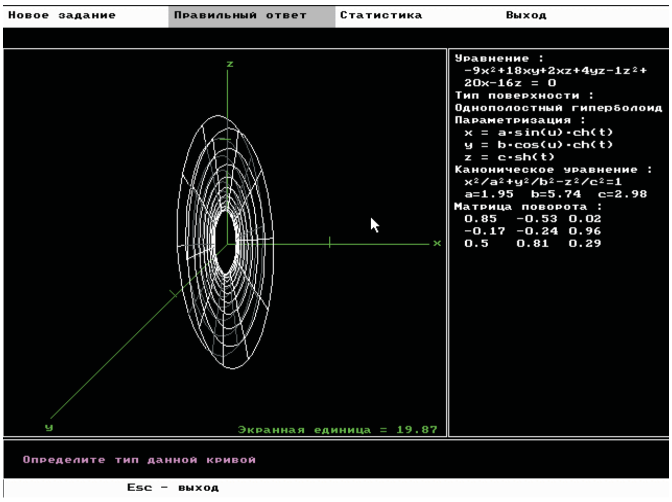


Fig. 3: Screenshot 3

Fig. 3 illustrates control mode is presented (the presentation of the right answer, which has the view of the information presented in the previous Fig. 2). The student is asked to answer step by step questions concerning the object, the equation of which is generated in in random way and presenting at the screen. An in-screen calculator is available as well as a way to transfer data between calculator and the program. The student can skip the difficult question (in this case the program itself gives the right answer). The statistics about the students' work is stored in a file and can be used for self-assessment as well as for teacher evaluation. In control mode the main menu has such devices: New Task, Right Answer, Statistics, Quit.

Problem 4. Two squares $A_1B_1C_1D_1$ and $C_1B_2C_2D_2$ with centers A_3 and C_3, and common vertex in the same orientation are given. Let B_3 and D_3 be the midpoints of the segments B_1B_2 and D_1D_2. Prove that the quadrilateral $A_3B_3C_3D_3$ is a quadrate as well. How can this fact be generalized? (Cabri-Geometry, Derive).

For such kind of problems the package Cabri-Geometry offers the user possibilities to build «dynamic geometrical models». By playing with such models the student (or researcher) can generate productive ideas about the properties of the exploring objects.

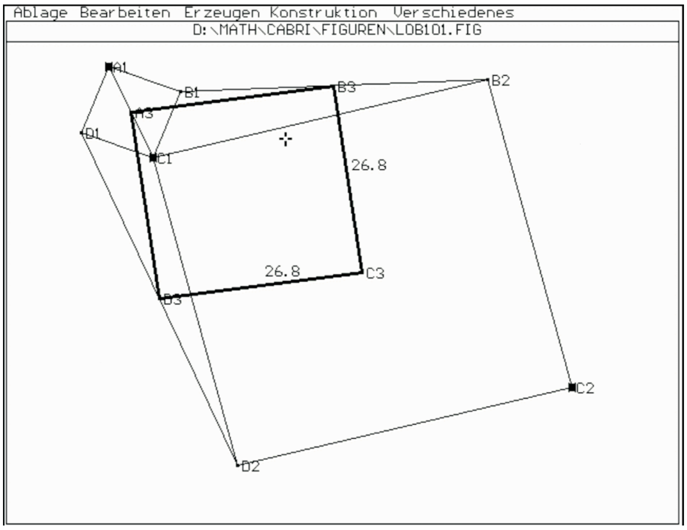


Fig. 4: Screenshot 4

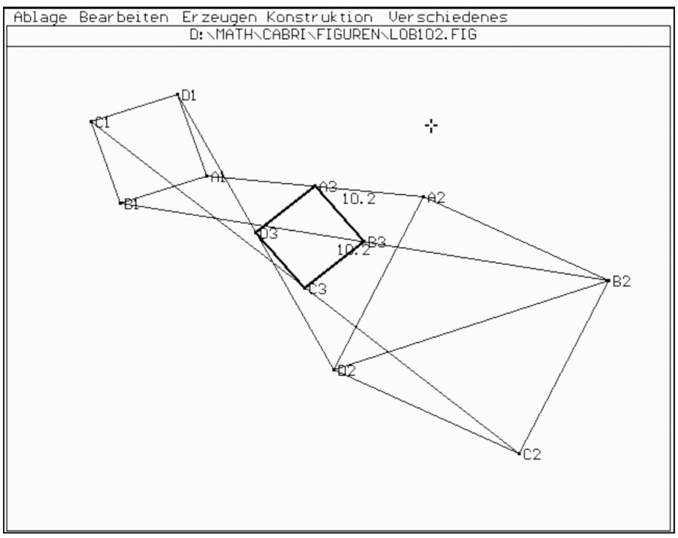


Fig. 5: Screenshot 5

In Fig. 4 the variant of a dynamic model to this problem is shown in Cabri-Geometry. The three based points A_1, C_1 and C_2 can be moved at the screen with the mouse, changing the sizes and the positions of the original squares. The parameters (the length of its sides and the size of the angles) of the resulting quadrilateral $A_3B_3C_3D_3$ are displayed on the screen.

The generalized experiment of this problem is shown in Fig. 5. The two original squares now have no common vertex. Yet the resulting quadrilateral is still a square! The next generalizing idea may be the avoiding of the particular case of the ratio (1/2) in which the vertices of the resulting quadrilateral divide segments joining correspondent points of two original squares. Dragging indicates that the resulting quadrilateral really is a square in all cases. Nevertheless a logical proof is needed - the only way for proving a result in mathematics! It is remarkable that the proof by coordinate analytic method can be done in Derive. The proof of the second discussed above generalization is given below as a protocol of work in Derive with appropriate comments.

Proof of the generalization of the Problem 4 in the Derive environment:

#1: «Let O_1(A, B) and O_2(C,D) are centers of the squares $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$.»
#2: O1:= [A, B]
#3: O2:= [C, D]
#4: n1:= [P, Q]
#5: m1:= [-Q, P]
#6: n2 := [R, S]
#7: m2 := [-S, R]#8:«Express the coordinates of the vertexes of the two original squares»
#9: A1 := O1 + N1
#10: B1:=O1 + M1
#11: C1:=O1 - N1
#12: D1:=O1-M1
#13: A2:=O2 + N2
#14: B2 := O2 + M2
#15: D2 := O2 - M2
#16: C2 := O2 - N2
#17: Divide (x , y , t) : = (x + ty) / (1 + t)
#18: «Define the coordinates of the vertices of the dependent quadrilateral»
#19: A3:= Divide (A1, A2, t)
#20: B3:= Divide (B1, B2, t)
#21: C3:= Divide (C1, C2, t)
#22: D3:= Divide (D1, D2, t)
#23: «The proof that the quadrilateral A3B3C3D3 is a parallelogram»
#24: (A3 - B3) - (D3 - C3) «The simplification of the #24 gives:»

```
#25:   [0, 0]
#26:   «The proof that the quadrilateral is a rectangular:»
#27:   (A3 - B3).(A3 - D3)
#28:   0
#29:   «The proof that the rectangular A3B3C3D3 is a square:»
#30:   ModV(x): = x.x
#31:   SModV(A3 - B3) - SModV(D3 - B3) «By simplification of #31 obtain:»
#32:   0 «OK»
```

Remarks

- Discussed generalizations of the Problem 4 are not the end in the chain of generalizations which could be conjectured as a result of experiments in Cabri-Geometry environment or in any other way. The reader could play in this Math game.
- Analytic proofs done in CAS (in Derive or any other CAS environment) rise very interesting question – are they really proofs? Or they are only experiments, explorations of particular cases and thus they are only preparatory work (as the dragging) and even in these cases the strong logical (deductive) proof is still needed. Maybe the tracing mode of computer transformations which shows transformations step-by-step could be sufficient? All this is the subject of very important and interesting discussions about the matter of Math proof in Technological Age as well as using CAS in Math education.

The most interesting result of our work in preparation of the described above guidebook is that the special selection of problems for solving in computer environment is not needed – almost each of them could be effectively solved in suitable environment with suitable activities.

The discussed guidebook is planned to be issued in 1999.

3. References

Rakov S.A. & Gorokh V.P. (1996): *Computer experiments in geometry*. Kharkov, RCNIT.

Rakov S.A., Olejnick T.A. & Sklyar E.V. (1996): *The package DERIVE in mathematics courses*. Kharkov, RCNIT.

Rakov S.A., Nikolaevskaya M.I. & Nikolaevskiy Y.A. (1994): *Geometry-A – Learning Experiments in Analytic Geometry course*. Kharkov, RCNIT.

Rakov S.A. & Garmash A.Y. (1994): *TRAGECAL – Computer Microworld in Plane Geometry*. Kharkov, RCNIT.

Notes

1. Defined as the geometric transformation of a plane which maps each point of the plane P to P' which lies on the line passing through C and P and the ratio of distances d(P',C) and d(P,C) equals k.
2. We define the 'oblateness' as the geometric transformation of a plane which maps each point P of the plane to P' for which the line PP' is perpendicular to line l and the ratio of distances of these points from the line l equals k.

SEMITRANSPARENT MIRRORS AS TOOLS FOR GEOMETRY TEACHING

Luciana Zuccheri
Department of Mathematical Sciences, University of Trieste, I-34127 Trieste, Italy
zuccheri@univ.trieste.it

Abstract: *The author presents and discusses an experience in Geometry teaching, made in several years in co-operation with teachers of the Didactic Research Group of Trieste. This experience, based on the use of semitransparent mirrors, led to the realization of a Mathematics exhibition with didactic purposes, which is also described.*
Keywords: *-*

1. Laboratory activities and didactic tools

The Italian ministerial programmes for primary and middle school Mathematics (for 6-14 years old pupils) contain many geometrical topics, one particular topic being geometric transformations. As a first approach to this topic, the programmes suggest stimulating the pupils' curiosity and intuition, encouraging them to operate on and manipulate physical apparatus, the recommendation being to use a laboratory for Mathematics activities. Such a Geometry laboratory can be realized without necessarily using computers, by giving the pupils papers, scissors, ink, mirrors and everything necessary to make the first experimental observations about symmetries, folding and cutting papers and making ink spots. Following this suggestion, the teachers of the *Didactics Research Group of Trieste* (which is active in the Department of Mathematical Sciences of the Trieste University since 1970s and is financially supported by the Italian Ministry for University and Scientific and Technological Research and by the Italian National Council for Research) use, in laboratory activities, both electronic instruments (from the simple pocket calculator to the personal computer) with different kinds of software (mainly Cabri for Geometry teaching), and didactic tools made using structured or commonly used materials. In fact the concept of didactic tool is very general: any object may be a didactic tool, if it is used in appropriate way and if it is integrated in a well-planned teaching path. This “definition” of tool is restricted only to material objects, but we could also consider some learning environment as a “didactic tool” in an extended meaning, when this one really improves the learning: this is the case of the Geometry exhibition illustrated in the following, in which many factors (among them, the emotional factor to be exceptionally “out of the school”) stimulate and motivate the pupils to learn.

2. A didactic tool for geometry teaching: the Simmetroscopio

The primary school teacher Bruno Giorgolo, member of the *Didactics Research Group of Trieste* (DRG), often realizes interesting didactic tools, uses them in his classroom activities and proposes them to the Group. Some of these tools are even commercially marketed. The main characteristic of the tools realized by Giorgolo and of the activities which are related with them is that they stress both the affective and playful aspects and the operative ones. One very interesting didactic tool created by Giorgolo around 1987 and continuously developed by him is called the *Simmetroscopio* (see: Giorgolo B. 1992); its first production and marketing in Italy was made by M.C.E. (“Movement of Educational Co-operation”). The *Simmetroscopio* has been used by the DRG since the school year 1987/88. It was presented for the first time in a Italian meeting in Torino in 1989 and since that was illustrated several times in many parts of Italy and to researchers of other countries. The *Simmetroscopio* is a simple and flexible instrument, easily adaptable to didactic itineraries, formed by a set of two or more semitransparent mirrors (made by treated glass) that can be assembled, rotated and shifted in various ways on a support. The support is also a work-desk, with other useful accessories. By such means we can treat a range of geometrical concepts and develop various didactic itineraries about Geometry, to study the fundamental objects and properties of Euclidean Geometry, and various geometrical transformations of the plane or of the space. Of course, it is also possible make significant links with other disciplines, such as Physics, and use this instrument in other activities, for instance the expressive ones. There exist many types of Simmetroscopio, studied for different aged pupils. It is used at any level, from 5 years old pupils to adults. In fact, many teachers of our Group tested it at primary and middle school, we have used it also in courses for teachers and Giorgolo has also adapted it to kindergarten level (see: Giorgolo B. 1997); in this last form it was used in experiments in some municipal kindergartens of Modena (Italy). Below some pictures of the Simmetroscopio are shown; the semitransparent mirror permits not only the reflected image to be seen but also the objects which lie beyond itself. This is useful for many didactic purposes. To stress the semitransparency, in figures 1 and 2 symmetric objects of different colours are posed on opposite sides of the mirrors and we see in each mirror an object coloured by composition. There exist other tools that use the semitransparent effect; see for instance: Lott J.W. 1977.

Fig.1: The basic version of the Simmetroscopio

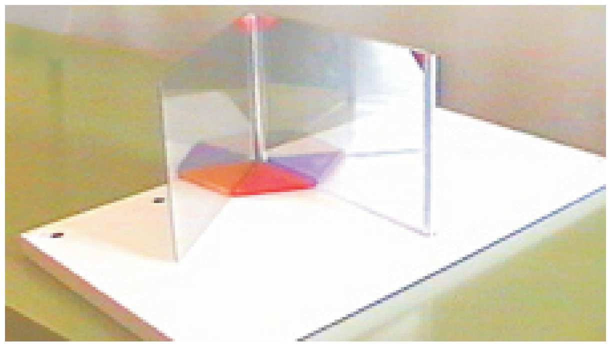


Fig. 2: A version for learning rotations

3. The exhibition "Oltre Lo Specchio" ("Beyond the mirror")

The experiences of the Group using the Simmetroscopio led Giorgolo and me to the planning of an exhibition structured as a laboratory, based on its use (see: Zuccheri L. 1992, II), that is the exhibition "*Oltre lo specchio*" ("*Beyond the mirror*"). It was realized in 1992 in co-operation with the *Laboratory of Scientific Imaginary of Trieste* and from 1992 until 1997 it was visited by many thousands of people. It is a *hands-on* exhibition (the British psychologist Richard Gregory coined this word), that means it is a learning environment, placed outside the school, in which everything shown must incite interest and curiosity and, in contrast to what happens in most traditional museums or exhibitions, everything must be touched (for references about this type of museums and exhibitions, see: Salmi H.S.1993).

"*Beyond the mirror*" was developed as a teaching aid for primary and middle school teachers about geometrical transformations, therefore its contents are the ones contained in the primary and middle school programmes and it is structured by pre-established learning paths. It deals with various geometrical subjects, e.g. reflections, rotations, similarities, projective transformations, and each of them, for a methodological belief tied to the exhibition context and linked to practical problems (such as length of visits, and pupils' attention span), is treated in various deepening levels. The methodology used is that of the guided discovery through the opportune development of activities (see: Zuccheri L. 1996): the exhibition is in fact a Mathematics laboratory in which visitors find paths of various levels of difficulty that conduce them to discover and learn particular Mathematical concepts, by means of observation, object manipulation and drawing. The activities are progressive and they go on from a very concrete operative level to a more abstract one. They are individual, two by two, or group ones. The two by two activities often request that the two pupils interact and that they exchange each other questions, observations and reflections. The various activities are grouped in sections, each of them has a structure conceived to motivate the pupils to learning. Each section is composed of various work stations, with different activities; here there are the instructions cards, explaining the work to do, and all the necessary materials (semitransparent mirrors, structured materials, various objects of common use, operative cards with printed drawing, writing-materials). Usually, a very simple task to do with an amusing individual or group activity (the play element is an essential component of the methodology) is proposed as the first one; very often, during this first activity the pupils make observations about their body, and about phenomena that will be analysed later, re-proposed in concrete objects or drawing. A poster summarizes the essential points, which the section work aimed at, with the aim to fix the fundamental concepts. To achieve the most effective results, various communication levels are used: representative iconic, verbal written, verbal spoken. The last one is the task of the appropriately prepared assistants, who guide exhibition visitors, giving them the necessary directions and suggestions. Some of the main contents and some paths developed in the exhibition are described as follows (see: Zuccheri 1992, I).

- *Reflection symmetries path*: this starts with practical experiences with giant mirrors in three-dimensional space, and then in the plane, observing also the clockwise/anticlockwise rotation change; the concepts are later formalized in two deepening levels (one for primary and one for middle school); the most difficult level conduces the visitors to solve geometrical problems.

- *Rotations and angles path*: this starts with practical experiences in three-dimensional space with a giant kaleidoscope, and then in the plane; the concept of angle measure and the composition of two axial symmetry with intersecting axis are also treated.
- *Translations path*: this is carried out starting from three-dimensional space with the observation of the effect of two big parallel mirrors, and later considering the plane; the formalization of the concepts is given then in two different levels; the most difficult one leads to the proof of the fact that the composition of two symmetries with parallel axes is a translation.
- *Similarities transformations path*: similarity is treated at the operative level for younger pupils, giving them the instructions to reduce or magnify some figures; a deeper level conduces the pupils to understand the underlying geometrical concepts.

4. Some examples

In this section we describe some exercises among those presented to the Thematic Group “Tools and Technologies in Mathematical Didactics” and during the Poster Session. They belong to the set of exercises from the exhibition “*Beyond the mirror*” (see: Zuccheri L. 1995). Only the kernel concepts of each exercise are sketched.

4.1 An exercise that mathematicians like

Given drawings like in figure 3 the pupils have to complete them with respect to their symmetry axis, on both the sides, using the semitransparent mirror. The piece of paper must be arranged below the mirror, overlapping the axis of symmetry with the bottom border of the mirror, and cannot be moved. Then the pupils operate on both sides of the mirror, copying the images that they see in the mirror. From psychomotorial point of view, we observe that it is easy to copy a drawing on the side beyond the mirror, whereas it is difficult to copy a drawing on our side of the mirror. To make this task easier, we could permit the pupils to rotate the paper under the mirror and to draw always beyond it. The mathematicians like this exercise, especially in the version described before, because this type of activity stresses the mathematical model of symmetry as transformation of the whole plane (i.e. one-to-one correspondence of the plane, as set of points, into itself); other implementations of this model (i.e. paper folding, normal mirrors) don't work so well, also from psychological and perceptive point of view.

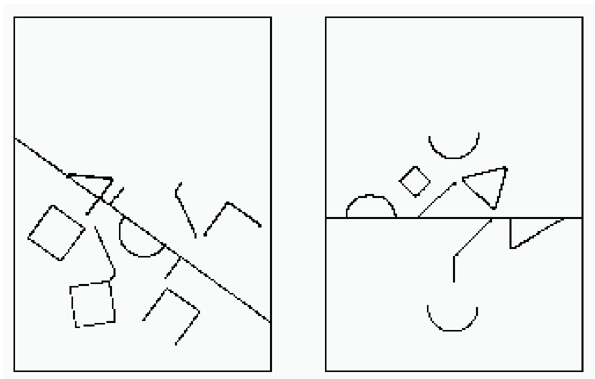


Fig. 3: Some operative cards from the exhibition "*Beyond the mirror*"

4.2 A path for describing the composition of two planar axial symmetries with parallel axes

The pupil proceeds starting from step 1 and he or she stops when the selected difficulty level adapted is reached. The difficulty level is linked to the age and to the previous knowledge of the pupil. Usually steps from 1 to 4 are for primary school pupils (steps 3-4 only for the last year), whereas the pupils of middle school can continue up to the end.

- *Step 1: Observation in three-dimensional space*. The pupil observes the effect of reflection of some object or of the body of a classmate in two giant parallel mirrors.

- *Step 2: Observation in the plane*. The pupil observes the effect of the reflection of a drawing representing a car (like figure 4.a), which is posed on the table, between two parallel mirror; it seems to be riding toward one of them; the pupil looks in this mirror and observes the various different images which are formed by the reflections. The first image seems to be a car which is going toward the real car figure, whereas the second image seems to ride in the same direction of the real car figure. The bottom borders of the two mirrors are depicted in different colours (red and green) and this is useful to represent the symmetry axis in the plane of the table.
- *Step 3: Concrete experience to link the empirical notion of translation as motion with that of translation as geometrical transformation of the plane*. The pupil repeats the previous observations using semitransparent mirrors and a cardboard model which represents a short motor racing track (see figure 4.b). A transparent figure of a car can be moved along the track, starting from the initial position, in which it overlaps an identical car figure. First the pupil puts the cardboard model on the table, between the mirrors (and the two cars overlap each other), then he moves the transparent car passing under one of the mirrors and looking in it. When the moving car passes over the first reflected image, he observes that the car doesn't overlap it (the reflection is an opposite congruence). When the moving car passes over the second image, the pupil observes that the car perfectly overlaps it (the reflection of the first reflected image is the translated image of the car in the initial position, and the translation is a direct congruence). Finally the pupil measures by a rule the distance between the two mirrors, and the distance between the starting and ending lines of the track, which are drawn in the model. He or she compares them and observes that the track is twice as long as the distance between the mirrors.
- *Step 4: First generalization*. To avoid the conceptual error of confusing the translation direction with some preferential direction suggested by the figure itself (the simplified previous model uses a car which is directed in the translation direction) the pupil considers a drawing like figure 4.c, puts it on the table posing the car figure between the two semitransparent mirrors as before, and copies the first two reflected images which he sees in one of the mirrors. Then he or she measures by the rule the distances between various points of the printed drawing and the correspondent ones on the second drawing that he has made. The pupil

observes that the distance is always the same (and that it is the same as found before). The idea of this strange movement is given to the pupil saying that the car is moved by a strong gust of "Bora" (the strong wind of Trieste).

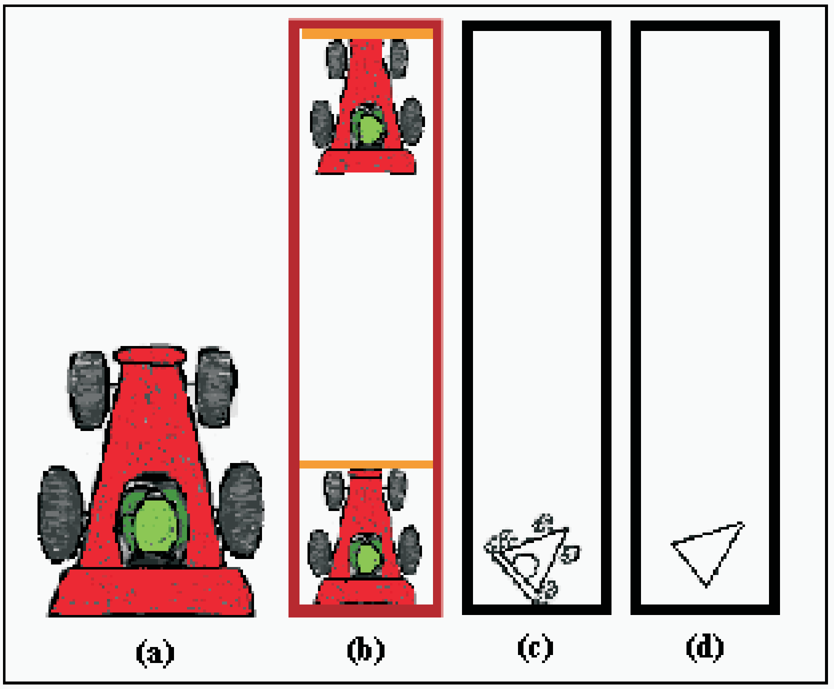


Fig. 4: Various steps of the translation paths

- *Step 5: Formalization*. The figure 4.c becomes more schematic, like figure 4.d. The previous exercise is repeated with figure 4.d, using semitransparent mirrors and drawing the two first triangles reflected into one of the mirrors. The vertices are labelled writing A, B, C on the printed figure, A',B',C' on the correspondent points of the first reflected image, A", B", C" on the correspondent points of the second image. The symmetry axes are depicted copying the bottom border of the mirror (red, we suppose) and the first green line which appears beyond the mirror.
- *Step 6: Problem posing*. The pupil observes that, if the paper sheet (on which he or she has just drawn) is moved in a direction perpendicular to the mirrors, then the first image does not overlap the triangle A'B'C', whereas the second figure always overlaps the triangle A"B"C". We ask the pupil why this fact happens.
- *Step 7: The solution*. The guided discovery path ends with the explanation of the phenomenon just observed (from mathematical point of view). The pupil reads on

the instruction card the correct explanation, i.e. the proof of the fact that the composition of two axial symmetry with parallel axes is a translation, with given direction and length. The pupil can compare his or her own reasoning with this explanation. The explanation is given using the formal symbols we have introduced before and geometrical arguments which involve the general properties of axial symmetries. This is a real mathematical proof because the reasoning is valid for any point of the plane.

5. Conclusion

In the experiences carried out by the Didactics Research Group of Trieste using the Simmetroscopio, it seems clear that the semitransparent mirrors offer a pleasant and immediate approach, both to children and to adults. Such mirrors give the pleasure of discovery, make the intuition of a range of geometrical concepts easier and, especially, can be use by students in an autonomous way, since they allow self-correction, with a remarkable feedback effect. The didactic power of the semitransparent mirror of Simmetroscopio is that it provides the possibility of seeing a reflected image and, at the same time, acting over this image, copying it beyond the mirror, or modifying it, drawing on our side of the mirror. This tool, integrated with the didactic paths of the Geometry laboratory "*Beyond the mirror*", revealed itself very effective for many purposes, and not only for the basic ones for which the exhibition was carried out. In fact the exhibition was also visited by classrooms of higher school level, was used fruitfully to up-date teachers of primary and middle school, and to prepare future teachers in a University Didactics course. The courses for teachers allowed the teachers to continue, in their classrooms, the development and the deepening of the geometrical concepts illustrated in the exhibition and various written materials were prepared for this purpose (see references below). The exhibition assistants often made reports about their work in the exhibition and this was fruitful to improve some exercises. The most meaningful practical result achieved was that many teachers integrated their Mathematics curricula with various visits of the same class, in different times, to the exhibition itself. The exhibition was appreciated from experts of this field; it was mentioned in the report about Italian Research in Mathematics Education presented at ICME 8 (Seville, 1996) (see: Fiori C, Pellegrino C. 1996; M.Marchi, A.Morelli &

R.Tortora 1996).

Acknowledgement

I'm grateful to Keith Jones for his helpful suggestions concerning this paper.

6. References

Fiori C. & Pellegrino C. (1996): The conceptual and the Popular Images of Mathematics. In N.A. Malara, M. Menghini & M. Reggiani (eds.), *Italian research in Mathematics Education 1988-1995*, 176-191, Roma: Consiglio Nazionale delle Ricerche.

Giorgolo B. (1992): Il simmetroscopio e le sue utilizzazioni didattiche. *Atti del Convegno Media e Metodi 3°* (Trieste, 6-7 maggio 1992), Trieste: Tipografia Moderna, 151-156.

Giorgolo B. (1997): *La fiaba della geometria*. COOPEDIT, Ronchi dei Legionari (GO)

Lott J.W. (1977): What can be done with a Mira. *Journal Math. Teacher*, V. 70(5), 394-399.

M. Marchi, A.Morelli & R.Tortora (1996): Geometry: the Rational Aspec s. In N.A. Malara, M. Menghini & Reggiani M. (eds.), *Italian research in Mathematics Education 1988-1995*, 55-73. Roma: Consiglio Nazionale delle Ricerche.

Salmi H.S. (1993): *Science Centre Education. Motivation and Learning in Informal Education*. Research Report 119, Department of Teacher Education, University of Helsinki.

Zuccheri L. (1992): *Guida alla mostra-laboratorio "Oltre lo specchio"*. Quaderno Didattico n.14, Dipartimento di Scienze Matematiche, Università di Trieste

Zuccheri L. (1992): "Oltre lo specchio", storia e motivazioni di un'esposizione didattica. *Atti del Convegno Media e Metodi 3°* (Trieste, 1992), Trieste: Tipografia Moderna, 143-150.

Zuccheri L. (1995): *Rapporto tecnico: le schede di lavoro della mostra-laboratorio "Oltre lo specchio"*. Quaderno Didattico n.29, Dipartimento di Scienze Matematiche, Università di Trieste

Zuccheri L. (1996): *Note per gli animatori della mostra-laboratorio "Oltre lo specchio"*. Quaderno Didattico n.32, Dipartimento di Scienze Matematiche, Università di Trieste (with enclosed videotape).

GROUP 3
FROM A STUDY OF TEACHING PRACTICES TO ISSUES IN TEACHER EDUCATION

KONRAD KRAINER (A)
FRED GOFFREE (NL)

Special Group-Publication:

Krainer, Konrad & Goffree, Fred (eds.):
On Research in Mathematics Teacher Education. From a Study of Teaching Practices to Issues in Teacher Education.
Proceedings of the First Conference of the European Society for Research in Mathematics Education, Special Volume.
Osnabrück: Forschungsinstitut für Mathematikdidaktik: Osnabrück.
Internet-Version: ISBN: 3-925386-52-1, (pdf-file, 810kB),
Paper-Version: ISBN: 3-925386-55-6.

On the following pages you will find "A Short Preview of the Book".

ON RESEARCH IN MATHEMATICS TEACHER EDUCATION

Konrad Krainer[1], Fred Goffree[2]
IFF/University of Klagenfurt, Sterneckstrasse 15, A-9020 Klagenfurt, Austria
konrad.krainer@uni-klu.ac.at
fi-home, bremlaan 16, NL-3537 kj Bosch en Duin, Netherlands
fred@fi.ruu.n

Abstract: *In this paper we briefly discuss the terms "practice", "teacher education", and "research in teacher education" and reflect on different areas of research which are relevant in the field of teacher education. We give an impression of the diversity of our work within our Thematic Group and provide a short preview of the chapters of the Special Volume (see previous page).*
Keywords*: teacher education, research practice, teaching practice, classroom practice, teachers' practice, teacher research, investigations*

The following text includes the preface and a short preview of all chapters of the book “On Research in Mathematics Teacher Education. From a Study of Teaching Practices to Issues in Teacher Education”. This book is a product of Thematic Group 3 “From a Study of Teaching Practices to Issues in Teacher Education” of the First Conference of the European Society for Research in Mathematics Education (CERME 1), held in Osnabrück, Germany, from 27th till 31st August, 1998. The editors gratefully thank Elmar Cohors-Fresenborg and Inge Schwank from the University of Osnabrück for their collaboration.

1. Some remarks on “practice”, “teacher education”, and “research in teacher education”

A central point of discussion in teacher education is “*practice*”. We often speak about teaching practice(s), classroom practice(s), teachers’ practice(s), etc., sometimes meaning the same and sometimes seeing differences. Similarly, we have different understandings of “*teacher education*”. In both cases, our interpretation is influenced by our own working background: for example, it makes a difference whether someone mainly works in pre-service education for primary teachers (e.g. being a teacher educator in a country where much more emphasis is laid on pre-service education than on in-service) or someone mainly works in in-service education for secondary teachers (e.g. being involved in a national teacher professional development programme). Whereas, for example, in pre-service education the creation of learning environments where student teachers get involved with *practice* (in order to learn more than "the big theories" about their future practice), is very essential, the challenge in in-service education is often more to use classroom practice as a learning environment where the teachers get involved with *theories* (in order to learn a "language" to speak about their practice). Different links between theory and practice will appear several times as a motif throughout this book. In the following, we briefly reflect on the terms "*practice*" and "*teacher education*" in order to make our understanding and usage of these terms explicit.

The most prominent part of teachers' professional work is done in *classrooms*, designing, managing, and evaluating content-related and social learning processes of students, dealing with a broad range of interactions, communications, assessments, pedagogical situations, etc. For that, in most cases, the terms "*teaching practice*" and "*classroom practice*" are used. Sometimes the plural "teaching practices" is used to indicate that different teachers have different ways to design their teaching. In its singular form, both, "teaching practice" and "classroom practice" not only refer to an individual teacher's way of teaching, but more generally indicate the *practical* field and context where teachers interact with their students – in contrast to the *theoretical* field and context where e.g. mathematics educators reflect on those practices. However, some people see an important difference between "teaching practice" and "classroom practice": whereas "teaching practice" more clearly refers to the teacher as a *person* (maybe mainly expressing an interest in his/her teaching or an interest in studying different teaching styles), "classroom practice" more neutrally refers to the classroom as a *system* that includes more than the teacher's actions (e.g. expressing an interest in the interaction process among students or between the teacher and the students). We tend to prefer to use "classroom practice" for two reasons: firstly, in the context of teacher education it is clear anyhow that the teacher plays an important role, and secondly, it expresses the point of view that we are more inclined to see the whole system in which learning and teaching processes are embedded. To some extent, this understanding expresses a shift from viewing classroom processes as mainly determined by the teacher to viewing classroom processes as complex features, where knowledge, meanings, norms, etc. are socially constructed and influenced by a variety of general conditions, as e.g. the importance of education in the society, the curriculum or the school climate.

This leads us to the term "*teachers' practice*". Although the most prominent part of teachers' professional work is done in classrooms, or is essentially related to it (preparation, reflection, etc.), it includes more than that. Increasingly the quality of schools has become a permanent issue of public discussion and of scientific investigations in the field of organizational development of schools. There is more and more awareness that a "good school" is more than the sum of "good classroom practices" of the individual teachers of a school, and that it is not only the principals' responsibility to contribute to the further development of a school, but also the joint responsibility of teachers (and students, parents, etc.). Therefore, "teachers' practice" cannot be confined to "classroom practice".

In addition, there is another systemic interconnection that should not be underestimated: it is our pre-service and in-service education that has a big influence on the learning process of teachers. For this reason, also our "teacher education practice" should increasingly become an object of (self-) evaluation and investigation.

This leads us to "*teacher education*". We understand teacher education as an *interaction process* (embedded in a social, organizational, cultural, ... context), mainly between teacher educators and (student) teachers, but also including systematic interactions among teachers aiming at professional growth. At the same time, we can see teacher education as a *learning environment* for all people involved in this interaction process.

The *overall goal* is the *improvement of teachers' practice* (in the case of institutionalized in-service education or other forms of promoting teachers' professional growth) or the adequate *preparation* for that practice (pre-service education). In both cases it is combined with an *improvement* of a complex network of (student) teachers' knowledge, beliefs, etc. The most prominent part of mathematics teachers' practice deals with interaction processes between the teacher and the students, focusing on students' learning of mathematics. Therefore we see the goal of mathematics teacher education as *promoting (student) teachers' efforts to establish or to improve their quality of teaching mathematics*, and the task of teacher educators *as designing adequate learning environments* to reach this goal in a joint effort with the (student) teachers. *Designing* and *evaluating* mathematics teacher education courses is one important part of mathematics educators' professional work. However, all over the world there are big differences concerning mathematics educators' responsibility for pre-service education, in-service education, administrative and management work or research activities.

Summing up, the title of *Thematic Group 3* can be interpreted in two ways. Firstly, it indicates a progress in the following sense: research in teacher education no longer confines itself to investigating teaching practice, but increasingly includes investigations on the broader complexity of teachers' practice and on investigations into our own teacher education practice.

Secondly, it indicates a progress in quite another sense: the study of teaching practice is no longer only a domain for academic researchers, but increasingly also one of student teachers and practicing teachers; learning environments where they can investigate classroom practice are more and more seen as effective elements of teacher education. The new possibilities of multimedia systems support this development.

A major focus of this conference was *research*. Concerning teacher education one might ask the question: Should we e.g. speak about research *in* teacher education or about research *on* teacher education? The situation is even more complex than that as we will see below.

Research in teacher education can be interpreted as a general term for investigations carried out within the framework of teacher education or at least with the goal or a clear potential of using its results in teacher education.

A kind of research coupled relatively loosely with teacher education we call *research in the perspective of teacher education*. The investigations are not done in the context of teacher education and do not focus on interaction processes within teacher education. In the past, most research projects in teacher education, and even a considerable part of recent initiatives, fall into this category. One prominent example is the investigation of (student) teachers' beliefs, knowledge, and practice (see chapters 2, 3, and 4). There is much diversity concerning researchers' inclination to draw explicit conclusions for designs of teacher education. It has to be added that a variety of research – not explicitly aiming at drawing conclusions for teacher education – is often the basis for creating powerful learning environments in teacher education.

Generally closer to drawing conclusions in the field of teacher education is *research in the context of teacher education*. Here investigations (e.g. on teachers' beliefs, knowledge, practice, etc.) are done in the context of teacher education (e.g. within the framework of an in-service education course) but they do not focus explicitly on the interaction process (within this course). In general, researchers are inclined to draw explicit conclusions for designs of teacher education, in particular concerning e.g. the course in which the investigations take place. Increasingly, (student) teachers are supported to do investigations on teachers' beliefs, knowledge, practice, etc. (see e.g. subchapter 4.3). We tend to subsume such investigations in the field of "research in teacher education" (here the question is less whether it fits to teacher education but more whether it can be titled as (traditional) "research", as "alternative research", as "action research", as carrying out "mini-research-tasks", etc.). For the purpose of this book, we use the term "investigations" as a generic term that includes all kinds of research, inquiry, systematic reflection, etc. However, it is a future challenge for teacher education to discuss this question in more detail.

Very close to drawing conclusions is *research on teacher education* courses, programmes or other forms of promoting teachers' professional growth. Here the focus is directly on teacher education and means investigations on the interaction between e.g. teacher educators and (student) teachers, the achievements of the participants, side-effects for the school, etc. One prominent reason to do that kind of research is *to evaluate the success* of teacher education courses or programmes, mostly with the intention to draw consequences from that internal or/and external evaluation (e.g. improving the course or stopping the programme). A special way of evaluation is self-evaluation on the basis of teacher educators' action research in order to improve their (teacher education) practice. Another reason is *to understand better the interactions* between teacher educators and (student) teachers, e.g. investigating which kind of teacher educators' interventions or teachers' beliefs influenced the process. In this latter case, the intention is – at least not explicitly – the improvement of one particular course or programme, but it is assumed that the results – because they really focus on teacher education – are a good basis for drawing conclusions for designing teacher education.

A direct connection to the improvement of teachers' practice is given in the case of *research as teacher education* (or teacher education *as* research, see chapter 6). A concrete form is (teachers') action research, understood e.g. as the systematic and self-critical reflection of practitioners into their own practice (see chapter 5). Research as teacher education means a close interconnection between *understanding* and *improving* teachers' practice. The joint reflection on the learning process (e.g. interactions, improvement of practice, ...) plays a crucial role. In principle, teachers' action research can be done among professional teachers themselves, everyone being the teacher educator (and "critical friend") for the others. In most cases, however, action research is initiated and promoted in the context of a teacher education course or programme where teacher educators act as "facilitators" of action research but also have more traditional roles in their interaction with the participants (teacher education *with* research, see chapter 6). The borders between teacher education *with* research and teacher education *as* research are fluid.

A very special kind of research is *meta-research on teacher education* which means analyses of research activities or general conditions in the field of teacher education (or parts of it), for example working towards "state of the art"-reports.

All these kinds of investigations can be subsumed under *research in teacher education*. They also can be combined: for example, embedding investigations of (student) teachers in learning environments of teacher education courses might be a starting point to investigate how successful these investigations are concerning (student) teachers' beliefs, knowledge, practice, etc.

2. Some remarks on the work of Thematic Group 3

The *work of Thematic Group 3* (TG 3), "From a study of teaching practices to issues in teacher education", was characterized by a considerable amount of *diversity* that distinguished it from the other six thematic groups of the First Conference of the European Society for Research in Mathematics Education (CERME 1). We confine ourselves to sketch *four aspects*.

Firstly, the theme of the group covers a very *broad field of relevant issues*. As expressed by the title, teacher education includes much more than the study of teaching practices, for there are many factors influencing those practices, e.g. teachers' beliefs and knowledge, students' abilities and motivation, school climate and professional communication among teachers, general conditions of teaching (class size, space, time, ...), curriculum and assessment, structure and political orientation of the school system, internal and external support systems for the improvement of teaching (through different kinds of in-service education) or teachers' socialization process, starting from their own classroom experience, over participating in pre-service education and school practice to a fully responsible work as a practicing teacher at a specific school.

Secondly, progress in teacher education not only means that we (as mathematics educators) confine ourselves to investigate more intensively teachers' practices (and beliefs, knowledge, ...) and make suggestions for its improvement but we also have to take *our own* (teaching, research, ...) *practice* etc. *as a matter of reflection and investigation*. Teacher education has to do with self-application of our theories, and therefore has another quality of challenge for our field. Our activities in teacher (pre- and in-service) education and our research activities are embedded into the whole system of further development of teachers' practice. We are not (only) external investigators and observers *of* this system, but we are (also) responsible actors *within* this system. We are not only trying to understand better teachers' practice (in order to give advise on how teachers, their practice, the curriculum, etc., could improve) but we also have the duty to understand our interventions into teachers' practice (e.g. through in-service education) or the implications of our pre-service education practice on student teachers' development (in order to improve our teacher education practice). Given this fact, teacher education – if we understand it as an interaction process between teacher educators and teachers – demands that the processes of understanding and improving (teachers' and our own practice) get more interwoven than in fields where we investigate situations and processes that do not include such a high personal involvement of ourselves.

Thirdly, teacher education has not the same *research tradition* as many other issues of mathematics education (like e.g. didactics of algebra or geometry). It is assumed that this is to a great extent a consequence of the aspect mentioned above: the high degree of involvement influences our approach to investigation and research. There are at least two approaches that play a prominent role in discussions in teacher education (conferences, journals, ...) all over the world – and also in our Thematic Group: One approach puts an emphasis on research on teachers (in a non teacher education context) in order to understand better teachers' practice (beliefs, knowledge, etc.), but often such research doesn't make clear connections to the improvement of (teachers' or teacher educators') practice. In this case the question is less whether we speak about research, but whether there is a close link to teacher education. Another approach takes – to some extent – the other direction: the authors (mostly teacher educators) tell – more or less – "success stories" about pre- or in-service projects at their institutions. Here, the improvement of teachers' practice, their professional growth etc. is in the foreground, more and more there are also reflections on teacher educators' learning processes and consequences for improving the project. Often, the weak point here is that we cannot easily find a systematic reflection on the research question, the criteria for success and a presentation of data that helps the reader to duplicate the improvement (e.g. to understand the main relevant factors that are due to the project and less to other influences). Here, the topic is surely teacher education, but the challenge is really to understand and to describe theoretically the processes which lead to the improvement. Both approaches are important, but in both cases we need a closer relation between understanding and improving. It is clear that an emphasis on *research* demands an emphasis on understanding, and it is clear that *teacher education* mainly aims at improving (learners' knowledge, practice, beliefs, ...). *Research in the field of teacher education* means to meet *both challenges*.

Fourthly, and this seems to be an outcome of the broadness of the field, the demand for self-applicability and the challenge of doing research in this field, *Thematic Group 3* not only had the largest number of participants, but also had the fewest number of accepted papers of all Thematic Groups of CERME 1. The group was characterized by a broad geographical and cultural diversity which was accompanied by a considerable heterogeneity of participants' research and development traditions, mother languages and English abilities. This means a context where communication and collaboration during the conference and the process to achieve a joint product (in particular, this book) was a really tough task.

Given the time pressure to finish our chapter, we tried our best to find a compromise between realizing the ERME-principles of communication, co-operation, and collaboration, aiming at scientific quality and coping with the deadline for publication. A lot of arguments could have been expressed more clearly and more interconnections could have been realized. Nevertheless, we are sure that our product marks a good starting point for deeper reflection on teacher education among mathematics educators in Europe and all over the world.

It would not have been possible to produce this chapter without the enormous motivation and self-discipline of all of its contributors, based on the extraordinary good working and social climate we jointly established at our meetings in Osnabrück. The following 22 people (in alphabetic order), coming from 13 different countries, participated in our Thematic Group, contributing to the quality of processes and products of our work: *Michele Artaud* (France), *Peter Berger* (Germany), *Lucilla Cannizzaro* (Italy), *José Carrillo* (Spain), *François Conne* (Switzerland), *Luis Carlos Contreras* (Spain), *Moisés Coriat* (Spain), *Fred Goffree* (The Netherlands), *Barbara Jaworski* (United Kingdom), *Konrad Krainer* (Austria), *Razia Fakir-Mohammed* (Pakistan, at present United Kingdom), *Maria Korcz* (Poland), *Hélia Oliveira* (Portugal), *Wil Oonk* (The Netherlands), *Marie-Jeanne Perrin-Glorian* (France), *João Pedro da Ponte* (Portugal), *Ildar Safuanov* (Russia), *Manuel Saraiva* (Portugal), *Andrei Semenov* (Russia), *Maria de Lurdes Serrazina* (Portugal), *Julianna Szendrei* (Hungary), and *Elisabeth Thoma* (Austria). It was a pleasure to co-ordinate efforts within this both professionally and personally fruitful learning community.

Yves Chevallard (France), *Fred Goffree* (The Netherlands), *Konrad Krainer* (Co-ordinator, Austria), and *Erkki Pehkonen* (Finland) were asked by the Programme Committee to lead the group. Unfortunately, Yves Chevallard and Erkki Pehkonen (whom we thank for his contributions during the planning process) had not been able to attend.

We gratefully thank *José Carrillo*, *Moisés Coriat*, *Barbara Jaworski*, and *João Pedro da Ponte* for their excellent preparation work for their meetings and the co-ordination of the corresponding chapters of the book. Many thanks we express to *Peter Berger* for his valuable contribution to manage the layout and his help to increase the readability of the text.

3. Short preview of the chapters 1 to 6

As mentioned before, as a result of the papers and posters submitted to *Thematic Group 3*, the meetings were structured along the following research *interests* in teacher education: *What, how, why, etc. (student) teachers believe, know, act/do, and* *reflect/investigate?* However, it was also intended to reflect on the corresponding results as being useful for teacher education practice. Therefore, *chapters 2 to 5* aim at finding a bridge between teacher education and investigations into teachers' beliefs, knowledge, practice(s), and reflections. We use the term "investigations" – which is broader than research – in order to be able, for example, to include student teachers' inquiry into teachers' practice.

- *Chapter 1*, "Teacher Education and Investigations into Teacher Education: A Conference as a Learning Environment" (written by Konrad Krainer), firstly describes teacher education as a complex field, highlights fundamental shifts, recent research foci and innovative forms of teacher education. This is followed by a description of how the CERME 1 context was used as a learning environment in Thematic Group 3, focusing on its first meeting.
- *Chapter 2*, "Teacher education and investigations into teachers' beliefs" (co-ordinated by João Pedro da Ponte, with contributions also from Peter Berger, Lucilla Cannizzaro, José Carrillo, Nuria Climent, Luis Carlos Contreras, and Ildar Safuanov), firstly gives an introduction to the broad field of research on teachers' beliefs, highlighting "beliefs" and "conceptions" as foundational topics in teacher education. Then two empirical studies on teachers' beliefs are presented. The first one looks at teacher's beliefs about problem solving and its relation to beliefs about mathematics teaching and learning in general, indicating a clear interconnection. The second one investigates teachers' beliefs concerning the computer, as a technical, personal, and pedagogical object, working out the importance of affective components of teachers' beliefs. The chapter concludes with a brief survey of methodological approaches and necessary research tools and sketches some directions for future work in this field.
- *Chapter 3*, "Teacher education and investigations into teachers' knowledge" (co-ordinated by José Carrillo and Moisés Coriat, with contributions also from Hélia Oliveira), firstly gives an introduction to the topic and some challenges, and then sketches different approaches to a characterization of teachers' knowledge (e.g. expert and prospective teachers knowledge, components of knowledge). This is followed by the question of how teachers' knowledge can be promoted through teacher education, pointing out the importance of action research, situated learning, the use of narratives and the need for an integration of

knowledge. The chapter concludes with open questions and a plea for viewing teacher education as an open process and argues for more communication, co-operation, and collaboration among teacher educators and researchers.

- *Chapter 4*, “Teacher education and investigations into teachers’ practice(s)” (co-ordinated by Fred Goffree, with contributions also from Marie-Jeanne Perrin-Glorian, Hélia Oliveira, Will Oonk, Maria de Lurdes Serrazina, and Julianna Szendrei), firstly gives an introduction to “good practice”, sketching different examples from Hungary and Portugal, where good practice in different contexts is realized or used for reflection. This is followed by a study of five teachers’ practices within the framework of the “theory of situations” and the anthropologic approach of “didactic transposition”, investigating the organization of content and the related didactical approach in secondary classrooms in France. Finally, the chapter presents a multimedial interactive learning environment (MILE) which is used in pre-service teacher education where student teachers construct practical and theoretical knowledge through investigating experienced teachers’ practice.
- *Chapter 5*, “Teacher education through teachers’ investigation into their own practice” (co-ordinated by Barbara Jaworski, with contributions also from Konrad Krainer, Razia Fakir-Mohammed, and Elisabeth Thoma), gives an insight into the work of *Thematic Group 3* in the meeting on teachers’ action research. Among others, it highlights the complexity of the teaching process and the variety of influences put on it (society, culture, ...) and presents answers of participants to questions, for example, concerning starting points, ways of involvement and contexts of action research, the theoretical background, or reflects on how action research fits with norms of established educational research. Some brief conclusions and directions for future work close the chapter.
- *Chapter 6*, “Investigations into teacher education: trends, future research, and collaboration” (written by Konrad Krainer and Fred Goffree), sketches some trends of investigations into mathematics teacher education, discusses the complexity of investigations into this field, reflects on learning from investigations, and points out some issues of future research and collaboration.

Konrad Krainer and Fred Goffree
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GROUP 4
SOCIAL INTERACTIONS IN MATHEMATICAL LEARNING SITUATIONS

GOETZ KRUMMHEUER (D)
JOAO MATOS (P)
ALAIN MERCIER (F)

INTRODUCTION

Götz Krummheuer
Freie Universität Berlin
Fachbereich Erziehungswissenschaft, Psychologie und Sportwissenschaft WE 02
Habelschwerdter Allee 45, D - 14 195 Berlin
goetzkru@mail.zedat.fu-berlin.de

Evidently, on international conferences one is confronted with a variety of different theories about a topic that appears like the intersection of these theories. Scrutiny as it necessarily emerges in the discussions during our group sessions, however, partly reveals fundamental conceptual differences according to the presumed common topic "social interaction in mathematical learning situations". Neither the understanding of "social interaction" nor the understanding of what "mathematical learning" might look like was common sense. One of the major results of these group-meetings was the increasing clarity about the different use of same and/or similarly looking concepts. The reference to social and personal constructivism was far to global as to call it a common basis of the papers of this group.

The reader might be aware of the changes of the definitions of similar concepts with regard to

- micro-sociological approaches based on interactionism (Blumer) and ethnomethodology (Garfinkel),
- the theory of didactic situations based on Brousseau's work,
- system-theory based on Luhman's approach,
- a theory of situated learning,
- a theory of social interaction as developed by post-Piagetian scientists, and
- a theory of project learning.

Birgit *Brandt* (Recipients in Elementary Mathematics Classroom Interaction) exposes aspects of participation in elementary (mathematics) classroom interaction. She considers classroom interaction as a multi-party-interaction with more than one "interaction unit" at the same time. Linked to micro-sociological approaches and with respect to the particular character of her observed classroom she describes reception roles and forms of parallel interaction in the multi-party-interactions of primary classrooms.

Jean-Philippe *Drouhard* (Necessary Mathematical Statements and Aspects of Knowledge in the Classroom) is interested in the concept of mathematical necessity and how it emerges in mathematical classroom processes. For this and with regard he addresses some specific issues like the intersubjectivity of the knowledge, the subject's experience of mathematical necessity, and the role of time in order to understand the "construction" of the necessity of the necessary mathematical statements.

Götz *Krummheuer* (The Narrative Character of Argumentative Mathematics Classroom Interaction in Primary Education) presents some results from two related research projects about processes of argumentation in primary mathematics classroom. His central research interest is to examine the relationship between the participation of students in argumentative processes and their individual content-related development in regular classroom settings. Based on a micro-sociological perspective he describes the narrative feature of these processes.

Alain *Mercier*, Gérard *Sensevy*, Maria-Luisa *Schubauer-Leoni* (How Social Interactions within a Class Depend on the Teacher's Assessment of the Pupils' Various Mathematical Capabilities. A Case Study) address from a clinical, mainly post-Piagetian point of view the issue of the interrelations between the knowledge acquiring processes and the social interactions within a class of mathematics: a) how can knowledge determine the kind of social relationship established during a didactic interaction, and b) reciprocally, how can the social relationship already established within the class influence one and each pupil's acquisition of knowledge?

The subject Natalie *Naujok*'s paper (Help, Metahelping, and Folk Psychology in Elementary Mathematics Classroom Interaction) is part of an investigation on student cooperation and its functionality with respect to learning opportunities. Her aim is to reconstruct on base of micro-sociological theories how students interactively construct the cooperation-form of helping. These students' taken-as-shared ideas will be explained by applying BRUNER's concept of folk psychology.

Alison *Price* (It Is Not Just About Mathematics, It Is About Life: Addition in a Primary Classroom) discusses of a classroom session transcript with respect to the question how the teacher uses language and examples from everyday social life to teach the children about addition and at what effect this process of situating the mathematics has on the children's learning and understanding. At issue is the social nature of learning, as the child tries to make sense both of the mathematics and of their life experiences in this situation.

Catherine *Sackur*, Teresa *Assude* & Maryse *Maurel* (The Personal History of Learning Mathematics in the Classroom. An Analysis of Some Students' Narratives) examine written narratives of students (high school, university or preservice teachers) about the students' personal memory on a mathematical subject. The purpose is to reconstruct students' experience in mathematics and their personal relation to mathematics. The authors relate their results to the concept of "didactical time" as developed in the theory of didactic situations.

Heinz *Steinbring* (Mathematical Interaction as an Autopoietic System: Social and Epistemological Interrelations) focuses on the problem of what are essential characteristics of mathematical teaching interaction. He uses the concept of "social communication as an autopoietic system" (Luhmann) as one general theoretical perspective and combines this approach with an epistemological analysis in order to clarify some characteristics of mathematical interaction in contrast to general social interaction.

Marie *Tichá* and Marie *Kubíova* contribution (On the Activating Role of Projects in the Classroom) deals with the approach of project learning. They present concrete examples from school practice in order to show the areas in which their approach of project learning was profitable for students. They identify main obstacles for implementing project-learning in classroom, and develop the future direction of their research.

RECIPIENTS IN ELEMENTARY MATHEMATICS CLASSROOM INTERACTION

Birgit Brandt
Freie Universität Berlin, Fachbereich Erziehungswissenschaft, Psychologie und Sportwissenschaft WE 02,
Habelschwerdter Allee 45, D-14195 Berlin, Germany
bbrandt@zedat.fu-berlin.de

Abstract: *In this paper I report about aspects of participation in elementary classroom interaction. The work is part of an empirical study of mathematics education and investigates elements of an interactional theory of learning (see first section). We see classroom interaction as a multi-party-interaction with more than one "interaction unit" at the same time. A participant of the event "lesson" can "switch" her or his attention within a selection of several "units", being "part" of one or more of them concurrently. A short episode of our data on partnerwork as example of multi-party-interaction is presented (second section). Within this multi-party-interaction I describe reception roles and forms of simultaneous interaction linked to Goffman 1981 (third section). Some remarks for the recipient's competence in participating conclude the paper.*
Keywords: *participation in classroom interaction, recipients' roles, social condition of learning*

1. The research project

At the Free University of Berlin there is a team for interpretative classroom researchers. This includes the research project "*Reconstruction of formats of collective argumentation*".[1] This paper results from close collaboration with my colleagues Götz Krummheuer and Natalie Naujok.

Following the idea of combining *cognitive constructivism and social interactionism* as outlined by Cobb and Bauersfeld 1995, the research project can be seen as a contribution to a sociologically orientated interactional theory of learning and teaching in mathematic classrooms. Considering that we are investigating content related interactions, the aim of the project is to elaborate the comprehension of collective processes of argumentation (Miller 1986) in elementary mathematics classroom interaction. In particular

- the (re)accomplishment of *formats of argumentation* in interactional processes (Krummheuer 1992, 1995),
- the *autonomy and authenticity* of the arguing persons (Brandt 1997) and
- the kind of *reception* of (and in) such processes.

The empirical basis of our research are videotaped lessons. We have observed two classes, for two weeks each: a first grade class (21 children, average age 7 years) and a multi-aged class of first through third graders (27 children, 6-10 years old). The analyses were done from transcribed episodes, which were selected according to the research purpose and methodological considerations. First we *reconstruct* the learning situations seperately by (a) interaction analysis, (b) argumentation analysis and (c) participation analysis.

The next step is the comparison of analyses of different interaction processes. The selection of episodes in the ongoing research process is guided by this comparison, which causes modifications of our analytical methods and the construction of theoretical elements, as well. In this paper, I concentrate specially on the participation analysis and on the reception in processes of argumentation (see Vollmer 1997 for our way of interaction analysis and Krummheuer 1995 for more details of argumentation analysis). Therefore, I first mention our theoretical background of participation and then present an example of interaction (in form of a transcript).

2. Participation in multi-party-interaction

As I mentioned above, our learning concept is based on the ideas of cognitive constructivism and social interactionism. We assume that the way in which a student participates in a classroom lesson initiates and limites her/his individual learning process. The learner invents her/his individual construction of new mathematical meanings by participating in mathematics classroom (Wood 1996). For learning by participating in social interaction it seems to be appropriate to refer to Bruner's (1983) concept of *participation in formats.* He describes for language acquisitation "formats" as

> "... standardized, initially microcosmic interaction pattern between an adult and an infant that contains demarcated roles that eventually become reversible." (p. 120)

The participation in such formats including the change of roles forms the learning process. The learning process can be described as the increasing autonomy of the child in stable interactional structures. In such situations the individuals involved generate a participation model in which the learning process takes place indirectly (Bauersfeld 1995, p. 280). The mutual attention which is typical in such dyads cannot be assumed for classroom interaction with 20-30 children and one or two teachers. So, we have to modify the concept of learning by participating in formats with respect to the peculiarity of multi-party-interactions.

A useful approach to multi-party-interactions is given by Goffman 1981 (see Sahlström 1997 for a detailed examination of the relevant literature). He demands in his general criticism of the dyadic model for conversation:

- the dissolution of the speaker-hearer dyad, and furthermore
- the decomposition of the everyday terms *speaker* and *hearer*:

> "It takes global folk categories (like speaker and hearer) for granted instead of decomposing them into smaller, analytically coherent elements." (p. 129)

For the aspects of speaking he comments

> "Plainly, reciting a fully memorized text or reading aloud from a prepared script allows us to animate words we had no hand in formulating, and express opinions, beliefs, and sentiments we do not hold." (p. 145)

He invents the term *production format* (see Brandt 1997 for production formats in classroom interaction). For the aspects of reception, he describes the different relations between the participants in a gathering as *participation framework*:

> "The point of all this, of course, is that an utterance does not carve up the world beyond the speaker into precisely two parts, recipients and non-recipients, but rather opens up an array of structurally differentiated possibilities, establishing the participation framework in which the speaker will be guiding his delivery." (p. 137)

The participation framework in the classroom interaction will be illustrated by a short scene of partnerwork taking place in a mathematics lesson of first graders. The interaction example is related to mathematics, but this part of the analysis is not the place to discuss the pupils' mathematical constructions of the interaction.[2]

2.1 Partnerwork in a multi-party-interaction: an empirical example

The teacher sends Woil to "table four" where Ekrem and Volkan need help for the same worksheet.[3] Although Ekrem and Volkan are sitting close together, Woil decides to explain it each of them separately, so we can observe two partnerwork sequences. First, Woil turns to Ekrem (please find signs of transcription at the end of the paper. [...] stands for leaved out utterances):

1	Woil	*standing with his worksheet behind Ekrem* here are ten, sure \ *showing the squares in the chart on Ekrem's sheet* and here is one . so you must /. **do look** \ (*inaudible)*
2	Ekrem	yes\ a **one** \ yes [...]

A longer dispute (with simultaneous talking) about the working process emerges. At the end Woil shows the squares of the chart and Ekrem notes the corresponding number in his sheet.

3	Woil	.. okay / *pointing the next square* and now here / . *looking at his own worksheet*
4	<Ekrem	a ten / *writing*
5	<Woil	*short look at his own sheet* yes / ten \ .. and then you must - *(inaudible)*
6	<Woil	*turning to Viola* *(inaudible)*+
7	<Viola	*turning to Woil (inaudible)*
8	Ekrem	and a two /
9	<Woil	*nods* uhm / are you able to do it /

10	<Ekrem	writing u h m \
11	Woil	look \ here is *showing the corresponding squares* twelve thirteen fourteen fifteen sixteen seventeen + and here / in these squares every time ten *(inaudible)*
12	<Ekrem	and in these squares always three four five six too
13	<Woil	yes \ yes \ *knocks his pen on the table* so \

Ekrem works alone. Woil goes to Volkan.

14	Woil	Volkan can you / *standing behind Volkan* You can't either / look *(inaudible)* [...] *pointing an empty square in the chart* what's coming here/
15	<Volkan	one \ *writing*
16	<Woil	yes \ then two / .. *pointing the squares in the third row* now you must here
17	Woil	*(inaudible)* count the red squares only
18	Ekrem	*working on his sheet* I can all*, looking to Marieke and Viola* Woil taught me \ *smiling* I can all by myself now\
19	<Woil	*showing the red circles* no here must be a **twelve** \
20	<Volkan	*looking to Woil, chewing his pen* *writing*
21	Woil	*looking around (inaudible)* .. then you must [...] +. okay /
22	Volkan	*nods, whispering* okay
23	Ali	*in the background* Woil \ Woil \
24	Woil	*turning to the teacher* I have explained it to Volkan and thingamy Ekrem \
25	teacher	*wonderful* \ do you want to explain more / ...

The whole sequence can be divided into two scenes: "Woil helps Ekrem" <1-13> and "Woil helps Volkan" <14-22>.While Ekrem decides to be an "active helping receiver", Volkan is more passive. Ekrem has a different access to the sheet and we can observe a convergence of Woil's and Ekrem's ideas in the end of their conversation (see Krummheuer 1992 for the convergence in interaction processes). So we can analyze different interaction formats and different production formats in these two scenes.

3. Participation framework

> "The process of auditing what a speaker says and following the gist of his remarks - hearing in the communication-system sense - is from the start to be distinguished from the social slot in which the activity usually occurs, namely, official status as a ratified participant in the encounter." (Goffman 1981, p. 131)

So, in an encounter, one can distinguish ratified participants and unratified participants - who never the less may be following the talk closely. In our example, all children and the teacher (and we, too) are ratified participants of the lesson, whereby the lesson as "social encounter" is demarcated by the bell and/or ritual greetings. But in a multi-party-interaction it is necessary to investigate "smaller units" of interaction. This is mentioned in Goffman 1981 and carried out in more detail by Levinson 1988. He introduces the "utterance-event",

> "[...] that unit within which the function from the set of participant roles to the set of individuals is held constant." (p. 168)

In such units it is possible to ascertain the participant roles for different recipients. The roles are "movable" in two kinds:

- The participant roles are "movable" in time. E.g. Ekrem's roles (in relation to Woil) in <1-13> and <18> are different.
- With the "'schism' into a number of parallel conversations" (Sahlström 1997, p. 21) in multi-party-interactions, a participant of the whole event can have different participant roles in different simultaneous conversations. E.g. in line <5-7>, Woil is engaged in two conversations.

Sahlström 1997 emphasizes the interactional process of the allocation of the roles by verbal and non-verbal communication, and carries out the relation of the participation role to a single utterance. We are interested in the thematical development of interaction processes. Therefore an analytical reduction to single utterances is not suitable. So we are looking for sequences with utterances referring to each other (by working out the interactional analysis). We introduce the term *focussed talk* for stable conversation situations in this sense.

3.1 Recipients' roles in focussed talks

On one hand *focussed talks* or *focussed utterances* are terms for the researcher's view at the analyzing process. The investigation of content-related interactions allows the discrimination of focussed talks by the interaction analysis. On the other hand for focusing a dialogue, there must be a stable interaction. So participants in the interaction process must focus on a topic and on one another, too.

Woil goes to Ekrem and with the words "here are ten, sure [...] do look" <1> and his physical behavior he addresses to Ekrem. Although there is a (inaudible) verbal exchange between Viola and Woil <6-7>, Ekrem and Woil focus each other in the sequence <1-13> without changing the topic. So, we can interpret this partnerwork as a focussed talk. For this focussed talk, the other children in the classroom are no longer ratified recipients as they are in a classroom discussion. One may assume that for these unratified recipients it is not reasonable to listen to the focussed talk of Woil and Ekrem attentively. (However, they can do so in a hidden way like a spy.) But it is reasonable for the teacher in her function as instructress and "client" (she sent Woil to help Ekrem) - and that is what Woil mentions in <24>. The teacher can be seen as ratified. So, in focussed talks, one can distinguish ratified recipients (here Woil, Ekrem and the teacher) from unratified recipients. Woil and Ekrem address to each other in their utterances as "next speaker". We call them *interlocutor* (see below). The teacher is ratified in another way: It is possible for her to listen carefully and (in her function as "client") to intervene, but she is not addressed as a "next speaker" by Woil or Ekrem. We describe her recipient role as *listener*. Then, Woil and Ekrem (or Woil and Volkan respectively) act in a way, such that coincidental and casual listening without effort is allowed for the unratified children in the classroom. So the two boys allow *overhearing*:

<table>
<tr><td>speaking person</td><td colspan="4">recipients</td></tr>
<tr><td rowspan="2"></td><td colspan="2">ratified</td><td colspan="2">unratified</td></tr>
<tr><td>addressed
interlocutors</td><td>unaddressed
listeners</td><td>tolerated
overhearers</td><td>not tolerated
eavesdroppers</td></tr>
</table>

In a focussed talk, there is an interplay of the speaking person and its interlocutors (see the shade in the table): The speaking person of the focussed utterance addresses to the interlocutors as *potential* next speakers. So looking at a focussed talk, one of the interlocutors is expected to be the speaking person of the following utterance while the speaking person of the current utterance will be an interlocutor.

3.2 Interaction lines

The whole event "lesson" as a multi-party-interaction is divided into lines of simultaneous conversations. Especially with respect to the opportunity of cooperation in small groups during not-teacher-centered periods, it is not possible to distinguish between foreground and background conversation. Here, once again, it is useful to look at focussed talks or from focussed talks, respectively. What listeners or overhearers of the focussed talks utter forms the incidental interaction lines. The roles of the participants in incidental interaction lines with regard to the focussed talk can help to describe the relationship between this incidental lines and the focussed talk. These different roles are shown in the second column of the table:

incidenta int. lines	*roles of the part. in inc. int. lines (with regard to the foc. dial.)*	*example*
byplay	listeners (ratified recipients in the focussed talk)	two children talking about new pencil while a classroom discussion takes place
sideplay	overhearers (unratified recipients in the focussed talk)	a simultaneous interaction to the dialogue "Woil and Ekrem" between children (Ekrem, Marieke, Viola in <17>)
crossplay	listeners *and* overhearers (passing the border of ratified / unratified)	a simultaneous interaction to the dialogue "Woil and Ekrem" between the teacher and a child
borderplay	listeners/overhearers *and* interlocutors (passing the border of addressed / unaddressed)	the teacher (listener) or a child (overhearer) ask a question to Woil or Ekrem (interlocutors) (Woil and Viola in <6-7>)

For the first three terms we refer to Goffman (p. 133-134). But in his term "byplay" there is no difference between listener and interlocutor. Concentrating on focussed talks (as a kind of stable interaction situations in a multi-party-interaction), we stress the relationship between interlocutors. So we complete the interaction lines by the term borderplay. Summarizing: the relationship of interaction lines in a multi-party-interaction cannot be described by looking at the whole event, but by looking from the perspective of a focussed utterance or a focussed talk, respectively.

3.3 Participants competence

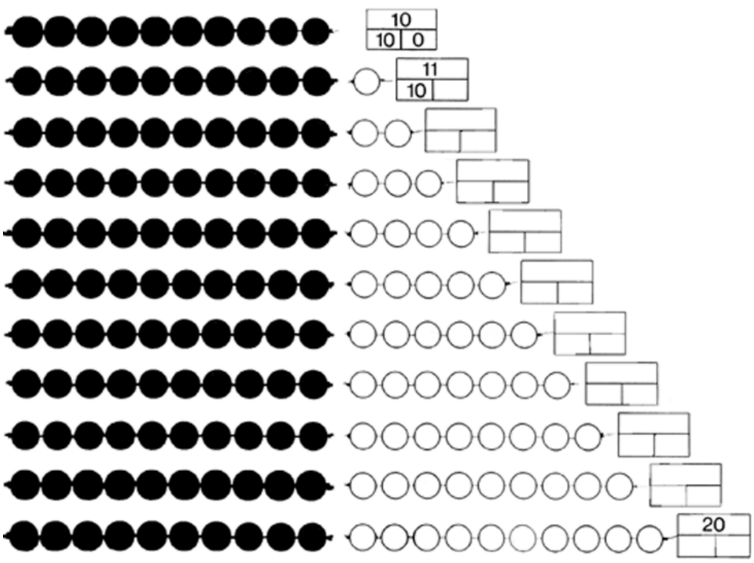
Although Woil is involved in a verbal exchange with Viola (<6-7>, borderplay), he is still the interlocutor of Ekrem. Ekrem’s writing act in <5> can be seen as utterance in the focussed talk. So the participants are able to produce stable conversation situations, which we can interpret as focussed talks. One can suppose that the competence of the participants in producing such structures is relevante for the possibility of learning in multi-party-interaction:

1. On the one hand, Woil in the role of an interlocutor <6-7>, is simultaneously involved in additional interaction lines. Taking Volkan’s writing process <20> as utterance in the focussed talk <14-22>, his interlocutor Woil is looking around. So Woil is aware of his (immediate) surrounding as overhearer. And again, the teacher is open to hear Woil’s utterances in <23-24>: She is aware of him and she is willing for a borderplay to the focussed talk, in which she is involved as interlocutor.
2. But on the other hand, Woil still is in contact with Ekrem (Volkan, respectively) as interlocutor of the partnership. This remaining attention to a focussed talk facilitates stable conversation structures with a thematic development in multi-party-interactions.

So the participants “switching” (by *oscillating attention*[4]) forms the interweaving of the interaction lines for the participants in the speech event. The participants “knowing” about how to *align* and to *weight* their attention enables focussed talks - the interactional unit for content related interactions in multi-party-interactions. At this point there is a link to the work of Natalie Naujok: Woil’s “knowing” concerning help and his competence in “remaining” in focussed talks (by *aligning and weighting* his attention) can be interpreted as aspects of a/his folk pedagogy (see Naujok 1999 for aspects of folk pedagogy in classroom interaction).

Notes

1. It is based at the Free University of Berlin and supported by the German Research Foundation (DFG).
2. A deeper examination of the content-related aspects is done in the analysis of argumentation and of production format. The detailed analyses of the following scenes can be read in the interim report of our research project (Brandt & Krummheuer 1998; in German, unpublished).
3. In the classroom, there are four (numbered) "big tables" arranged by three smaller ones, each for six children. Woil, Ekrem and Volkan are boys. Their mother tongue is not German, just as for many of their classmates. The worksheet is about numbers from 10 to 20. The children have to reduce them in the ten and the unit and note down the numbers in a special kind of chart. The numbers appear in their natural order:



4. Markowitz 1986 describes "attentionale Oscillation" as the addressing part in classroom interaction. We use "oszillierende Aufmerksamkeitsfokussierung" for the reception part.

4. References

Bauersfeld, H. (1995): "Language games" in the mathematics classroom: Their function and their effects. In P. Cobb & H. Bauersfeld (eds.), *The emergence of mathematical meaning. Interaction in classroom cultures*, pp. 271-292, Hillsdale, NJ: Lawrence Erlbaum.

Brandt, B. (1997): Forms of participation in elementary mathematics. Classroom interaction. *CIEAEM 4*9, Setubal, Portugal. (in press)

Brandt, B. & G. Krummheuer (1998): *Zwischenbericht zum DFG-Projekt: Rekonstruktion von "Formaten kollektiven Argumentierens" im Mathematikunterricht der Grundschule.* Fachbereich Erziehungswissenschaft, Psychologie und Sportwissenschaft, Institut für Grundschul- und Integrationspädagogik. Berlin: Freie Universität Berlin.

Bruner, J. (1983): *Child´s talk. Learning to use language*. Oxford: Oxford University Press.

Cobb, P. & H. Bauersfeld (1995): Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (eds.), *The emergence of mathematical meaning. Interaction in classroom cultures*, pp 1-16. Hillsdale, NJ: Lawrence Erlbaum.

Goffman, E. (1981): Footing. *Forms of talk*. ders. pp. 124-159, Philadelphia: University of Philadelphia Press.

Krummheuer, G. (1992): *Lernen mit "Format". Elemente einer interaktionistischen Lerntheorie. Diskutiert an Beispielen mathematischen Unterrichts.* Weinheim: Deutscher Studienverlag

Krummheuer, G. (1995): The ethnography of argumentation. In P. Cobb & H. Bauersfeld (eds.), *The emergence of mathematical meaning: interaction in classroom cultures,* pp. 229-279, Hillsdale, NJ: Lawrence Erlbaum;

Levinson, S. (1988): Putting linguistics on a proper footing: Explorations in Goffman's concepts of participation. In P. Drew & A. Wooton (eds.), *Ervin Goffman - Exploration the Interaction*, pp.161-227, Oxford: Blackwell Publishers. Reprint 1996.

Markowitz, J. (1986): *Verhalten im Systemkontext. Zum Begriff des sozialen Epigramms. Diskutiert am Beispiel des Schulunterrichts*. Frankfurt a. M.: Suhrkamp.

Miller, M. (1986): *Kollektive Lernprozesse*. Frankfurt a. M.: Suhrkamp.

Naujok, N. (1999): Help. Metahelping, and Folk Psychology in Elementary Mathematics Classroom Interaction. In I. Schwank (ed.), *European Research in Mathematics Ecucation*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Sahlström, F. (1997): Classroom interaction and "footing". *International Communication Association 47th Annual Conference*, Montreal, Canada.

Vollmer, N. (1997): How helpful is help among students? Theoretical basis of and methodological consideration for an interaction theory of learning in school. *CIEAM 49*, Setubal, Portugal.(in press).

Wood, T. (1996): Events in learning mathematics: Insights from research in classroom. *Educational Studies in Mathematics* 30, pp. 85-105.

Transcription		*italic*	(facial) expression, gestures, action
,	breathing space	+	end of the last expression, gestures, action
. .. ...	break (longer break → more points	/	pitchraising
bold	stressed spoken words	\	pitchdropping
b r o a d l y	lengthened words	-	floating pitch

NECESSARY MATHEMATICAL STATEMENTS AND ASPECTS OF KNOWLEDGE IN THE CLASSROOM

Jean-Philippe Drouhard[1], Catherine Sackur[1], Maryse Maurel[2], Yves Paquelier, Teresa Assude[3]

[1]IREM de Nice – UNSA, Parc Valrose, 06108 Nice Cedex 2, France

drouhard@unice.fr

sackur@unice.fr

[2]GECO (IREM de Nice)

maurel@unice.fr

[3]DIDIREM (Paris 7) et IUFM de Versailles, 336E, ch. Barthélémy Florent, F-83200-Toulon, France

assude@gauss.math.jussieu.fr

Abstract: -

Keywords: *social constructivism, necessity, experience*

1. Introduction

In mathematics, most statements may be called necessary. They are not just true or false, in the same way as the statement ”Osnabrück is the birthplace of Erich Maria Remarque”, but they are necessarily true or false, like the Pythagora’s Theorem. Until now, few studies in mathematics education have been focusing straightforwardly on the questions of when and how do students become aware of the necessity of such statements. To address, at least theoretically, this question we were led to introduce into the usual didactic ‘models’ (as for instance the Brousseau’s (1997) theory of didactic situations) some specific elements:

- the Inter subjectivity of the knowledge (i. e. the central role of the dialogues between the student in the very nature of the mathematical knowledge - obviously in the theory of didactic situations, the dialogue between students already plays an essential role!- Our point here is to make more explicit the relationship between the dialogue and the (epistemological) nature of the mathematical knowledge itself.)

- the subject's Experience (of contradiction, for instance)
- the crucial role of Time to understand the 'construction' of the necessity of the necessary statements (this latter point is mainly developed in the paper by Sackur, Assude et al. in the same Thematic Group)
- the notion of aspects of a knowledge

2. The *CESAME* research group

Within the research field of Social Constructivism (Ernest, 1995), we are working on a project (supported by a grant from the DIERF of the IUFM de Nice and in a second phase, in collaboration with the CEFIEC, Buenos Aires (M. Panizza and G. Barallobres)) called *CESAME*. This word, "*CESAME*", is an acronym of the following French words: " "Construction Expériencielle du Savoir" et "Autrui" dans les Mathématiques Enseignées ", which could be more or less well translated as: " the Subject's Experience of Dialogues with Interlocutors ("Others") in the Social Construction of Taught Mathematical Knowledge " (unfortunately the acronym is, by far, less euphonic in English…).

The first outcome of the CESAME group has been a set of questions:

- What is actually the subject experiencing, whilst constructing his/her mathematical knowledge?
- What are the various roles that "Others" (other students, the teacher etc.) can play (according to E. Goffman's perspective, 1959) in this subject's experience?
- Is it possible to experience the *necessity* of necessary mathematical statements (we shall often use "to experience sth" in the meaning of to have a (subjective, intimate) experience of sth.).
    - If so, what is the very nature of such an experience?
    - And, are actually some students experiencing it, and how?

In this paper, we shall not address the question of the roles of the ”Others” (that we previously described in terms of a ”double didactic pyramid”, see Drouhard 1997a, 1997b). We rather shall focus here on the notions of *necessity* of necessary mathematical statements (part 3: Necessary Statements) and on the *nature* of the related knowledge (part 4: ”Aspects of Knowledge”).

3. Necessary statements

3.1 Institutionalisation in mathematical discussions

Our starting point is the following. What is the teacher supposed to do, at the end of a *Scientific Debate* (Legrand, 1988) or more generally of a *Mathematical Discussion* (Bartolini-Bussi, 1991, see also Hoyles, 1985), in order to turn the many statements yielded by the discussion into ”official” mathematical statements, which is ”Institutionalisation” in the terms of the Theory of Didactic Situations (Brousseau, 1997, Margolinas 1992, 1993)? And what is s/he actually doing? On the one hand, in a strict Social Constructivist point of view, the group is supposed to ” construct ” the whole mathematical knowledge by itself, the teacher’s role being just to lead the discussion. On the other hand, the teacher actually says something about the new common knowledge. Well, what does s/he says? How what s/he says may change the status of the statements? In most cases, s/he summarises; we think, however, that summarising is not just saying the same things shorter; but has an effect on the *meaning* of the statements. In brief, our question is the following: *what* is institutionalised in a mathematical discussion, and *how? What* is institutionalised must be obviously related with what is specific to mathematics. Then, a first point is to determine what is *specific* to mathematics in a discussion, in other words to what extent a mathematical discussion differs from, say, a political discussion. A first answer could be the following: in a *mathematical* discussion*, individuals are experiencing contradictions*. We use here the words ”to experience” and ”contradiction” in a strong sense; it is not just to be engaged in some kind of contradictory debate, it is to have the intimate experience that others (”Autrui”, in French) can be certain of opposite ideas, and cannot be convinced by authority arguments, for instance.

3.2 Resistance

The simple fact that contradictions exist - and are perceived as such by the participants! - involves that in mathematics, some arguments are particularly strong. In other words: in mathematics, statements do *resist*. A wall resists to your efforts to break it, too; but, if you have an axe big enough, you can make a hole in it. On the other hand, there is no axe to allow you to consider true a false mathematical statement. Of course, you *can*, but you cannot avoid the fact that anyway it is an error. This ‘resistance’ of mathematical statements is, for us, characteristic of their ‘necessity’. Unlike statements like ”CERME-1 is held in Osnabrück”, most mathematical statements are *necessarily* true or false, as:

“for any a and b belonging to a commutative ring, $a^2 - b^2=(a+b)(a-b)$ ”

What is the *nature* of this necessity? It is not easy to characterise such an ‘epistemic value’ (Duval, 1995) of statements. A first (logicist) answer can be found in terms of mathematical logic: a *necessary* statement is the result of a valid inference (Durand-Guerrier, 1995; 199?). From this point of view, the problem of what is institutionalised at the end of a mathematical discussion (and more generally at the end of a lesson- actually, institutionalisation, Margolinas said (1992, 1993), is not a moment (at the end of a lesson) but rather a long-time process. Here, ”the end of… (a lesson, a discussion)” is just a way of speaking.) becomes simple and easy to understand: students have just to learn the content of the statements *and* that these statements result from valid inferences; in other words, that they have to be demonstrated.

This approach however remain for us somewhat unsatisfactory. We think, indeed, that it does not take into account the aspect of *resistance* of statements, whereas the experience of this resistance appears to be crucial in mathematical discussions (if it would not be the case, discussions would be just pedagogical ‘tricks’, to help students to find and refine ideas but not specific of a mathematical content).

3.3 Wittgenstein

We found in Wittgenstein (1978) a more subtle characterisation of necessity, related to resistance, although apparently paradoxical: the idea (expressed here in a very sketchy way) that mathematical objects resist us to the very extent that we *will* that they resist. The mathematical objects do not resist by themselves, as walls resist according to their physical nature. Mathematical objects resist because we *want* them *to resist us*. But, what is the origin of this willpower? It seems that, for Wittgenstein, mathematicians need objects that resist them because *doing mathematics* is precisely *working on such resisting objects*. Surely they *could* work with 'weaker' mental objects but then, what they would do, could no longer be called ”mathematics” (this is a turning point).

Wittgenstein (1986) uses often a metaphor of games. We could illustrate it as follows. In chess game, a pawn goes forward only. Obviously, you *can* move a pawn backwards; but you *may* not. Well, more precisely, you even *may* decide to authorise this movement indeed; but in this case *you no longer play the same game*. Wittgenstein (speaking here of a ”grammar”) stressed the importance of the dual notions of arbitrary rules of the game, and of the *definition* of an activity (you may break the rules but in this case you play another game, i.e. a game *defined* in an other way).

This (anti-Platonist) point of view appears often questionable indeed. However, we think that a concrete example may be found in the G. H. Hardy's book of Memoirs. In this book, Hardy speaks about the Indian mathematician Ramanujan (author of incredibly difficult formulas, some of them still used in the very recent computations of π) who learned alone the advanced mathematics in an old-fashioned exercise book (where solutions were given *without any demonstration*). Ramanujan, Hardy said, was a pure genius in mathematics but at the same time he did not really understand some fundamentals, for instance the meaning of a demonstration! We could consider that Ramanujan's mathematics are an example of 'different' mathematics, where most knowledge is similar to usual, except for some high-level touchstones as the role of the demonstration. Likewise, we could address the difficult question of the nature of non-occidental mathematics. Are (classical) Chinese mathematics mathematics? The computations are the same, but likely not the proofs...

In the CESAME group, we called this type of necessity (related to the rules of the mathematics considered as a game) the ”epistemic necessity” (we use this word, ”epistemic”, not to give to our study a pedantic style but we just need to avoid confusion between the intrinsic nature of mathematical statements (”epistemic necessity”) and the usual ”pragmatic” necessity (if you want this, you must *necessarily* do that)).

3.4 Awareness

From a learning-oriented point of view, an individual has many possibilities to be aware of this epistemic necessity. For a given mathematical statement, s/he may *know* (that the statement is necessarily true), know it *and* know *why*, know it and *remember* that s/he has been knowing why (but ignore why at present); s/he may even know that s/he forgot why precisely the statement is necessarily true, but be persuaded that *s/he could retrieve his/her experience* of the epistemic necessity of the statement, etc. The question of the various ways for a subject to be aware of the epistemic necessity of a statement still remains open. However, it is clear for us now, that one of the specific features of mathematical discussions is to give students a chance to have an experience of the necessity of the statements (in terms of resistance to contradiction) and that this epistemic necessity is a part of what is to be *institutionalised* by the teacher.

4. *Aspects* of knowledge

Let us come back to the initial question: *what* is institutionalised in a mathematical discussion? We just saw that the epistemic necessity is one of the answers. It is clear however that a much larger amount of things may be learnt through such discussions: for instance, in algebra, the know-how and also what we could call the ”know-why” of the rules. More generally, something *about* mathematical statements is institutionalised (by using the word ”institutionalise” we do not suppose that it would be direct or explicit! We just want to say that since the mathematical knowledge is useless if the ”know-how” and the ”whys” are not available, these higher level knowledge are a part (sometimes hidden) of the aims of a lesson), which is more than their strict content. In order to give a framework to this idea, we propose to consider that a (taught) knowledge present three aspects:

I. The *first aspect* is made of the mathematical *content*: of the knowledge, the *semantics* of the related statements, according to the Tarsky logic (in which the *meaning* of ”2+2 = 4” is precisely that two plus two are four; this is what Quine calls the ‘denotation’ (in a sense totally opposite to Frege), saying that ”denotation is “ de-quotation ”” (removing the quotation marks)).

II. The *second aspect* contains the rules of the mathematical game, the know-how, but also that, for instance, algebraic expressions have a denotation (Drouhard, 1995), true statements are necessarily true, definite statements are either true or false, demonstrations are (the) valid proofs etc.

III. The *third aspect* contains the most general believes about mathematics, as for instance ”mathematics is a matter of understanding” or in the contrary (as many students say), ”in mathematics there is nothing to understand” (our point of view here is descriptive and not prescriptive: we try to say *that* there are various aspects of knowledge, and not *what* they *might* be).

We chose to call these components of a mathematical knowledge ”aspects” to express the idea that these components could hardly be defined and studied separately one from another. A mathematical theorem (whose ”content” is its aspect I) is mathematical (aspect III) indeed, inasmuch as it is necessarily true (aspect II)! The distinction between these aspects is just a question of ”point of view”. We could imagine an optical metaphor: a given mathematical knowledge is like a White light, obtained by superposition of many coloured light; and a colour (corresponding to one aspect in the metaphor) can be viewed only if we observe the light through a coloured filter, which suppresses all the other colours.

Of course, that the expert knows the rules of the mathematical games, and that anybody has general believes about mathematics is in no way original. For us, the interest of focusing on such aspects is that now we are able to address the question of *how* and *when* are these various aspects are *taught* and *learnt*. Actually, we feel that many studies in didactics focus mainly on the teaching of the *aspect* I of the knowledge (its ”content”) or more precisely that aspects II and III are seldom in the foreground but instead often remain in the shadow (the Scientific Debates about the ”pétrolier” (tanker) or the (electric) ”circuit” devised and studied by Marc Legrand (1990) are amongst the few ones that address explicitly what we call aspects II and III of the mathematical knowledge).

5. Teaching and learning

Let us consider now the second facet of the question: *how* is knowledge, with its three aspects, institutionalised? In particular, what is the role of the teacher related with aspects II and III of a mathematical knowledge? What does s/he do? Clearly, the teacher cannot just say about the epistemic necessity of a statement *S* that ”*S* is necessary”! We think that, a priori, this role is rather different than the teacher's role about the aspect I of knowledge. In particular, we think that his/her role is often *performative* (utterances that allow us to do something by the simple fact they are uttered: Austin, 1962).

The teacher puts the label ”mathematics” on what the student does and so s/he gives the student a hint on what mathematics are. Let us consider the sentence:

“Using denotation to invalidate a rule or a result is a mathematical action”

From our point of view such a sentence could illustrate the way we wish to use the aspects of knowledge while teaching:

(a) (the rule is invalid - it is false - in the same way that ”CERME-1 is held in Berlin” is false,

(b) (to see (demonstrate, prove, check) that it is false a mathematician has a method, tries with values of *x*.

(c) (to know that this method is valid and to use it, is ”doing mathematics”.

We believe that (a), (b) and (c) above do not play the same role in the learning of mathematics and cannot be told in the same way. Similarly, as teachers, we sometimes find work of students which are written with words and symbols commonly used in mathematics. These works look like mathematics and someone who doesn't know mathematics could be mistaken. What we feel then is that ”this mess has nothing to do with mathematics”: assertions are incoherent, the result is taken for granted and used in the proof, notions are mixed one for the other… not only are the theorems and definitions false (aspect I), but above all we do not recognise the ”rules of the mathematical game” (aspect II).

It seemed useful to us to point these three aspects of knowledge so as to keep in mind how learning and consequently teaching mathematics is a complex action. Mathematical statements are mathematical if and only if they actualise the three aspects of knowledge. A consequence of this, is that the students' conceptions about mathematics (aspect III) must be constructed in the process of learning mathematics. It is a sort of paradox: one learn something which is not yet defined and learning it, this person learns what is the nature of what s/he is learning (maybe, this is not specific to mathematics: learning piano is not just learning how to move one's fingers on a keyboard so as to produce the correct sounds).

In brief, we can consider that to learn mathematics (I and II), is also to learn *what are* mathematics (III). On the other hand, to teach that a statement is mathematical (II) (this led us to rehabilitate the famous ”Effet Jourdain” (Brousseau, 1997)), is also to teach that mathematics are made (III) of statements like the one which is taught (I)!

6. Provisional conclusion

We are fully aware that in this paper most ideas are mainly tentative and expressed in a quite abrupt way. We tried our best (and authors like Wittgenstein are not of a great help to allow us to express ideas in a non ambiguous, clear and detailed way…), but the research is still ongoing and to express our ideas in a more detailed and concrete way is our work to come. We tried however to find some empirical data to support these ideas, and this can be found in the paper of Sackur & Assude (presented in the same CERME Thematic Group). An other ongoing research on inequations is precisely addressing the question of to what extent are students aware of the necessity of algebraic statements when they solve inequations. A first presentation of the outcomes of this study will be done at the SFIDA seminary (Séminaire Franco-Italien de Didactique de l'Algèbre),

which will held in Nice in november, 1998:
(http://math.unice.fr/~iremnice/sfida/index.html)

We would like to stress however the point that our research takes place without ambiguity in the domain of social constructivism, not just for anecdotal reasons as the many references to Wittgenstein but since we think (following this author!) that the question of the epistemic necessity has no sense beside the point of view of the social construction of mathematics: all our keywords (rules of a game, definitions, beliefs...) are *socially* constructed, shared and taught.

7. References

Austin J. L. (1962): *How to do things with words*. Oxford University Press.

Bartolini-Bussi, M. G. (1991): Social Interaction and Mathematical Knowledge. *Proceedings of the Fourteenth Annual Meeting, International Group for the Psychology of Mathematics Education.* Genoa, Italy: University of Genoa.

Brousseau, G. (1997): *Theory of Didactical Situations in Mathematics*, Edited and translated by N. Balacheff, M. Cooper, R. Sutherland & V. Warfield. Dordrecht: Kluwer.

Drouhard J-Ph. (1995): "Blind Calculators in Algebra: 'So What?' Attitude". In E. Cohors-Fresenborg (ed.), *Proceedings of the European Research Conference on the Psychology of Mathematics Education*. (ERCME '95). Osnabrück: University of Osnabrück.

Drouhard J-Ph. (1997a): "Communication in the Classroom with a CAS: the Double Didactic Pyramid Model", In B. Kutzler & J. Monaghan (Dir.), *The State of Computer algebra in Mathematics Education*, Chartwell-Bratt.

Drouhard, J-Ph (1997b): Communication in the Classroom on Necessary Mathematical Statements: The Double Didactic Pyramid Model. In *M. Hejny´ & J. Novotna (eds.), Proceedings of the European Congress on Research in Mathematics Education, Podebrady*, Prag, Czech Republic: Charles University.

Durand-Guerrier V. (1995): Place de la logique formelle comme outil d'analyse des connaissances mises en œuvre dans le raisonnement mathématique dans une perspective didactique. In G. Arsac, J. Gréa, D. Grenier & A. Tiberghien (Dir.), *Différents types de savoir et leur articulation*, Actes du Colloque de Francheville, Grenoble: La Pensée Sauvage.

Durand Guerrier V. (1996): *Logique et raisonnement mathématique: Défense et illustration de la pertinence du calcul des prédicats pour une approche didactique des difficultés liées à l'implication*. Thèse de Doctorat, Université Claude Bernard, Lyon I.

Duval R. (1995): *Sémiosis et Pensée Humaine, Registres sémiotiques et apprentissages intellectuels*, Bern: Peter Lang.

Ernest P. (1991): *The Philosophy of Mathematics Education.* London: Falmer Press.

Goffman E. (1959): *The Presentation of Self in Everyday Life*. Garden City, New York: Doubleday.

Hardy G. H., (1985): *L'apologie d'un mathématicien*, Belin.

Hoyles C. (1985): What is the point of group discussion in mathematics?. *Educational Studies in Mathematics*, 16.

Legrand, M. (1988): Genèse et étude sommaire d'une situation codicactique: le débat scientifique en situation d'enseignement. In C. Laborde (ed.) *Proceedings of the Premier Colloque Franco-Allemand de Didactique des Mathématiques et de l'Informatique*, Grenoble, France: La Pensée Sauvage

Legrand M. (1990): "Rationalité et démonstration mathématiques, le rapport de la classe à une communauté scientifique", *Recherches en didactique des mathématiques*, 9/3.

Margolinas Cl. (1992): *De l'importance du vrai et du faux dans la classe de mathématiques*, Grenoble: La Pensée Sauvage.

Margolinas Cl. (1993): "Le rôle du maître dans la gestion des phases de conclusion en situation didactiques", *Recherches en Didactique des Mathématiques*, 12-1.

Wittgenstein L. (1978): *Remarks on the Foundations of Mathematics*. MIT Press (Cambridge, Mass.).

Wittgenstein L., Anscombe G.E.M. & Von Wright G.H. (1986): *On Certainty/Uber Gewissheit*, Harper Collins.

THE NARRATIVE CHARACTER OF ARGUMENTATIVE MATHEMATICS CLASSROOM INTERACTION IN PRIMARY EDUCATION

Götz Krummheuer
Freie Universität Berlin, Fachbereich Erziehungswissenschaft, Psychologie und Sportwissenschaft WE 02
Habelschwerdter Allee 45, D - 14 195 Berlin, Germany
goetzkru@mail.zedat.fu-berlin.de

Abstract: *Results from two related research projects about processes of argumentation in primary mathematics classrooms will be presented. The central research interest is to examine the relationship between the participation of students in argumentative processes and their individual content-related development. Hereby, the focus is on mathematics teaching and learning situations in regular classroom settings. Illustrated by the interpretation of a joint solving process of third-graders this paper shows the narrative character of these processes. The theoretical relevance and some practical implications of this approach will be outlined, finally.*
Keywords: *social interaction, argumentation, narrativity*

1. Introduction

Some results from two related research projects about processes of argumentation in primary mathematics classrooms will be presented. The projects are “Arguing in primary mathematics classrooms” sponsored by the state of Baden-Württemberg, Germany in 1994 and 1995, and “The reconstruction of ‘formats of collective argumentation’ in primary mathematics education” sponsored by the German Research Foundation (DFG) since October 1996.

The central research interest is to examine the relationship between the participation of students in interactional classroom processes and their individual content-related development. Hereby, the focus is on mathematics teaching and learning situations in regular elementary classes. The theoretical orientation of these projects is based on ethnomethodology (Garfinkel 1967), symbolic interactionism (Blumer 1969) and cultural psychology (Bruner 1996). From these perspectives classroom situations are seen as *processes of interaction*: Students and teachers contribute to their accomplishment according to their insight into the sense and purpose of these events. They act as it seems *sensible* and *tenable* to them. For this, they *interpret* these classroom situations: they reflect, set up and review hypotheses, and make rational decisions; common features are found which temporarily enable them to cooperate. In such a classroom situation they develop their content-related understanding in order to be mutually regarded as responsible and capable and in order to participate in the joint creation of this interaction. Thus, in such a situation their mind is challenged, which they employ and develop by the way.

In both projects the research approach concentrates on (collective) argumentation (see Krummheuer 1995, Krummheuer & Yackel 1990). This paper stresses on the narrative character of this process. In chapter 2 the view on learning as a social phenomenon will be outlined. Here, the concepts of "culture" and "classroom culture" play a crucial role. The chapter 3 deals with the narrative feature of those classroom cultures. In the fourth part an example of a collective task solving process among three third-graders will be presented, in which the narrative character of their joint argumentation process will be explained. The last chapters summarizes the results, integrates them in the larger research-context and shows one practical implication.

2. The social constitution of learning

In the discussion about conditions of learning it is more frequently stressed that the entire phenomenon of human learning is not spanned if one confines one's studies to the interior, mental processes of the learner. Learning is also a social process which takes place in the interaction between human beings:

> ... where human beings are concerned, learning ... is an interactive process in which people learn from each other (Bruner 1996, p.22; see also Bauersfeld 1995, Bruner 1990, Erickson 1982 & Krummheuer 1992).

Thus, learning is socially constituted, or in other words: interactive processes are vital parts of the "nature" of learning. This insight led Bruner 1986 to formulate the feature of a psychology which constitutionally considers social elements for psychic processes, as for example in the course of learning. He defines this as *cultural psychology* (p. 35) and explains that each individual development must be expressible in the particular symbolic system of a given culture. For this, the members of a culture not only have the general means of their language but, additionally, they can also employ specific culturally accomplished ways to interpret the psychological disposition of individuals. Bruner 1990 defines this as *folk psychology* (e.g. p. 33ff). Regarding the teaching- learning- process in a classroom situation he speaks of *folk pedagogy* (Bruner 1996, p. 46). These two concepts include the mostly implicit basic assumptions of a culture about the psychological functioning of its members.

> All cultures have as one of their powerful constitutive instruments a folk psychology, a set of more or less connected normative descriptions about how human beings 'tick', what our own and other minds are like, what one can expect situated action to be like, what are possible modes of life, how one commits oneself to them, and so on. We learn our culture's folk psychology early, learn it as we learn to use the very language we acquire and to conduct the interpersonal transactions required in communal life (Bruner 1996, S. 35).

In this quotation it is also stated that such folk psychological insights are acquired at an early age and that they are already linked with the learning of one's own native language. The folk psychology is acquired through narrative interaction, in other words, their insights are learned from stories. Bruner designates this way of "learning from stories" as an independent mood. On the one hand he contrasts this mood with a logical and scientific way of thinking, on the other hand, he regards it as a learning mechanism through which a child develops fundamental views and perspectives of the world it lives in (s. 1986, p. 11ff and 1996, p. 39f). In this sense, learning is not only the appropriation of culture, it is implemented in its co-creation. Especially with regard to primary education it is often stated that basic cultural techniques such as reading, writing and arithmetic are taught and acquired, here. From the developed perspective of cultural psychology, this seems to be an insufficient point of view: Children do not only learn the contents of culture. Rather, through their contributions in reading, writing, and calculating they also create "a" or "the" culture. Concatenating these two aspects one arrives at what could be described as *classroom culture*. Participation in this double sense integrates the social constitution of learning. Thus, *classroom culture is a culture of subject matter and a culture of learning*.

3. The narrativity of classroom culture

Obviously, in primary education children like to listen to and to tell stories. The argument presented here goes beyond this empirical evidence and claims that the children learn by these stories. Hereby they learn the content of different school subjects. The classroom culture of our primary schools is characterized by narrativity: frequently, the different contents are presented in a narrative style, and the social constitution of classroom-learning can be described in related models of participation in situations of story-telling. This is also relevant for mathematics classes, and the analysis of processes of interaction concerning this subject matter can demonstrate the importance of this thesis in general.

In the following, firstly, some characteristics of such narrations in the observed mathematics classes will be presented. The aim is to clarify which aspects of narrativity are most relevant in mathematics classroom-interaction. It will be shown that it is of an argumentative nature and that argumentation dissolves in a narrative presentation (see also the concept of “reflexive argumentation” in Krummheuer 1995). This means: *The narrative classroom culture of primary education is based on rationality, and the social constitution of classroom-learning is the participation in the interactional accomplishment of argumentative, narratively structured sequences of actions*. This thesis does not imply that in classroom-situations “stories” will be told endlessly and that beyond educational goals in native language classes, classroom-education intends to teach children the telling of stories. Rather it is that the negotiated theme in a classroom-interaction emerges more frequently in such a way that one can reconstruct aspects of a narrative process. Thus the concept of “narration” is used here in order to describe a specific phenomenon of everyday classroom-conversation. It is not meant in the sense of literary science.

According to Bruner 1990 one can identify four characteristics for narrative accomplishments: its

1. “sequentiality”
2. “factual indifference between the real and the imaginary”
3. “unique way of managing departures from the canonical”, and
4. “dramatic quality” (p. 50).

Here, the first and third points are of special interest. The claimed narrativity of classroom-culture is seen in the patterned sequentiality of classroom-interaction. The specifity of an event, such as the elaborated solving process for a new mathematical task, is presented in relation to the canonical management of such events or problems. Classroom-processes display some specifities in which they differ from usual narrations:

- Frequently, students and teacher complement each other in the role of the story-teller. Thus, there are no definite roles of the “listener” and the “teller”. Usually, several persons are engaged generating a story.
- Not only stories about the past are told, but also stories about something new emerge. Usually one associates with the concept of “narration” the image that something that is already over comes to the fore. One can describe this as the *presentation* of facts (Kallmeyer & Schütze 1977, p. 159f). In our observed group-work about mathematical problems we see also a narratively characterized interaction, during which the children accomplish their single steps of calculation and at the same time express what they do: in such cases they “tell” or “narrate” how they come to their solution, or better: how one can come to a solution. One could rather call this the *constitution* of facts. In this constitution one can identify typical elements of a narration, which is the ordered presentation of a concrete case, in which a problematical situation is managed (see e. g. Bruner 1986, p. 16ff; 1990, p. 47ff; Krummheuer 1997).

According to the basic theoretical and methodological assumptions these exemptions are not claimed in a quantified statement of a high grade of percentage. In contrary a momentum of classroom culture in primary education assessed as typical is attempted to conceptualize. In the following more evidence will be given to this theoretical approach by reconstructing in detail a classroom-episode.

4. An example

The boys Daniel, Slawa and Stanislaw from a third grade are confronted with the presentation of numbers at the back of T-shirts which represent the first parts of a number sequence. Their task is to determine the fifth element of this sequence which is: A: {3 - 8 - 15 - 24 - ?} For this sequence, Slawa can quickly give a solution:

47	Slawa	*(pointing at the picture)* Here comes five, here comes seven´..
49	Slawa	here comes *(.)* nine´
52	Slawa	He gets an eleven-
53	Daniel	Why eleven´
54	Stanislaw	Why´
55	Slawa	Well eleven. Look´, *(precariously whispering)* how much plus three,
56		look´, at this number. five-
57<	Daniel	Well´, from three to eight are five.
58<	Slawa	*(directed to Daniel and still pointing at the picture)* here comes already
59		seven´, seven-
60	Daniel	seven-
61	Slawa	nine´ *(.)* eleven.
62	Stanislaw	*(inarticulate)* well yeah.
63	Slawa	Eleven plus twenty-four. add it here. then one gets *(figures about 2 sec)* thirty-five.

From a mathematical stance, one can view in Slawa’s solution the thematization of the general concept of the sequence of differences and the first four numbers of a specific sequence of differences {5 - 7 - 9 - 11}. The boy cannot name them. He does not define them explicitly and in a certain way he is not talking about them, but through them. His two classmates cannot follow him. Slawa is obliged to explain; generally, he reacts in the way just described: He names the four elements of the sequence of difference. One short episode might demonstrate this:

77<	Slawa	This are five. here *(points at paper)* then seven´, here comes nine
78<	Daniel	five *(mumbles inarticulately)* from eight to fifteen are seven´
79	Slawa	add always two to it.
82<	Slawa	Thus here comes eleven´, Daniel *(points at number sequence)* here
83<	Daniel	seven´ yes from, yes
84		comes eleven to that number
85		from fifteen to twenty-four are nine.
86	Slawa	Thus here you get thirty-five . *(inarticulate)* thirty-five.
87	Stanislaw	Whoop-
88	Daniel	Yes, nine´

One recognizes how Daniel in <78, 84 and 88> agrees to the numbers 5, 7 and 9 as the difference between the given elements of the initial number sequence. He and Stanislaw as well do not conceptualize the numeration of these numbers as the elements of a number sequence which emerges by finding the differences. Even Slawa's meta-comment about the rule for this sequence of differences in <79> does not help. Slawa's finding of the solution, his presentation and his justifications are narratively oriented. In order to understand his solution one must, firstly, recognize the phenomenon of a sequence of difference and, secondly, the defining characterization $x_{n+1} = x_n + 2$ while repeating the numbers 5, 7, 9, 11. If somebody cannot infer this argument from the numeration of the numbers, he does not understand the sense of the story at all. Summarizing, the four following conclusions can be drawn from the interpretation of this episode:

1. The mathematical concepts which are necessary for understanding the approach of the solution are not introduced explicitly. In a narrative way, it is rather pointed at them opaquely. Not all students will be able to recognize it by this way of presentation. The plausibility of the related solving process might be inscrutable for them.
2. For the accomplishment of the different steps for the solution the boys need certain mathematical competencies such as addition and subtraction of positive integers.

3. No meta-commentaries about the sensibility and rationality of the solution-process is given. This will be seen to characterize narratively organized interaction.
4. The presentation of the solution steps proceeds mainly by verbalization. The boys do not use alternative presentations such as visualization or embodiments. This also is a characteristic for narratively organized processes of interaction in mathematical group work.

Considering these four points, it becomes obvious that at the communicative surface of such interactive processes of processing it is talked only about calculation. Certainly, this is not an unknown phenomenon about mathematical activities among students and happens also very often at higher degrees (Krummheuer 1982, 1983). The thesis is: *the verbally presented calculations represent only the surface of a deeper structured rational procedure which is typical for narratively organized processes of interaction*. Thus, this form of classroom interaction provides the rationality of problem related action which reveals the argumentation about the correctness of a solution for the student in as much as he/she is able to infer this argumentation from this specific sequence of actions of the accomplished narrative.

5. Social learning conditions in classroom interaction and an implication for enhancing the classroom culture

The basic insight of this research is that in classroom teaching and in group work as well a proved folk psychology of learning in this classroom culture becomes apparent which for sure is not given by nature or God but which has two very important features: It functions in everyday classroom situations and it has a rationality.

The rationality of actions expresses itself in the pursuing or novel creation of a sequence of working steps. With regard to Erickson 1986 it is called the “academic task structure” (ATS). This is a sequence of actions as it is accomplished *by the participants* in their process of interactional negotiation. In primary mathematics classes, this interactive realization occurs often in a narrative style: the conducted calculations are told according to the sequence of the ATS in as much as the necessary competencies can be integrated. Typically, the inner logic of the total approach within such narratively realized academic task sequences is not explicitly thematized, but it is expected as a specific achievement of the participants. They have to *infer* this inner logic from the specific presentation of the narrations (for more details see Krummheuer 1995, 1997). This does not usually happen successfully and often not in its entirety. Learning which is related to novel concepts and insights does not happen automatically. But, on the other hand, this kind of narrative classroom culture is characterized by a great stability in everyday primary school teaching and learning situations and there are many students who daily proceed successfully in their content-related learning development by participating in this classroom culture.

With regard to those students who do not proceed successfully, one issue will be emphasized, here: It is the fact that writing and application of other illustrating tools are missing. The observed interaction processes in the two projects are solely based on oral exchange. Bruner 1996 speaks with regard to this point of the necessity of an externalization tenet.

> Externalization, in a word, rescues cognitive, activity from implicitness, making it more public, negotiable, and 'solidary'. At the same time, it makes it more accessible to subsequent reflection and meta-cognition (Bruner 1996, S. 24 f.).

The starting point of my argument is that in the project episodes preferentially verbal productions can be observed. Generally, the quick evaporation and the situative uniqueness of verbal accomplishments impedes the reflection on such interactive procedures - at least for some, the so-called "weak" students. Complementing such reflections with a written presentation of not only the result, but especially of the process of working seems helpful. Bruner 1996 refers to the concept of the œuvre of the French psychologist Ignace Meyerson. Œuvre does not mean a somehow standardized scientific presentation. It rather means that the children by themselves find a productive form of written presentation of their thoughts. Œuvres, produced in such a way, facilitate easier listening and possible repetitions, if necessary.

"Creative and productive writing" in such a sense is not only a category of native language classes, but in general a platform for reflection on classroom related processes of symbolization. It is not the question if the children should write down something that is correct in the sense of the subject matter, but rather that the children are supposed to find means of a presenting their thoughts which lasts over a longer span of time.

Such classroom culture provides *all* participants with well-founded possibilities to negotiate meaning productively and to produce shared meaning. The specific problem might be to identify forms of externalization which enable all students and (not only) the teacher or researcher to pursue a specific solving process. With regard to arithmetic one can refer here to standardized iconic ways of presentation. However, they need to be assessed and enhanced for this special use of providing reflection for the students.

6. References

Bauersfeld, H. (1995): "Language games" in the mathematics classroom: Their function and their effects. In P. Cobb & H. Bauersfeld (eds.), *The emergence of mathematical meaning. Inter-action in classroom cultures*. Hillsdale NJ: Lawrence Erlbaum.

Blumer, H. (1969): *Symbolic interactionism*. Englewood Cliffs, NJ: Prentice-Hall.

Bruner, J. (1986): *Actual minds, possible worlds*. Cambridge, MA.: Harvard University Press.

Bruner, J. (1990): *Acts of Meaning*. Cambridge, MA.; Harvard University Press.

Bruner, J. (1996): *The culture of education*. Cambridge, MA.: Harvard University Press.

Erickson, F. (1982): *Classroom discourse as improvisation*. In L. C. Wilkinson (ed.), *Communicating in the classroom*. New York: Academic Press.

Erickson, F. (1986[3]): Qualitative methods in research on teaching. In M. C. Wittroch (ed.), *Handbook of research on teaching*. New York: Macmillan.

Garfinkel, H. (1967): *Studies in ethnomethodology*. Englewood Cliffs: Prentice-Hall.

Kallmeyer, W. & Schütze, F. (1977): Zur Konstitution von Kommunikationsschemata der Sachverhaltsdarstellung. In D. Wegner (ed.), *Gesprächsanalysen*. Hamburg: Helmut Buske.

Krummheuer, G. (1982): Rahmenanalyse zum Unterricht einer achten Klasse über "Termumformungen". In H. Bauersfeld, H. W. Heymann, G. Krummheuer, J. H. Lorenz, & V. Reiss (eds.), *Analysen zum Unterrichtshandeln*. Köln: Aulis.

Krummheuer, G. (1983): *Algebraische Termumformungen in der Sekundarstufe I*. Abschlußbericht eines Forschungsprojektes. Bielefeld: IDM der Universität Bielefeld.

Krummheuer, G. (1992): *Lernen mit "Format". Elemente einer interaktionistischen Lerntheorie. Diskutiert an Beispielen mathematischen Unterrichts*. Weinheim: Deutscher Studien Verlag.

Krummheuer, G. (1995): The ethnography of argumentation. In P. Cobb & H. Bauersfeld (eds.), *The emergence of mathematical meaning: interaction in classroom cultures.* Hillsdale, NJ.: Lawrence Erlbaum.

Krummheuer, G. (1997): *Narrativität und Lernen. Mikrosoziologische Studien zur sozialen Konstitution schulischen Lernens*. Weinheim: Deutscher Studien Verlag.

Krummheuer, G. & Yackel, E. (1990). The emergence of mathematical argumentation in the small group interaction of second graders. *14th conference of the International Group for the Psychology of Mathematics Education*, Oaxtepec, Mexico.

HOW SOCIAL INTERACTIONS WITHIN A CLASS DEPEND ON THE TEACHERS'S ASSESSMENT OF THE PUPILS VARIOUS MATHEMATICAL CAPABILITIES: A CASE STUDY

Alain Mercier[1], Gérard Sensevy[2], Maria-Luisa Schubauer-Leoni[3]
[1]UFM Aix-Marseille, 63 La Canebière, F-13001 Marseille, France,
mercier@romarin.univ-aix.fr
[2]UFM Bretagne, 26 rue du docteur Roux, F-29800 Landerneau, France
gerard.sensevy@bretagne.iufm.fr
[3]APSE Genève, Switzerland

Abstract: *In this contribution[1], we will address from a clinical point of view the issue of the interrelations between the knowledge acquiring processes and the social interactions within a class of mathematics: a) how can knowledge determine the kind of social relationship established during a didactic interaction, and b) reciprocally, how can the social relationship already established within the class influence one and each pupil's acquisition of knowledge?*
Keywords: *social interactions, didactical contract, teacher's assessment*

1. Framework

The second part of the question (b) has been amply debated upon, since Rosenthal & Jacobson (1968) who showed the influence of the teacher's expectancies upon pupil's cursus. Many other studies have gone further, emphasising that such expectancies may refer either to presupposed capabilities or may concern more precisely social stereotypes (Mehan 1979) within a labelling theory. The question goes beyond both teacher's expectancies and the labelling theory since some studies (Sirota 1986, Broccolichi 1995) clearly showed how both the teacher's questioning and the pupil's ensuing answers or his/her subsequent speaking initiative depend hugely upon the pupil's position within the school ratings, even more than upon their parents' social backgrounds. However, whether in those latest studies or in the previous ones, the very object of the interactions i.e. how expertise and knowledge might depend on the teacher's expectancies has not been precisely addressed. We never found any study showing their effective weight, hic et nunc within the class, upon the pupil's acquisition of a piece of knowledge.

That's why we are addressing the question from its primary perspective (a): how can a pupil's expertise give him/her such or such a social position within the class? Specifically, does the teacher's assessment of a pupil's mathematical capabilities influence the expectancy process within a social interaction which stakes are and remain 'studying the piece of knowledge taught'?

The french speaking didacticians and social psychologists are currently using a fundamental notion: the didactical contract (Brousseau 1984, 1986, 1997, Chevallard 1988, 1995, Schubauer-Leoni 1986, 1996). According to us, this notion can contribute to raise in a convincing way the issue we want to address. Brousseau defines the didactical contract as a system of reciprocal expectancies between teacher and pupils, concerning knowledge, which contract is setting both pupil's and teacher's acts, and can explain them thereby. Using a sociological theorisation, some didacticians (Schubauer-Leoni 1988) could then claim that this covenant shaped a group habitus (Bourdieu 1990) within the class. Reay (1995) shows how the notion of habitus helps to make visible the taken-for-granted inequalities embedded in social processes; we emphasize their conversion into inequalities in the process of teaching, setting studies, and learning. While the notion of didactical contract seems quite momentous to us, to understand cognitive phenomena within the didactic space, such a notion seems to be limited as well. We will point at two issues: a) although emphasizing the didactic standards and categories which are suitable within a given class of mathematics - considered then as an institution (Douglas 1986), the notion of didactical contract stricto sensu fails to help us understand how one and each pupil inhabits (privately) those standards (Sarrazy 1996, Schubauer-Leoni & Perret-Clermont 1997), and b) the notion of a didactical contract refers to the class as a whole, and fails to make us understand how different pupils are offered different didactic standards within the same class.

So, in order for us to bring out those phenomena, we must first show in an empiric way how, according to this position in the school hierarchy, one and each pupil occupies specific niches within the didactical contract. The following lines may be considered in this respect as a first approach. We will base our contribution on a recent research (Mercier 1997, Schubauer-Leoni & Leutenegger 1997, Sensevy 1997, Salin 1997).

2. Social interactions and acquisition of knowledge within the class of mathematics: the case of Jerome and Louis

It is a lesson on numeration in a primary school class (4th grade, in France CM1) during which the pupils work upon how to write large figures, through a number's dictation. The study was based on the observation of the class, and we developed several series of analysis levels of the videoregistered interactions, a transcription of which was formalised: so that we gave the public utterances of each actor of the lesson together with their public graphic gestures when they come to the blackboard to explain themselves out.

We wanted to achieve a representation of the didactic space, modelling the sequence of events through our theory. The mathematical analysis of the problem gave us the didactical problem, from the teacher's and from the pupils' point of view (Mercier 1997). First level of a conversational analysis led us to the teaching problem for the teacher; then the analysis of the teaching team preparation led us to the teacher's project (Schubauer-Leoni 1997); and the sequencing of the lesson, for which we analysed some pupils' acts and the teacher's teaching gestures, enabled us to understand some of the problems those pupils wondered about and the way the teacher conducted her project when meeting a didactically delicate teaching situation (Sensevy 1997). Through this observatory method, we had the opportunity to offer a description of this mathematics class, seen as a social space set by didactics stakes (i.e. collective production and personal acquisition of mathematical knowledge) which determine social stakes (i.e. increased acknowledgement of one's capability to acquire or produce mathematical expertise): a space where one and each pupil follow his/her proper route. So, as to give a bird's eyeview of these analyses, we'll introduce a few moments of two selected pupils' public works at the blackboard. Each of them spent more than a quarter of the teaching period there; Jerome came without any question, and he learned mathematics while staying at the blackboard, when Louis came with a good question and did not found any answer: we want to explain the difference.

At this moment during the lessons upon large figures, all the pupils were ordered to write «seventeen million two thousand and fifty eight». The teacher is aware that some pupils have written «17 200 058»: referring to the teaching team preparation, this is the expected mistake. The teacher chooses one of these pupils to go and write his solution onto the blackboard. Jerome is one of the best in mathematics, one can therefore deduce that the teacher takes it for granted that he is capable to explain away the strategy he uses: in her lesson's preparation, the teacher wrote that she should choose "a fairly good pupil". So Jerome would seem to come as a teaching auxiliary.

Jerome

Here is a summary of the whole set of interrelationship between teacher, class, and Jerome, when this latter is in front of the blackboard. First, Jerome fails to see where his mistake arises from (notwithstanding his teacher's asking the whole class for help) and he tries to prove that he is right. Let's look at Christian's help for example, in front of what Jerome has written on the blackboard :

seventeen	|million|	two	|thousand|	fifty eight
17		2 00		

Christian: this (*showing the* «2») are the units… and this (*showing the «*0*»*) are the tens and these (*showing the two «*0*» from the right hand to the left hand side*) are [the hun

Jerome: [but no… this (*showing «*2*»*) is the hundreds…

Christian: this (*showing the* «2» *then «two thousand»*) are the thousand…

(*several pupils are speaking at the same time*)

Teacher: hush… hush…

Jerome: (*while writing at the blackboard*) there we write an «h»… there we write a «t»… and there the units…
(*on the blackboard we can see)*

seventeen |million| two |thousand| fifty eight

h t u

1 7 2 0 0

A pupil: he is wrong…

(*the teacher rubs out the « h », « t », and « u » Jerome has just written*)

Teacher: then... (*taking Christian by the shoulders*) they still disagree… (*one of her hands on Jerome's shoulder, while pushing Christian back to his seat with her other hand*) they can't find a common ground (*she has rubbed out «*h*», «*t*», and «*u*»,*)… who can come and give them a clear solution which both can agree on…

Some pupils: I… I...

Teacher: Fatia… okay… come… let her come...

Minute 24

Jerome: oh yes… (*whispering*)

One can see, at the end of that series of turns that Jerome, whose mistake everybody tries to explain, shows a new understanding (oh yes), a dawning appearing almost by itself.

As for Fatia, the pupil called to help Jerome, she develops her explanation: she leads. However, noticing her first hesitation (in fact, Fatia was looking for a piece of chalk), the teacher gives back the leadership to Jerome (who had kept the piece of chalk). This decision attests the fact that the teacher had noticed Jerome whispering (oh yes), so that the whole scenario (lasting ten minutes) of the successive presence of five pupils in front of the blackboard seems to have had but one function: give Jerome some respite, so as to enable him to look for a solution to his problem.

Which he does… despite the teacher's rubbing out the marks for hundreds, tens and units (the three orders in the class of thousands): Jerome keeps thinking with them. The mathematical analysis of Jerome's writing actions shows that Jerome, when making the mistake and writing «17 200 058», used two rules which dialectics he had been unable to use in a higher efficiency structure encompassing and furthering both of them. Here are these implicit rules :

R1: put the figures into the proper class;
R2: put three figures for a class;

but for Jerome, both these rules remain unconnected, so to speak « self centered », devoid of any dialectic unit. Jerome's mistake may have been produced in such a way: he wrote figures into the proper class, 17 for seventeen (million), 2 for two (thousand), and 58 for fifty eight (units), writing then three figures for a class (completing with three 0). In the episode we have just briefly introduced, he finds out a new rule R3, encompassing and furthering the previous ones so that he can articulate them.

R3: give an order a figure

Jerome obtains this new rule by reconstructing ”the large figures' numeration system”, starting from the small figures' one, as the following dialogue shows, when he is writing ”… fifty eight”:

Minute 27

Jerome: as three are left there… there… three are left… (*shows on the blackboard*) so

Teacher: wait… wait… I think [that

Jerome: [zero hundred (*writes «0»*) zero hundred… there is zero hundred…

Teacher: and [then…

Jerome: [or else we would have said one… one hundred and f… fifty eight… and as there are none we write this (*points at the last «0»*) then we write the fifty (*writes «5»*) and eight (*writes «8» so that we can read «17 002 058»*)…

Jerome managed to solve the problem, thanks to his present numbering knowledge. For example, he writes 058 because 0 must occupy the order that would be occupied by 1 if he had had to write one hundred and fifty eight, so that the rule R3 (give each order a figure) is achieved. This mathematical analysis of Jerome's utterances and writing gestures shows that he has really performed a creative work, which has enabled him to acquire the knowledge at stake in an original way. So he seems to have fullfilled the teacher's expectations, which would confirm his staying long minutes in front of the blackboard, « protected » by the teacher as in a kind of airball where he could « snatch » here and there what suited him from other pupils' help, while keeping thinking aloud. This conjunction of favorable elements enables him to build a new personal knowledge which he displays. However the teacher does not repeat, for all the other pupils: studying her preparation shows, by the way, that she never had the least intention to rely on the small figures writing rules to describe and prove how large figures' writing rules work: every rule in that field remains implicit.

Louis

Jerome's error was expected from some of the pupils, but Jerome's mathematical learning was not expected from any one of them: the teacher expects only. He does. But Louis' idea disturbs the teacher: « Why write the 0, he says ? Either we put all of them or we put none »: he suggests the writing 17 2 58, such intervals showing here the successive class names million and thousand. When asking Louis to come at the blackboard, the teacher seems to expect that the class will help Louis. Thus starts a very long episode (from minute 27 to minute 45) where the teacher, helped by pupils, endeavours to bring the heretic Louis to reason. But to no avail. We can then notice this remarkable fact : the teacher together with the pupils has but one strategy to attempt ensuring Louis' agreement at hand, which consistes in asking him to read the number a new.

Teacher's failure with Louis allows us a better understanding of teacher's success with Jerome.

Minute 45

Teacher: (*to one pupil*) no you… shut up… enough is enough… (*to Louis*) look Louis… honestly… that number over there… were you told that this seventeen was somewhere when it was written that way… read this number for me… if I give you the name of this number read good god… hush… what is this number here come on read… shut up the class…

And Louis reads up correctly ! As the study of numbering is based on no mathematical reference which mights enable the pupils to check that their conjecture is correct, the teacher has no solution available but to ask the pupil again and again to give his answer, until the expected answer appear. In this didactical environment, Jerome's position in the class can then become absolutely dominant. This is obvious through the following exchanges (see the minutes 30-32) when Nathalie, who came to back Louis in his aberration, is grilled by Jerome, who makes a brilliant comeback.

Jerome: (*from his seat*) no… but because if we remove the « 0 »… the two zeros overthere… well it isn't anylonger the same number…
Teacher: okay… come here and show this to us…
Nathalie: yes… it still does give the same number…
Teacher: hush…
Jerome: still the same number… I ask you… read it to me and then you see…
Teacher: okay… go ahead… do it… explain it to them…
Jerome: wait… here is the chalk… (*to Nathalie*) so we are going to do like you… aren't we…
Teacher: (*to Jerome*) well… wait… because they aren't going to see… there I am afraid they won't see… yes write it for us there… write for us there… and move a bit… wait for us… (*to the class*) you'll see all the same at last… well we are going to see what he wants to show us… (*to Jerome*) don't write in a too large a hand… I'm afraid that there won't be enough space there (*Jerome writes on the right side of the blackboard*

17 2 58

(background noises)
Teacher: hush… hush…
Jerome: (*to Nathalie*) well come on… read it to me now…

Minute 31
Nathalie: it is still the same number…
(*protests can be heard*)
Teacher: shush… shush… well… here I think that Jerome has… as for the others well… do you think like Nathalie that this (*shows 17 2 58*) is the same number as that (*shows 17 002 058*)…
(*one can hear yes ! yes ! no ! …*)
Teacher: tell us Louis… hush…

Louis: because here there are intervals… so this always means millions…

One can see how Jerome plays the part of deputy teacher, who shuffles pieces of chalk and who behaves with Nathalie as a teacher with a pupil. The teacher completely agrees with this behavior, she may hope that Jerome, as a didactic collaborator, will manage to clear up the situation. He has the status of deputy teacher, which is proved by this teacher's question (minute 38) to him, as he is standing up near the blackboard while two other pupils are sent back : « Come on and ask Louis something which would enable him to become aware of his mistake »

3. The mathematical work in didactical game and classroom game

> «Modelling a teaching situation consists of producing a game specific to the target knowledge, among different subsystems: the educational system, the student system, the milieu, etc. There is no question of precisely describing these subsystems except in terms of the relationships they have within the game.» (Brousseau, 1997, p. 47).

These few minutes of the classroom's work shape not only such a didactical game, but a social game. A rapid study of them show, according to us, how closely knit they are: that is to say, the ranking and categorisation process of an individual does not seem to be generated by any outside circumstance but primarily from its particular position as a pupil, in the class' didactical covenant. Thus Jerome is strong, leading enough to a) build at the blackboard a new personal knowledge in mathematics in an almost private way though helped by all the other pupils, and b) interfere within the class in the capacity of a deputy teacher who is given the task to convert the dissidents and to explain Louis his mistake while nobody, even the teacher, managed to do.

What would have happened if Jerome had made the same mistake as Louis ? A very different interactional network, doubtlessly. One must therefore understand how within this context didactical interactions are social ones as well. The pupils are assessed by the teacher, keeping in mind the background of a common didactic experience which gives them, or fails to do so, some sort of didactic legitimacy, which seems to produce most of social power in the classroom. In this episode, instead of being their own personal mathematical job, pupils' work is nothing but the strict obedience to behavioural rules, since no adidactic situation for a precise mathematical problem has ever been designed by the teacher (Brousseau, 1986, 1997). In this class' didactical contract, pupils' solving numbering exercises presupposes harnessing the rules but that is not enough. Social legitimacy is given to those assessed pupils whom the teacher can trust, and so who can have (one more time) the upperhand over the actual mathematical work.

The analysis of the here and now of the class then shows that some pupils such as Jerome don't have to respect the same working methods as the whole class, and that the didactic insight they were able to develop in the past is a capital which they can invest in the present situation. The interaction analyses thus show that a few pupils are not corseted by the rules and the didactical contract, and may allow themselves to think by themselves, as Jerome does. Most pupils, like Louis, cannot refuse to « play by the rules » because their position is impossible to maintain since it questions the very legitimacy of what is learnt: indeed, why accept one convention rather than another one if the one chosen can't be explained in a more convincing way ? But such a claim does not afford any legitimacy.

In the didactic interaction that we were observing, only those pupils who are conversant enough with the school work to accept its social and didactical rules may sometimes bypass them for their own mathematical use. Referring to the observed episodes, Jerome is one of them, Louis or Nathalie are not. They learn mathematics: their legitimacy and domination is thus increasingly justified as "the teacher's assessment of their mathematical capabilities influence the expectancy process". For example, Jerome mathematical production (min. 26-27) is not taken for granted, as the teacher does not repeat for the class; nevertheless Jerome is expected to explain the rules (min. 47), which legitimates him: then, he uses the very knowledge that he produced twenty minutes ago. His mathematical legitimacy is a didactical effect of his social legitimacy, which gives him back more didactical power. This is not the case for Christian or Fatia: have they less social or mathematical legitimacy than Jerome? Louis question asks for the rationality of the implicit rules, taking these rules for a mathematical knowledge. So that his didactical power is not reinforced during the competition with Jerome: and his social and mathematical legitimacy fall, just as Nathalie's ones do.

Notes

1. We must thank Jackie Macleod for helping us friendly to translate in good UK English a rough French version.

4. References

Bourdieu, P. (1990): *The logic of practice*. Cambridge: Polity Press.

Broccolichi, S. (1995): Domination et disqualification en situation scolaire. In Cours-Sallies, (dir.), *La liberté au travail*, Paris: Syllepse, 83-98.

Brousseau, G. (1984): The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H.G. Steiner (ed.), *Theory of mathematics Education*, pp. 110-119. Occasionnel Paper 54, Bielefeld, Germany, University of Bielefeld, Institut für Didaktik der Mathematik.

Brousseau, G. (1986): F*ondements de la didactique des mathématiques*. Thèse d'Etat, Bordeaux 1.

Brousseau, G. (1997): *Theory of didactical situations in mathematics*. Dordrecht: Kluwer Academic Publishers.

Chevallard, Y. (1992): Fundamental concepts in didactics: perspectives provided by an anthropological approach. In A. Mercier, R. Douady (eds.), *Research in Didactique of Mathematics* (selected papers published with the participation of ADIREM), pp. 131-167 Grenoble: La Pensée Sauvage.

Douglas, M. (1986): *How institutions think*. New York: Syracuse University Press.

Mehan, H. (1978): *Learning Lessons, Social Organisation in the Classroom*. Cambridge: Harvard University Press.

Mercier, A. (1997): La relation didactique et ses effets. In C. Blanchard-Laville (ed), *Variations sur leçon de mathématiques à l'Ecole «l'écriture des grands nombres»*. Paris: L'harmattan.

Reay, D. (1995): "They Employ Cleaners to Do that": habitus in the primary classroom. *British Journal of Sociology of Education*, Vol 16, N°3.

Rosenthal, R.A. & Jacobson, L.F. (1968): *Pygmalion in the Classroom. Teachers Expectations and Pupil Intellectual Development*. New-York: Holt, Rinehart & Winston.

Salin, M-H. (1997): Contraintes de la situation didactique et décisions de l'enseignante. In C. Blanchard-Laville (ed), *Variations sur leçon de mathématiques à l'Ecole «l'écriture des grands nombres»*. Paris: L'harmattan.

Sarrazy, B. (1996): *La sensibilité au contrat didactique. Le rôle des arrière-plans dans la résolution des problèmes d'arithmétique au cycle 3*. Thèse. Université de Bordeaux 2.

Schubauer-Leoni, M.L (1986): Le contrat didactique: un cadre interprétatif pour comprendre les savoirs manifestés par les élèves en mathématiques. *Journal européen de psychologie de l'éducation*, 1/2, 139-153.

Schubauer-Leoni, M-L. (1988): Le contrat didactique: une construction théorique et une connaissance pratique. *Interactions didactiques*, n° 9, Université de Genève et de Neuchâtel.

Schubauer-Leoni, M.L (1996): Etude du contrat didactique pour des élèves en difficulté en mathématiques. In C. Raisky et M. Caillot (eds.), *Au delà des didactiques, le didactique*, pp.159-189. Bruxelles: De Boeck Université.

Schubauer-Leoni, ML. & Leutenegger, F. (1997): L'enseignante, constructrise et gestionnaire de la séquence, In C. Blanchard-Laville (ed.), *Variations sur leçon de mathématiques à l'Ecole «l'écriture des grands nombres»*. Paris: L'harmattan.

Schubauer-Leoni, M.L & Perret-Clermont, A.N. (1997): Social interactions and mathematics learning. In T. Nunes & P. Bryant (eds.), *Learning and teaching mathematics*, pp. 265-283. An international Perspective.Psychology Press .

Sensevy, G.. (1997): Désirs, Institutions, Savoir. In C. Blanchard-Laville (ed.), *Variations sur leçon de mathématiques à l'Ecole «l'écriture des grands nombres»*. Paris: L'harmattan.

Sensevy, G. (1998): *Institutions didactiques, étude et autonomie à l'école élémentaire.* Paris: Presses Universitaires de France.

Sirota, R. (1986): *L'école primaire au quotidien*. Paris: Presses Universitaires de France.

HELP, METAHELPING, AND FOLK PSYCHOLOGY IN ELEMENTARY MATHEMATICS CLASSROOM INTERACTION

Natalie Naujok
Freie Universität Berlin: FB 12, WE 02, Habelschwerdter Allee 45, D-14195 Berlin, Germany
vnatasch@zedat.fu-berlin.de

Abstract: *The subject is part of an investigation on student co-operation and its functionality with respect to learning opportunities. In this paper, the aim is to reconstruct how students interactively construct help as a form of co-operation. The students' commonly negotiated ideas will be explained by applying Bruner's concept of folk psychology.*
First, the working context and the broader project are presented. Then, the author's research interest and the aim for this paper are outlined. Thirdly, the students' negotiation of helping is illustrated with an example of classroom interaction, before, fourthly, this is linked to Bruner's concept of folk psychology or pedagogy, respectively. Summarizing considerations conclude the paper.
Keywords: *student cooperation, folk psychology and folk pedagogy, social condtions of learning*

1. Working context and project

This contribution is connected to two others in the thematic group "Social interactions in mathematical learning situations", namely to those of Birgit Brandt 1999 and Götz Krummheuer 1999. The three of us form the team of interpretive classroom research at the Free University of Berlin. The paramount aim of our work is to contribute to the development of an interactional theory of the social conditions of learning and teaching in classroom settings from a sociological perspective. Our research approach and with it our perspective on lessons is based on social constructivism and interactionism as outlined by Cobb and Bauersfeld 1995. We concentrate on the social aspects and therefore on interactive exchanges. In such exchanges, meanings become negotiated and stabilised between the interactants. So, interpretation and sense-making are not just individual but social matters. In the context of school, this is evident. It is for this reason that some teachers value student cooperation so highly: the students are more involved in interaction. Taking this theoretical perspective into account, we attend the regular lessons, take video-recordings and field notes, and collect students' papers. Later, we select episodes for transcription and analysis. This selection is based on reviewing of the field notes and on watching the video-tapes. It is guided by the theoretical purposes of the study and by the methodical commitment to comparative analysis. The selected episodes are interpreted in several steps, and finally we generate elements of a theory. Altogether, the approach is microsociological and microethnographical.[1]

We are working on a project sponsored by the German Research Foundation. The project is on the reconstruction of formats of collective argumentation in the interactions of primary mathematics classrooms. For this project, we observed two learning groups, each for a continuous period of two weeks during which all classroom activities were video-taped. Until now, we concentrate on the data of the first group. It is a more or less regular first grade class, so most of the twenty-one children are seven years old. The lessons are conducted in a rather "open", non-teacher-centred way. Many lessons per week are reserved for so-called weekly work: Every Monday each child is given a list of tasks which he or she is responsible to fulfil during the week. (They all receive the same tasks.) The students sit in groups of six, three tables building a big one. Usually some of the students talk to each other while working on their tasks. These interactions build the main part of the data which is analysed with respect to cooperation among students.

2. Help and metahelping as forms of student cooperation

Within the context outlined above, I investigate different forms of student cooperation, cooperation being understood as a generic term for any form of students' interaction related to their work. What functionality with respect to learning opportunities do these cooperations have? Where and how do disturbances occur and what is the relation between student cooperation, disturbances and learning in classroom interaction? [2] Altogether, the focus is more on how cooperation works than on what comes out of it on the cognitive side. The specific topic of this contribution is help, metahelping, and folk psychology, respectively folk pedagogy. Folk psychology and pedagogy are terms used by Bruner (1990, 1996) in order to refer to common ideas about how human beings work and learn (see below). In the data, there are indications for that the students have ideas of help or helping. They have ideas of what such processes look like or what they should look like, respectively. However, these ideas are seldom made explicit: maybe because they are quite diffuse, maybe because they are so obvious. However, sometimes the students make explicit remarks thereon. This shall be called metahelping. Metahelping means helping to help by explaining someone how help should be carried out; it is talking about help or "going meta" on help.

The paper's aim is to reconstruct how students interactively construct help as a form of cooperation. Reconstructing this means to reconstruct part of the classroom's folk psychology, respectively pedagogy. And this is crucial, because such ideas guide the students' acting (see Bruner 1990, 37 f.) and with it their learning about which we want to learn more.

3. An empirical example of metahelping

The scene takes place in the first grade class mentioned above. The students' task is to work on a mathematics paper. Some students were asked to help others. Wayne just finished helping a class mate and walks up to two boys who are engaged in another helping interaction. He critiques their procedure as not being allowed. Such utterances seem to indicate what Bruner (1996) calls folk psychology. Please note lines 1-5 above all[3]:

1	Wayne	*(goes to Grigori and Jarek; Jarek is helping Grigori)*
2		uh/ one is not allowed to copy like this\
3	Jarek	of course you may\
4	Wayne	no\ you are only to tell him how to do it\
5	Jarek	uhm/ *(inaudible)* hey I **know** how to do it\

6	Teacher	well Jarek \ it's like that\ . well . Grigori does **not** understand it when he
7		**copies**\ but Grigori **does** understand it and can do it on his **own**/ . when you tell
8		him **how** to do it\ do you understand Jarek\ when you tell **how to do it**\ then he
9		can do it on his own then he doesn't need to copy anymore\
10	Jarek	yeah -
11	Teacher	all right/ well try it . **how** one does it\ *(inaudible)* you'll manage\

Here, the students talk about helping someone explicitly. The scene allows to reconstruct two notions of helping. Wayne says to other students that it was not alright to copy or to let someone copy, respectively. He seems not to accept the act of letting someone copy as an act of helping but outlines that one should only tell the other person how to proceed. One must not give the solution but explain a possible way of solving the problem. This resembles a rule about help and helping although Wayne does not give any reason or theory-like explanation for his critique.

Jarek contradicts, of course, Grigori may copy. He does not show any doubt about it. And in line <5> he says that he knew how to do it. Probably this refers to the social task of helping and not to the mathematical task. Jarek seems to accept the act of letting someone copy as an act of helping. It is possible that this implies an idea of a rule as well. This rule might be not so different from Wayne's rule. For Jarek, copying could be a way of solving the problem, seeing the task to fill in the sheet as the problem. In this case, the aim of their helping activities could be a criterion to distinguish between the boys' ideas of the situation "Helping to fill out a work sheet" (see Goos, Galbraith et al. 1996).

However, the boys do not solve their differences. The teacher joins the group and explains some rules to Jarek concerning the relationship between helping and learning. She stresses the importance of enabling someone to comprehend and to act solely. Only in that case, the other became autonomous and did not need to copy any longer. Abstractly spoken, the teacher gives a theory which backs up Wayne's comment. In argumentation analysis one would say that she repeats Wayne's statement and adds a rationale or a warrant, respectively, to it. This supports the above interpretation that, in line <4>, Wayne words a rule - and that a folk psychological or pedagogical one.

Such elements of folk psychology seem to show in other scenes, too. Only one more situation shall be told briefly. Franzi asks Daria if she may copy of her folder. Nicole who is sitting at this group table, too, says she would tell the teacher. Franzi and Nicole have a quarrel. At the end of the scene, Nicole says: .. but if you copy you can't do it\. This statement seems to include the notion that copying does not foster learning. It resembles the teacher's utterance of the previous scene and can also be explained in folk pedagogical terms as will be shown in the next section.

4. Folk psychology and folk pedagogy

Folk psychology is sort of a theory of human mind and interplay or interaction. Bruner outlines folk psychology as a means to interpret one's experiences and to construct understanding as well as to anticipate one another:

> All cultures have as one of their most powerful constitutive instruments a folk psychology, a set of more or less connected, more or less normative descriptions about how human beings "tick," what our own and other minds are like, what one can expect situated action to be like, what are possible modes of life, how one commits oneself to them, and so on. We learn our culture's folk psychology early, learn it as we learn to use the very language we acquire and to conduct the interpersonal transactions required in communal life. (Bruner, 1990: 35)

Now, folk psychology concerns all the working of human beings. Bruner includes learning processes explicitly. For this special area of folk psychology, he creates the term "folk pedagogy":

> Not only is folk psychology preoccupied with how the mind works here and now, it is also equipped with notions about how the child's mind learns and even what makes it grow. Just as we are steered in ordinary interaction by our folk psychology, so we are steered in the activity of helping children learn about the world by notions of *folk pedagogy*. (Bruner 1996: 46)

What Wayne says when going meta on helping can be regarded as such a more or less normative description. The rule "One shall not let copy but explain how to do the task!" or, from the other interactant's point of view, "One shall not copy but get explained how to do the task!" is interpreted as part of the classroom's folk pedagogy. It is like a rule which implies an intuitive theory of helping to learn.

According to Olson and Bruner (1996: 10), such theories are omnipresent but seldom made explicit. As mentioned above, also in the project's data, there are only few scenes in which pupils go meta on rules or theories. In the first scene it is the teacher who makes the theory explicit, while in the second, it is the student Nicole who verbalises her interpretation of the classroom's folk psychology.

Folk psychology, on the one hand, is something owned by a culture, on the other hand, it is something 'learnable' by an individual; and it is changing - just as language. For the acquisition of one's culture's folk psychology as well as for language acquisition participating in interaction is crucial (see Markowitz 1986; on different forms of participation see Brandt 1997). This participation is always bound to a local context and thus specific to the culture (Bruner 1995: 205) - whereby culture can refer to subgroups as well. A classroom can be such a cultural setting. Looking at the transcript in total, part of the process of establishing classroom culture becomes visible. The teacher with her ideas is engaged in this process as well as the students are. She brings in her notions, and the students more or less include them into their folk psychological constructions and reconstructions, in their interpretations and negotiations.

The concept of folk pedagogy does not only apply to adults as the quote seems to suggest (see Olson & Bruner 1996: 20; Bruner 1996: 46; and below). As illustrated by the example above, already children have ideas of how help should be provided. And it is in school, where many of these ideas get constructed and reconstructed, where folk pedagogy gets negotiated and established.

It is crucial about folk psychology and folk pedagogy (not only) in classroom interaction that

> teaching (...) is inevitably based on notions about the nature of the learner's mind. (Bruner 1996: 46)

I comprehend teaching as a role. It is not bound to the person of the teacher or to anybody else. Although, in a common article, Olson and Bruner do include the teacher in this sentence:

> teaching (...) is inevitably based on teachers' notions about the nature of the learner's mind. (1996: 11)

But students can teach as well. Surely, their teaching is based on their notions of the learner's mind, too. And as the elements of the classroom's folk psychology are interactively constructed, they are also based on the teacher's notion of learner's mind. This includes ideas of helping to learn.

Folk pedagogy is based on the idea that it is possible to help somebody else to learn. This leads to the further question, whether one can help oneself to learn, too. I think that it is possible to act in a learning-productive way. I showed elsewhere (Vollmer 1997a), for instance, how a student defends himself against another who wants to tell him a solution of a mathematical task. Thus, folk pedagogy is crucial for teaching and learning.

5. Summarizing considerations

Since the contribution in hand is a state-of-the-art report, all what can be offered for the time being is to present actual considerations.

1. Olson and Bruner argue that notions about other's minds are not necessarily verbalised (1996: 10). But it were not enough to explain what children do, one had to find out what they think they are doing as well (13). Only, how can one get to know more about such theories or assumptions about learners' minds? Olson and Bruner refer to the work on children's theory of mind (13).They describe theory of mind as one of four research lines (next to intersubjectivity, metacognition, and collaborative learning) which have in common that their subject is "the child's own folk psychology (and its growth)" (20) and which are based on seeing the child rather as a thinker than as a doer, a knower, or as "knowledgeable" (21). But they do not explain why they refer to this research line and not, e. g., to metacognition. In my

opinion, there is a difficulty, because a lot of research on children's theory of mind is carried out in laboratory settings. This does not match the cultural approach.

2. One can interpret a single students' utterances and ascribe ideas to this student which seem sufficiently plausible. For instance: Wayne might have an idea like that help aims at enabling someone to do something autonomously by telling him how to do it. This includes the idea that after one is told how to do something, one may have learned it (the learner as knower). This interpretation might match Wayne's folk pedagogy, but can only be speculative. Individual and psychological statements are not in the range of interactional analysis.
3. Assuming that folk psychology and pedagogy are bound to the local context and taking a classroom for local context, there must be classroom specific folk psychology and pedagogy. This orients students' activities during the lessons. Concurrently, folk ideas are getting developed and established in this classroom interaction, namely as part of a classroom culture. Thereby, folk psychology is not necessarily homogeneous. There are always 'subcultures'. Every individual, then, has his or her ideas of a more or less varying folk psychology. As illustrated by the first scene especially, this is at the same time a concerted construction and an individual one. In school, especially in the beginning, the point is to develop a common one.

   The aim of research is to reconstruct classrooms' folk psychologies. Because they get constructed, negotiated, and stabilised in interaction one must analyse this 'natural' interaction and focus on what becomes commonly construed. One possibility of reconstructing such a theory is to look at scenes in which the interactants argue. One may assume that in those, they appeal to the shared folk psychology explicitly in order to convince others of what they think or want - according to the motto: "I should be most convincing when I refer to elements of our shared convictions." Thus, students' going meta and arguing, e. g. on helping, and the analysis of these interactions can offer a special access to folk psychology or pedagogy.
4. To look at the interaction processes in total and to do comparative analyses is essential in order to reconstruct the social construction of classroom-specific folk pedagogy. Take a last look at the scenes. In the example with the boy students, they only give (rule-like) statements, no theory, and they do not agree on a common

view. It is the teacher who sets the standard by giving a theory-like explanation. The scene illustrates the process of (further) development of folk pedagogy impressively, because it shows how different interactants construe this subject of folk pedagogy commonly.

Notes

1. For more about the theoretical basis and methodological considerations see Vollmer 1997b.
2. This is of special importance when one thinks of a theory of learning or of practical purposes. In many classrooms, teachers encourage cooperation without taking the potentially problematic impacts on the learning process into consideration (see e. g. Vollmer & Krummheuer 1997; Vollmer 1997a; Vollmer 1997b).
3. For the rules of transcription see the end of the paper, please.

6. References

Brandt, B. (1997): Forms of participation in elementary mathematics classroom interaction. *CIEAEM 49*, Setúbal, Portugal.

Brandt, B. (1999): Recipients in Elementary Mathematics Classroom Interaction. In I. Schwank (ed.), *European Research in Mathematics Ecucation*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Bruner, J. (1990): *Acts of Meaning*. Cambridge, MA/London, UK: Harvard University Press

Bruner, J. S. (1995): Commentary. *Human Development* 38; 203-213.

Bruner, J. S. (1996): *The Culture of Education*. Cambridge, MA/London,UK: Harvard University Press.

Cobb, P. & H. Bauersfeld, eds. (1995): *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures*. Hillsdale, NJ/Hove, UK: Lawrence Erlbaum.

Goos, M., P. Galbraith, et al. (1996): When does Student Talk Become Collaborative Mathematical Discussion? In T. Clarkson. Merga (eds.), *Technology in Mathematics Education*: Proceedings of the Nineteenth Annual Conference of the Mathematics Education Research Group of Australia, Melbourne, 237-244.

Krummheuer, G. (1999): The Narrative Character of Argumentative Mathematics Classroom Interaction in Primary Education. In I. Schwank (ed.), *European Research in Mathematics Ecucation*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Markowitz, J. (1986): *Verhalten im Systemkontext. Zum Begriff des sozialen Epigramms. Diskutiert am Beispiel des Schulunterrichts*. Frankfurt a. M.: Suhrkamp.

Olson, D. R. & J. S. Bruner (1996): Folk Psychology and Folk Pedagogy. In D. R. Olson & N. Torrance (eds.), *The Handbook of Education and Human Development. New Models of Learning, Teaching and Schooling*. Cambridge, MA: Blackwell; 9-27.

Vollmer, N. (1997a): *Helping in Mathematics Group Work: Supporting and Disturbing Cooperation.* Microfiche: ERIC/CSMEE: ED 409 185.

Vollmer, N. (1997b): How Helpful is Help among Students? Theoretical Basis of and Methodological Considerations for an Interaction Theory of Learning in School. In *Proceedings of CIEAEM 49*, Setúbal, Portugal.

Vollmer, N. & G. Krummheuer (1997): Anfangen - Machen - Helfen. Zur Beziehung zwischen Arbeitsteilung und Aufgabenverständnis während einer Partnerarbeit im Mathematikunterricht. *Journal für Mathematikdidaktik.*

Signs of transcription	
Times	all utterances; no punctuation marks
bold	words spoken with emphasis
(inaudible)	the utterance of the speaking person is inaudible
/	pitchraising
\	pitchdropping
-	floating pitch
,	breathing break
. .. ...	longer breaks (the longer the more points)
italics	(facial) expression, gestures, action

IT IS NOT JUST ABOUT MATHEMATICS, IT IS ABOUT LIFE: ADDITION IN A PRIMARY CLASSROOM

Alison J. Price
Oxford Brookes University, Wheatly Campus, OX33 1HX Oxford, UK.
aprice@brookes.ac.uk

Abstract: *This paper is a discussion of a classroom session transcript. The children are in a reception class (aged 4 to 5 years) and in their third term of schooling. In mathematics they are currently learning simple addition with totals up to ten. Here I look at how the teacher uses language and examples from everyday social life to teach the children about addition and at what effect this process of situating the mathematics has on the children's learning and understanding. At issue is the social nature of learning, as the child tries to make sense both of the mathematics and of their life experiences in this situation.*
Keywords: *primary, addition, social interaction*

Children in Great Britain begin formal schooling earlier than many of their European neighbours. The role of the teacher in the reception class (two years before grade 1) is to manage an environment in which the children can learn to play and interact as social beings, to develop their social and language skills, to learn how to act as pupils, and to learn concepts and ideas that will be the 'building bricks' of their future careers in school and beyond. This current research attempts to describe and analyse the learning of mathematics in such classrooms environments. The focus is on the interactions between teacher, child, activity and situation, and on the evidence of learning that ensues.

1. Theoretical background

This paper describes a single teaching incident drawn from a wider collection of classroom observations which set out to look at mathematics learning from a social constructivist perspective. Through the work of Piaget and his colleagues (Inhelder & Piaget 1958; Piaget 1966) children are seen to construct their own understanding through experience with the environment. This view has been further developed in mathematics education, for example by von Glasersfeld (1991; 1995) who describes a radical constructivist view as having two basic principles :

- "knowledge is not passively received but built up by the cognizing subject;
- the function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality." (1995 p. 18)

von Glasersfeld does not however recognise a special and separate place for the role of social interaction in this construction process asking "how do these 'others', the other people with whom the child populates his or her experiential world, differ from the innumerable physical objects the child constructs?" (ibid. p.12). However this is to see these 'others' only from the viewpoint of the child and not of the adult, since the 'innumerable other physical objects' do not intent to 'teach' while the adults have intention to interact with the child and the environment to bring about learning. It is from this view of learning that I began the research. However my own view of social constructivism is also being socially constructed; it changes almost daily through interaction with others, personally or through the written page, with observations of children in learning situations and with my reflections on these. Through observation I have come to realise the extent to which children are constructing mathematics through teaching situations which include so much more than the mathematics in terms of content, context and interactions. The children in the mathematics lesson are not just learning mathematics but also learning to be children in a mathematics lesson. They are also bringing to the lesson not just mathematics previously learned but also their own ways of making sense of the world (Donaldson 1978). So analysis of classroom interactions is showing the importance of social interaction, broadening the theoretical background to include not only the role of language and communication (Pimm 1987; Durkin and Shire 1991) and the role of the adult in advancing learning through scaffolding the child's learning within the zone of proximal development (Vygotsky 1978; Bruner 1967; 1982), but also consideration of how the social world of the classroom affects children's mathematical learning (see for example Wood & Yackel 1990; Cobb, Wood et al. 1991; Cobb 1994; Cobb & Bauersfeld 1995) and the sociocultural aspect of the classroom echoing the work of Lave and Wenger (1991). Much of this latter work is relatively new to me and I look forward to the conference to further develop my thinking in these areas.

2. Data collection and analysis

The data is qualitative, collected as a result of observation by the researcher in the classroom. Extensive notes were taken at the time both of what was said and the actions, and sometimes demeanour, of the teacher and children. Some of the whole class interactions were also audiotaped, but when observing group work it did not prove possible to do so due to the quiet voices of the children being studied and the noise of other children working around them. Reception classrooms are very active places with considerable movement and noise. Notes were also taken of discussions with the teacher before and after the lesson. Analysis of the data follows procedures suggested by Strauss and Corbin (1990) who speak of "the discovery of theory from data systematically obtained". Each teaching episode (critical event) is analysed in depth in order to generate theory rather than to test existing theory.

3. The bus lesson

The session starts with the twenty-seven children sitting together with their classteacher. The teacher (Beth) introduces the session with a rhyme. Each time the rhyme is said, different numbers of children are said to get on the bus creating pairs of numbers for addition.

Beth *Here comes the bus it soon will stop,*
Hurry up children in you pop,
5 inside and 5 on top,
How many altogether?

The children raise their hands and one is chosen to answer. Each time, today, the total is ten.

Beth *Well done ...Now, can we all see the board?*

Beth draws a bus outline on the board. She has also prepared ten circles of card with children's faces drawn on them, and with blutack on the back so that they will stick onto the board.

Beth *Let's see if we have got ten children (counts the faces with the class). Now, all the children want to go on the bus. Do you think they are going to the Garden Centre, or perhaps they are going swimming.*

The purpose of this lesson was identified with the teacher as being "addition of pairs of numbers that make ten, ... but I also want to talk to them about 'the same', and 'more and less' ". The teacher does not identify this to the children but goes straight into the context, that of children travelling on a bus. This was a significant lesson since it was the first occasion where the mathematics was situated in a more abstract, though familiar, setting. Previously the children had used actual objects (beads, counters, bricks etc.) to represent numbers. Here the teacher has a set of pictures to help the children by representing the situation, but the idea is introduced as if it represents a real life setting not just as if the pictures are there to count. The children had very recent experience of travelling in buses and could bring that experience to their understanding of the lesson. The previous week the class had been on a visit to a large garden centre as part of their science work on plant growth. They also went swimming once a week during this term. For each of these they were taken in a bus.

Discussion of why the children are on the bus could be seen to be irrelevant to the mathematics but situates the mathematics in a familiar setting for the children. Although the teacher is trying to move the children into a less concrete more abstract understanding of number, it is still related to experience.

Beth *Now, let's put some on the bus (puts five faces onto the top deck counting) 1, 2, 3, 4, 5. So, how many inside (places rest of faces onto lower deck counting) 1, 2, 3, 4, 5.*

Charles *What about the bus driver?* *(this is ignored)*

The teacher moves into mathematics mode now. She places the 'faces' on the bus to represent children looking out of the windows. Charles, however is more concerned about the situation. His query "what about the bus driver", is a relevant one for him. If the children are going on a journey they will need a driver. Children cannot drive the bus. Beth ignores this since for her it is not relevant to the mathematics.

Beth *Who can tell me a number story about this?*
Emma *Five add five altogether make ten.*
Beth *Can anyone else tell me something about this story, some other little sentence? Has anyone noticed anything about the two numbers?*
Several children responded with statements like 'If we put one more on top it would be six and four.' None of these responses were accepted.
Charlotte *They are both equal.*

The children were used to the idea of a number story being a mathematical representation of the situation. Emma's response was therefore the expected one. The teacher's second question however is ambiguous. It relates to her second learning intention identified above, that of wanting to "talk to them about the same, and more and less". But the children are not aware of this and are still concentrating on creating number sentences. Charlotte's answer "they are both equal" comes closest to the teacher's intention and is therefore accepted. The teacher did not pursue this since her train of thought was interrupted by Charles' persistence.

Charles *(again) What about the driver?*
Beth *We're not worried about the driver. I think he has gone for a cup of coffee. Now, while he was having his coffee some of the children started to run about, so ... she went upstairs ... and so did he ... and so did she (moving a face each time.)*

Beth now recognises that Charles' query is of real concern to him and needs to be addressed. The mathematical answer could be that the driver is irrelevant, we are only interested in counting and adding up the children; yet it might be that Charles is concerned that there will have to be eleven, rather than ten, people on the bus, so affecting the mathematics. The teacher therefore reverts to the situation and makes up a reason for the driver not being present, "I think he has gone for a cup of coffee". This answer seems to satisfy Charles since he does not mention the driver again. It also gives the teacher a way into the next part of the story. "Now, while he was having his coffee some of the children started to run about". The absence of the driver allows the children on the bus to do something they would not normally be allowed to do, and results in a change in the situation mathematically.

Beth	*Who can tell me something about the numbers now?*
Ian	*It still makes ten*
Beth	*Very good. Ian, can you tell me about this? How many on top?*
Ian	*Eight.*
Beth	*And how many downstairs?*
Ian	*Two*
Beth	*So, can you tell me anything else?*
Ian	*Eight add two altogether makes ten.*

Ian's observation that it still makes ten shows an understanding of conservation of number that is not always present in children of that age. Piaget showed that young children did not always realise that when objects were moved around spatially their quantity did not alter. Ian had previously been observed in practical situations having to recount bricks to check that the total was still the same. However here he responded without counting. It is perhaps because of the socially defined situation that he is able to identify that, since the children only moved around on the bus, none got on or off, there were still ten present. Beth has constructed the situation which allows him to do something which he is unable to do alone, scaffolding his understanding of conservation. Beth congratulates him on this observation then requires him to interpret the new number combination using the language normally accepted in the classroom.

Beth	*Who can tell me something else? (there is no response) ... Charlotte told me that they were the same last time. Angela, can you tell me which section has more or less in it?*
Angela	*There's more upstairs.*
Beth	*David, can you tell me a sentence with the word less in it?*
David	*There's less downstairs.*
Beth	*Can you all say that? There's more upstairs and there's less downstairs (children join in).*

This was new learning for the class. Although the children used these words naturally in terms of comparisons in everyday situations, e.g. "I've got more buttons than you," they had not been required to use them in more formal mathematical situations. After the lesson Beth shared that she had hoped to move onto comparisons of the numbers ‘two is less than eight’, perhaps introducing subtraction ‘two is six less than eight’, however she now realised that the children were finding these comparisons difficult. None of the children offered comparative sentences without encouragement and their articulation was limited. She therefore asked all the children to repeat the sentences that had been offered and moved on. They would do further work on this later.

At this stage the class were split into groups to work on different tasks all related to the number ten. Some children worked with the classroom assistant and some unaided. All of the children would eventually do each of the tasks. One group continued to work with Beth. This consisted of Emma (5), Angela (5), Charles (4), Ian (5), Jacob (5).

Beth Right, you are getting very good at adding aren't you?
She rehearses the bus rhyme with them once more using 5 + 5 and 9 +1.
Beth We are going to do some sums together now. What is 8 and 2 more?
Jacob Ten
Beth 7 and 3 more?
Emma Ten
The children seem to be remembering these except for Ian who was seen calculating on his fingers.

This incident was noteworthy since it was the first occasion I observed Mrs. B. asking the children calculations out of context. Although the questions are obviously related to the bus situation they had just left, the pictures are no longer there to make the link. So, Ian who previously conserved the number ten in context did not make this link and was required to use his fingers to represent the numbers in order to calculate an answer. Jacob and Emma seem to have made the link since they answered immediately yet had not previously made sums to ten and could not later do these calculations out of context.

The group were each given their maths books with a drawing of a bus and some instructions written in.
Beth Right, if you look in your books you will see the bus. What does it say? 'Draw 10 children on the bus'.
Charles I can read that.

Beth *I thought you could. You are getting very good at reading.*

As noted previously (Price 1997) the teacher is not only teaching these children mathematics but also social relationships and all other areas of the curriculum. The written instructions were also new to this lesson. Previously instructions were given orally but Beth felt that the children's reading was improving and this gave her the opportunity to use writing with a purpose in context (Clay 1975; Hall 1989). It is perhaps unlikely that Charles would have been able to read all of this sentence out of context (Charles is still not yet five) but here the context seems to act as a scaffold for learning within his zone of proximal development.

The children open their books and begin to draw children in the bus.

Beth *Think about how many you are going to put on the top and bottom. They are a bit naughty these children running up and down the stairs.*

Beth here reminds the children of the task, again in context. She includes a moral element here; children should not be allowed to run up and down the stairs and are therefore being naughty. This is in no way an essential part of the mathematics task and yet it keeps the children on task in terms of relating the mathematics to the bus story.

The children drew faces in the windows of their bus as requested.

Beth *Right, when you have done that I want you to write a little number sentence underneath. Jacob, read out to me what your number sentence is.*

Jacob *Eight and two altogether makes ten. We're learning about ten.*

Beth *What do you know about ten?*

Jacob *If you are ten you can run faster than me, because you know Martin who comes and sees me sometimes, he's ten and soon he is going to be eleven.*

Jacob's observation 'we're learning about ten' was uninvited. It was also mathematically perceptive to the extent that he realised that was the object of the lesson. He did not offer 'we're learning about busses' or 'we're learning about children' both of which are equally true. His answer to the teacher's question 'what do you know about ten' now appears incongruous. The expected answer might be something like 'you can make ten in lots of different ways'.

Yet what Jacob is doing is relating back to other information that he already knows about the number ten. His relationship with the older boy Martin is important to him and at the age of five, ten seems a great age. Yet by explaining about Martin's age he also shows that he has knowledge of ordinal number: now ten, soon eleven. I feel that this incident gives some added insight into the workings of the child's mind, busy constructing understanding. The number ten is not just an abstract number to be learnt about in maths. It also conjures up relationships and feelings that could seem irrelevant to the teacher, yet are important to the child. I do not think it an accident that this lesson, so rich in situation about the children running around on the bus with their friends while the driver has gone for a cup of coffee, should elicit another, though very different, situation with meaning for the child.

4. Conclusion and implications

The young children in this class are learning about early addition concepts situated in concrete and social situations. The previous work has mainly been in the form of concrete apparatus, and I have observed the effect that such concrete apparatus has on their learning (Price 1997). That this lesson situated the addition in the more social context of a bus journey has allowed analysis of the effect of this on the children's learning. This was shown to have three effects:

First, the situation can give rise to ideas that are not part of the mathematics but which may distract the child's thought. So, Charles was not able to concentrate on the mathematics of children on the bus until he had resolved where the driver was. In a similar way, concrete apparatus used in a previous lesson had caused distractions which resulted from emphasis on the shape and spatial relationship between cubes, instead of the number of them. Mathematical concepts like addition are essentially abstract yet we cannot teach them to young children in a purely abstract way. We therefore want children to experience addition in a wide range of settings so that they can generalise the concept from these settings, but as teachers we also need to be aware of the distractions these cause and to ask how we can help the child see past the distractions.

Secondly, the situation may act as a scaffold to allow children to achieve in ways that a more abstract setting would not. Ian was seen to be able to conserve number in the context of children on the bus though he had previously been unable to do so with concrete materials in full view. He later needed to revert to counting when working in the small group, perhaps because at this stage he did not realise that the situation was the same and the bus and children pictures had been left behind. He needed both the social situation *and* the bus picture to give him a confident understanding of conservation. Whereas the standard Piagetian conservation of number task appears to confuse the child who cannot see whether to concentrate on the discrete or continuous nature of the line of counters, is it the discrete nature of children and the clear boundary of the bus that aid Ian's thinking?

Thirdly, situating the mathematics in a social setting appears to allow the children to reflect on other social settings with respect to mathematics. Jacob's story about Martin was not related to the social context of the bus in any way we could recognise, yet gives us insight into the links he is making in his mind. We do not see here the construction of mathematical concepts in the abstract, for the child is making links to other experiences of life which number words and concepts bring to mind. Perhaps it is the maintenance of such links that will allow children to use and apply their mathematics more effectively in the future.

The mathematics pupils in the classroom are not just 'doing maths'. They are trying to make sense of the world, making links between their present experience and those of the past. For these young children it is not just about mathematics it is about life. An understanding of the social world of children is essential to our understanding of how children learn mathematics.

Acknowledgement

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5. References

Bruner, J. S. (1967): *Towards a Theory of Instruction*. Harvard: Belknap Press.

Bruner, J. S. (1982): The organization of action and the nature of adult-infant transactions. In M. von Cranach & R. Harre (eds.), *The Analysis of Action*. Cambridge: CUP.

Clay, M. M. (1975): *What did I write?* Auckland, New Zealand: Heinemann.

Cobb, P., ed. (1994): *Learning Mathematics: Constructivist and Interactionist Theories of Mathematical Development.* Dortrecht: Kluwer Ac.

Cobb, P. & H. Bauersfeld, eds. (1995): *The Emergence of Mathematical Meaning.* Hillsdale, New Jersey: Erlbaum.

Cobb, P., T. Wood, et al. (1991): A Constructivist Approach to Second Grade Mathematics. In E. von Glasersfeld (ed.), *Radical Constructivism in Mathematics Education*. Dortrecht: Kluwer Ac.

Donaldson, M. (1978): *Children's Mind*s. London: Fontana.

Hall, N., ed. (1989): *Writing With Reason*. London: Hodder and Stoughton.

Inhelder, B. & J. Piaget (1958): *The Growth of Logical Thinking From Childhood to Adolescence*. USA: Basic Books Inc.

Piaget, J. (1966): *Biology and Knowledge*. Edinburgh: Edinburgh University Press.

Price, A. J. (1997): *Constructing an Understanding of Early Number*. British Society for Research into Learning Mathematics. Bristol, November 1997.

Strauss, A., & Corbin, J. (1990): *Basics of Qualitative Research: Grounded Theory Procedures and Techniques*. Newbury Park, California: Sage.

von Glasersfeld, E., (1995): *Radical Constructivism*. London: Falmer Press.

von Glasersfeld, E., ed. (1991): *Radical Constructivism in Mathematics Education*. Dortrecht: Kluwer Ac.

Vygotsky, L. S. (1978): *Mind In Society*. Cambridge, Mass.: MIT Press.

Wood, T. & E. Yackel (1990): The Development of Collaborative Discourse with Small Group Interactions. In L. P. Steffe & T. Wood (ed.), *Transforming Children's Mathematics Education: International Perspectives*. New Jersey: Erlbaum.

CESAME: THE PERSONAL HISTORY OF LEARNING MATHEMATICS IN THE CLASSROOM: AN ANALYSIS OF SOME STUDENTS NARRATIVES

Catherine Sackur[1], Teresa Assude[2], Maryse Maurel[3]

[1]IREM de Nice – UNSA, Parc Valrose, 06108 Nice Cedex 2, France

sackur@unice.fr

[2]DIDIREM (Paris 7) et IUFM de Versailles, 336E, ch. Barthélémy Florent, F-83200-Toulon, France

assude@gauss.math.jussieu.fr

[3]GECO (IREM de Nice)

maurel@unice.fr

***Abstract**: Our paper comes from the work we did in the GECO and more precisely in a project named CESAME supported by the IUFM (Institut Universitaire pour la Formation des Maîtres) of Nice. CESAME means: "Construction Expérientielle du Savoir, et Autrui dans les Mathématiques Enseignées" and our aim is to interpret what students say or write about their experience of mathematics and more precisely about their experience of the Necessity in mathematics (see the paper by Drouhard presented in the group of CERME). In this paper we shall present the part of our work which concerns the way to "reach" this personal experience, so we shall examine how one can make the students produce narratives about their experience. We shall develop some theoretical questions, examine what could be efficient methodologies and present and analyse some narratives we have collected. We make use of some concepts which have been developed by the French didacticians, mainly Brousseau and Chevallard. References of their works are given in the bibliography; moreover we give short illustrations at the end of this paper.*

***Keywords:** social constructivism, time, narrative*

1. First step

At the beginning of the year, we very often ask our students (high school, university or pre-service teachers) the following question: "Tell briefly a personal memory, something which happened to you last year or before, on a mathematical subject. This anecdote should have a mathematical content, and concern yourself (do not recall the day when you fell off your chair, and do not cite merely the title of a chapter ". This question may be formulated in different ways, for example "tell two memories concerning mathematics, one which impressed you positively, the other one negatively".

The purpose of such a question is to try to catch what students might say about their experience in mathematics and about their relation to mathematics. This is interesting for us, both as researchers and as teachers.

What is the importance for us of such a demand? To cope with this question it appears to be useful to introduce the notion of "didactical time". The "didactical time" means the time which regulates the way knowledge processes and the way in which knowledge is divided for teaching (Chevallard & Mercier, 1987). The didactical time gives the norms for the process of knowledge, and it organises the chronology of the students' learning. Its management is under the responsibility of the teacher. Generally the students cannot convert the didactical time into a personal history of their own knowledge. That means that in the math classroom there is almost no time for the students to express their own personal relation to time; in a more precise way, we can say that there is no time for them as individuals to say "I did not know; then I did this…, and now I know".

Asking a student to recall memories gives her/him the opportunity:

- to identify an event which is important for her/him in her/his personal history of mathematical knowledge or from the point of view of her/his relations to mathematics.
- to put words on it and thus to make it exist for her/him, interacting with others, since a narrative is not completely private.

For us researchers, the interest is double in the way that we believe that:

- the history of knowledge is part of the knowledge

- mathematical knowledge is not limited to an accumulation of definitions and theorems, but includes knowledge about the nature of mathematics (Assude & al. 1997). When recalling a math event the student identifies what is "mathematical" for him and this is a knowledge about mathematics which, for us, is part of an individual's mathematical knowledge.

2. Time and narratives

We shall examine here some theoretical questions about narratives, based mainly on the works of P. Ricoeur (1983). As we said before and as we shall see more precisely later, the time in the classroom is "blank" for the student. When they are asked to recall a personal memory concerning mathematics, they do not imagine that they may evoke the way they lived during the time of the class of mathematics, their own, personal history. There seems to be no capitalisation of this temporality as they are not able, or they don't allow themselves to tell the story of their personal knowledge.

The problem our students are facing when confronted with our demand is:

- What is the meaning of the question "what happened to me? " ?
- How could I say it?

These two questions are central in our preoccupation because we think that students certainly build stories about the time they are living and about the events in this time. What could be the content of such stories? It should not be a mere accumulation of anecdotes; one doesn't need to re-live or re-evoke the entire situation with all its details to understand the meaning of a mathematical object. Things are certainly more complex: when the story is stored in the memory, the different events are related to one another, some disappear being included into others, the chronology also may be changed. In fact one builds a new story which can be stored but also which can be recalled upon need. Roger Schank (1995) says (translated into English by the authors (sorry for the poor English!)):

> "The creation of a narrative is a process which implies the memory. Why do human beings tell stories? For a very simple reason; because the process itself of

> creation of the story creates the mnemonic structure in which the gist of the story will remain for the rest of our life. To talk is to remember. Certainly, psychologists have known for a long time that repetition helps memory. But the fact of "telling a story" is not repetition; it is creation, and the fact of creation is a full mnemonic experience."

Moreover stories which are most able to be told, and often the most interesting ones to listen to or to read are those related to frustrating experiences. To put our memories into shape, and to learn, we tell stories in which difficulties have been solved in a way or another. Thus we memorise the solutions, and build a new knowledge, which is stored to be used in the future."

Our hypothesis is that learning includes developing the ability to recognise in the *present* an "echo" of a *past* story which lies in our memory. This permits us in the *present* to anticipate, to know whay our expectations are, to understand and thus act for the *future*.

In other words, the theoretical bases of narratives lie in this triple representation of *past*, *present* and *future*. They answer the question of the signification of temporal experience. The *present* is both the time of *memory* and the time of *anticipation*, that is of attention for a project concerning the future.

We shall examine now if we can find in the memories of the students what we are looking for: the expression of temporal construction and use of knowledge and among others the experience of necessity.

3. Typology of memories

We asked a wide variety of students about their memories: high school students, university students, pre-service teachers (primary school) and mathematics pre-service teachers. Most of the narratives we talk about in this paper are collected at the very beginning of the year when the teacher first meets with the students. As one can easily understand by reading the present paper, we, in the CESAME group do think that memory of learning is an important issue in the process of learning mathematics; we then present the work in the math class emphasising this point. This is the reason why we first ask for the narratives and then start talking with the students about what we think is mathematics learning. Later, during the school year we ask for other narratives and one of the subjects of our work is precisely the role these could play in the teaching and learning of mathematics.

We have just began the analysis of some of these memories and we shall present here the very first results so as to illustrate both our aims as researchers and as practitioners. As a consequence of our theoretical background we have constructed a grid for this analysis which permits us to give a typology of the memories and is a tool for interpretation.

Grid for the analysis of the narratives

I. ACTORS

	who is in the story?	who is acting?	who is the interlocutor?	who is thinking?
the student				
another student				
the group				
the teacher				
“someone“				
another				

II. MATHEMATICAL CONTENT

- label
- precise content
- precise statement

III. NON MATHEMATICAL CONTENT

- evaluation
- test
- result of tests

- oral questioning

IV. WHAT ARE THEY EXPERIENCING?

- understanding
- not understanding
- break of “didactical contract“
- epistemological break
- positive experience
- negative experience

V. STYLE OF WORK

- classical
- discussion
- group

VI. CHRONOLOY

VII. LEVEL OF KNOWLEDGE

The analysis of the first narratives we collected (about one hundred) led us to three observations; the first is that most memories we studied involve institutional matters. We call them “non mathematical contents”, such as tests…(see above). Most of these memories relate negative experiences, unfairness or humiliation. A second observation is that no “other” is mentioned except the teacher who is perceived either like an outstanding person or like an humiliating person and very seldom like someone who helps in learning mathematics. The third one is that mathematical contents are rare: there are very few mathematical events.

Here are two examples of narratives produced by 12th graders:

Eric: *During the lesson on primitives which lasted one week, I used to mix up primitives and derivatives in almost all excises. For the test I managed not to make the confusion and I got a mark of 12/20.*

I. ACTOR

	who is in the story?	who is acting?	who is the interlocutor?	who is thinking?
the student	*	*		
another student				
the group				
the teacher				
“someone“				
another				

II. MATHEMATICAL CONTENT

- *label* *

III. NON MATHEMATICAL CONTENT

- *test* *

- *result of tests* *

IV. WHAT ARE THEY EXPERIENCING?

- *positive experience* *

- *negative experience* *

V. STYLE OF WORK

- *classical* *

VI. CHRONOLOY *not mathematical*

VII. LEVEL OF KNOWLEDGE *nothing*

Caroline: *When I was in grade 9, my ambition was to become a math teacher. My teacher proposed that I would teach the lesson on linear systems (2 equations, 2 unknowns). I never had had such a terrible experience: nobody was listening to me. Hence I don’t want to become a math teacher anymore.*

I. ACTORS

	who is in the story?	who is acting?	who is the interlocutor?	who is thinking?
the student	*	*		
another student				
the group				
the teacher	*	*	*	
“someone“				
another				

II. MATHEMATICAL CONTENT

- *label* *

III. NON MATHEMATICAL CONTENT

- *oral questioning* *

IV. WHAT ARE THEY EXPERIENCING?

- *negative experience* *

V. STYLE OF WORK *not classical*

VI. CHRONOLOY *not mathematical*

VII. LEVEL OF KNOWLEDGE *no*

As a conclusion, we may say that students produce memories without mathematical knowledge (although the demand was explicit on that point), involving no other people but the teacher, without any chronology. There is no “before”, no “ now”, no “after”. Now the question is: either the students never experienced any mathematical event, or they don’t remember them (that means that they didn’t identify them as mathematical), or they don’t feel allowed to talk about them.

These first results show us the importance of institutional liabilities which weight on the students when they are facing an unusual demand. This questions us from different points:

- Is there room in the math class for students to live a personal story about mathematics?
- What do students allow themselves to recall about their personal relationship with mathematics?
- What is the role of the history of the construction of knowledge in the knowledge itself?
- How could we lead students to talk about the experiences they were confronted to in the math class?

From the point of view of the practitioner, these observations give us a clue to make things change. We have to legitimate, from the point of view of the institution, the facts of talking about mathematics, of living mathematical events in the math class… A use of narratives, which would become frequent, could help us in this direction. They could become a didactical tool which would help the student to contact her/his own personal experience of mathematics and identify the nature of her/his relationship with mathematical knowledge. The remaining problem is how to make students produce the narratives and how to use them in the classroom.

4. Some examples of narratives

As we said before, students limit themselves to the institutional aspects, where time is not something to share and is not related to their own work. The school as an institution gives no importance to the way time flows for the students and to the way they live in the classroom.

As an extreme example of this non-existence of a lived and shared time we can cite this narrative of a mathematics pre-service teacher:

> "*positively: addition*
> *negatively: subtraction*
> *more seriously, thing that were really important have nothing to do with mathematics. I'm sorry.*"

We can observe here that someone who is almost a math teacher has nothing to tell about her/his experience of mathematics. S/he will not be able to use it while teaching, to have it as a resource when s/he encounters difficulties in her/his teaching. This narrative shows that the memories can be a sign of the nature of the relationship with mathematics. This is important for a practitioner.

We will oppose to this narrative that of another pre-service teacher:

> *"During my first year at the university I had once the satisfaction of solving a problem with a solution which I entirely constructed myself. Then the teacher worked for a whole hour to try to find where was the bug, but never managed to find any."*

Here we think that we find the existence of a mathematical memory: the student talks about an experience that s/he lived, which had to do with the necessity in mathematics, which involved others, the teachers and the group of students as interlocutors.

Rather than very dull stories about institutional liabilities which are what we most often gather, we shall cite two other narratives which have been collected in the middle of the year. In this class the teacher used to allow the students to express their personal temporality by proposing styles of work which are not classical which we shall not describe here and by discussing the role of memory. So after some months he asked the question: "two or three things which happened since September. I remember that…"

> *"When I was in grade 10, I used to think that if a unction was growing its limit had to be + ∞. This year I learned, while studying the chapter on functions, that a function could be growing and have an upper bound at the same time."*

Here one can find clearly the expression of a lived temporality. This 11th grader expresses the memory of a mathematical event.

Here is another narrative:

> *"I remember failing on the inequality $x < x^2$. I thought it was a true inequality, and now I know that, of course, it is false."*.

This student has, some months after the event took place, a sort of evidence, which is certainly not the result of a convention. This narrative leaves room to time, and we can find something which appears to us like necessity.

5. Conclusion

A majority of narratives do not include anything related to mathematical content, to other people than the teacher, and to mathematical events. Nevertheless we have shown that some narratives do give room to the expression of time and of necessity. We think that those illustrate a better relationship to mathematical knowledge than the others.

Our further work will tend to go further in the interpretation of these narratives and in the study of some questions such as: What is the role the narratives could play in the classroom? Could they lead to a better construction of mathematical knowledge? How could we make students produce them: how make students build stories for themselves about their life in the math classroom? How to ask to get these stories?

6. References

Assude T. & Paquelier Y. (1997): L' atelier de recherche mathématique au primaire. Problèmes liés à la gestion du temps et de la mémoire. *Actes de la CIEAEM 49*, Setubal.

Assude, Drouhard, Maurel, Paquelier & Sackur, (1997): Présentation des travaux du groupe CESAME. *Actes de la 9ème Ecole d' Eté de Didactique des Mathématiques*, C. Comiti et al. Eds, Houlgate.

Brousseau G. (1986): Fondements et Méthodes de la Didactique des Mathématiques. *Recherches en Didactique des Mathématiques*, vol. 7-2, pp. 33-115.

Brousseau G. & Centeno J. (1991): Rôle de la mémoire didactique de l'enseignant. *Recherches en Didactique des Mathématiques,* vol. 11-2.3, pp. 167-210.

Chevallard Y. & Mercier A. (1987): *Sur la formation historique du temps didactique*, Publications de l' IREM de d' Aix-Marseille n° 8.

Drouhard J-Ph. (1996): Communication in the classroom with a CAS: The Double Didactic Pyramid Model. In B. Kutzler & J. Monaghan, (eds.), *The State of Computer Algebra in Mathematics Education*, Chartwell-Bratt.

Drouhard & al (1999): Communication in the classroom: the notions of necessary mathematical statements and of levels of knowledge. In I. Schwank (ed.), *European Research in Mathematics Ecucation*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Ricoeur P (1983): *Temps et récit.* Seuil Points, Paris, 3 Tomes.

Sackur, Drouhard, Maurel & Pecal (1997): Comment identifier les connaissances cachées en algèbre et qu'en faire? *Repères-IREM*, 28.

Schank R. (1995): De la mémoire humaine à la mémoire artificielle. *La Recherche* n° 273, vol 26, pp 150-155.

Notes on didactical situation and didactical contract:

Didactical and a-didactical situations are situations of teaching in the classroom; in a didactical situation the intention of teaching some knowledge is explicit: the teacher says: "today we shall study the concept of derivatives". In an a-didactical situation the intention is not explicated; the teacher gives some problems to the students; to solve the problems the students will need to learn something which will be later identified as the concept of derivatives. In an a-didactical situation the need for the concept is better perceived by the students.

Didactical contract means the set of more or less implicit rules which regulate the relations between teacher and students in the classroom. Some are institutional and very general rules (the students must sit during the math class), others are very specific ones linked to mathematics. Breaking the didactical contract produces facts which permit to identify some of the rules: the teacher is not supposed to give a test on a lesson which was studied the year before; a problem should not lead to the answer: "one cannot know".

Such problems have been studied under the concept of "the Age of the Captain" at elementary school level. An example is: "in a classroom there are 12 girls and 10 boys, what is the age of the teacher?" Almost no child says: "I cannot know". More than that if the answer is obviously ridiculous the children will say: "it's your fault, you didn't give me the right numbers!". The effect of the didactical contract is that any question should have an answer using the data of the exercise and the computing capacities the child recognises her/himself.

MATHEMATICAL INTERACTIONS AS AN AUTOPOETIC SYSTEM: SOCIAL AND EPISTEMOLOGICAL INTERRELATIONS

Heinz Steinbring
Universität Dortmund, FB Mathematik / Didaktik der Mathematik, Vogelpothsweg 87, D–44221 Dortmund, Germany
heinz.steinbring@mathematik.uni-dortmund.de

Abstract: *What are essential characteristics of mathematical interaction? For trying to answer to this question, a broader question is raised: In which way general social communication functions as an autonomous system? Using the concept of "social communication as an autopoietic system" (Luhmann, 1997) an everyday teaching episode is described and analyzed. From this general theoretical perspective one can discover that in the course of this episode one teacher–student interaction fails whereas another interaction seems to succeed. Reasons for the failure or the success can only be developed from an epistemological point of view. An epistemological analysis of the episode clarifies important characteristics of mathematical interaction in contrast to general social communication. Correspondences between the epistemological constraints of mathematical interaction and the function mechanism of social communication as an autopoietic system are identified.*
Keywords: *mathematical interaction, epistemological analysis, social communication*

1. Mathematics teaching as a social process of communication

The fundamental separation between social and psychic processes has strongly been emphasized by N. Luhmann (1997).

> "Pedagogy could scarcely admit that psychic processes and social processes operate completely separately. But the consciousness of individuals cannot reach other individuals with its own operations. … But when communication shall come about, another, a likewise closed, a likewise autopoietic system has to be activated, that is a social system, that reproduces communications by communications and does not make more than this." (Luhmann 1996, p. 279)

The concept of “autopoietic system” has been introduced by Maturana and Varela (1987); it characterizes self-referential systems consisting of components which are permanently re-produced within the system for its preservation. Not only biologic processes but also social and psychic processes are investigated with this concept. What is the essential difference between a social and a psychic process?

> “A social system cannot think, a psychological system cannot communicate. Nevertheless, from a causal view there are immense, highly complex interdependencies.” (Luhmann 1997, p. 28)

The consequence for mathematics education is, that *direct* connections or *immediate* influences between the social system “mathematics teaching” and the psychological system “mathematics learning” of the students are definitely impossible. The effects from teaching on learning are highly complex interdependencies. Therefore teaching cannot automatically induce understanding in the consciousness of students.

> “Understanding is never a mere duplication of the conveyance in another consciousness, but within the communication system it is itself a connection condition for further communication, that is a condition of the autopoiesis of the social system.” (Luhmann 1997, p. 22)

For constructing possible “connections” between communication and consciousness language is a central means.

> “The specificity of the relation of communication and consciousness depends … on the fact that this coincidence thanks to the availability of language … does not happen by chance, but is expected and even can be partly planed. … [one] can … say, that spoken communication can treat the psychic systems as a medium, which is always ready to take on communicative forms. Consciousness on its part can use language for treating communication as a medium upon which it always can impress its forms; for a verbally expressed thought can always be communicated and in this way force the communication process to convert a psychic stimulus.” (GLU, 1997, p. 87)

Within (spoken or written) language one has to distinguish between »sign« (more exactly »signifier«, which might also be given by a sound) and »sense«. This distinction between sign and intended meaning is the starting point – the take off (Luhmann, 1997, p. 208) – for the autopoiesis of communicative systems. Here, Luhmann refers to the work of de Saussure (1967).

> "Signs are also forms, that means marked distinctions. Signs distinguish, following Saussure, the signified (signifiant) from the signifier (signifié). In the form of the sign, that means in the relation between signifier and signified, there are referents: The signifier signifies the signified. But the form itself (and only this should be named sign) has no reference; it functions only as a distinction, and that only when it is actually used as such." (Luhmann, 1997, pp. 208).

How the autopoiesis of communication is possible? In the course of interaction the participants mutually provide with their "*conveyances*" (or communicative actions) "signifiers" that may signify certain "information" (signifieds).

> "Decisive might be …, that speaking (and imitating gestures) elucidates an intention of the speaker, hence forces a distinction between information and conveyance with likewise linguistic means." (Luhmann, 1997, p. 85)

The conveyor can only convey a signifier, but the signified that is intended by the conveyor, which alone could lead to an understandable sign, remains open and relatively uncertain; in principle it can only be constructed by the receiver of the conveyance, in a way that he himself articulates a new signified.

> "… we start with the situation of the receiver of the conveyance, hence the person who observes the conveyor and who ascribes to him the conveyance, *but not the information*. The receiver of the conveyance has to observe the conveyance as the designation of an information, hence both together as a sign …." (Luhmann, 1997, p. 210).

The receiver must not ascribe the possible signified strictly to the conveyor but he has to construct the signified himself; the signified and the sign is constituted within the process of communication.

The possible detachment of the information belonging to the conveyance from the conveyor is the starting point of the autopoiesis of the communicative system:

"This increases the possibilities to expose oneself to certain environments or to remove from them, and it offers the chance to the selforganisation of participants to keep their distance from what is communicated. One remains perceivable, comprehensible, however, in what one contributes deliberately to the verbal communication. The consequence is a proper autopoietic system of verbal communication … ." (Luhmann, 1997, p. 211)

2. The characterization of mathematical interaction as an autopoietic system

In the following we will analyze a short episode of mathematical interaction. The first part (phases 1 to 3) contains the interactive constitution of the correct understanding of a task of written subtraction as a complementary task, taking the corresponding single place values:

```
  H T O
  6 2 3
- 3 5 9
-------
```

The expected algorithmic procedure of calculation should use strategies of filling up the subtrahend to the minuend at the single value positions. During phase 4 the student Tarik is asked to perform the calculation; the interaction fails. During phase 5 the student Svenja approaches the calculation with the expected strategies; the interaction with Svenja is successful.

The parts of interaction can be structured into phases and sub phases:

Phase / Subphase	Contribution	Theme
4.	16 - 29	The mathematical interaction with Tarik
4.1	17 - 19	Tarik calculates a subtraction task: 9 minus 3
4.2	20 - 23	Tarik calculates a complementary task: 9 + __ = 3
4.3	24 - 27	The "impossible" complementary task must be changed
4.4	28 - 29	Tarik's proposal: 3 + __ = 9 is refused

5.	29	-	37	The mathematical interaction with Svenja
5.1			30	Svenja calculates a complementary task: 9 + __ = 13
5.2	31	-	33	Svenja's proposal is accepted and a justification is asked
5.3	34	-	36	Justification: Both numbers become bigger by 10
5.4			37	Confirming summary by the teacher

Both interactions shall be analyzed by means of the notion of *sign as the distinction between signifier and signified*. This analysis *shows* the failure of the interaction with Tarik and the success of the interaction with Svenja; but this analysis of the *social* interaction cannot clarify the *reasons* for the failure or for the success. To uncover the reasons requires an *epistemological* analysis related to the mathematical knowledge and its meaning as it is interactively constituted; this epistemological analysis correlates to the conceptual structure of *sign as the distinction between signifier and signified*.

2.1 Analysis of the 4. phase: The mathematical interaction with Tarik

The teacher invites Tarik to begin to calculate; she conveys a signifier (“One, 9") within the complex signifier of the whole subtraction task (623 – 359) together with an intention.

16 T.: … So, then start, with the ones, 9, yes, speak once more.

17 Tarik: 9, …

18 T.: Go on,

First, Tarik repeats the signifier “9"; and then he transforms it into a new signified ”9 minus 3".

19 Tarik: … minus 3.

The teacher contrasts this with a new signifier: “complementary task”, which she does not want to be understood as a signified for “9 minus 3". Tarik's signifier ”9 minus

3" cannot be related to the signifier “complementary task”; thus the teacher intents the constitution of a another sign.

20 T.: We pose a complementary task.

Tarik formulates a new signifier (“… with plus …”), which is at the same time an accepted signified. The teacher expands the signifier, and Tarik makes it complete.

21 Tarik: … with plus …

22 T.: 9 plus how much equals?

23 Tarik: 3.

With the question “Does this work?” the problem is posed whether one can introduce a new signified / signifier for this signifier “9 + _ = 3".

24 T.: Does this work?

25 Tarik: No.

26 S.: Of course!

Tarik negates the question for a possible signified / signifier for “9 + _ = 3"; another student declares it would function. The teacher, too, seems to confirm that for the signifier ”9 + _ = 3" there is no adequate new signified / signifier.

27 T.: 9 plus how much equals 3? Doesn’t work, but, what works?

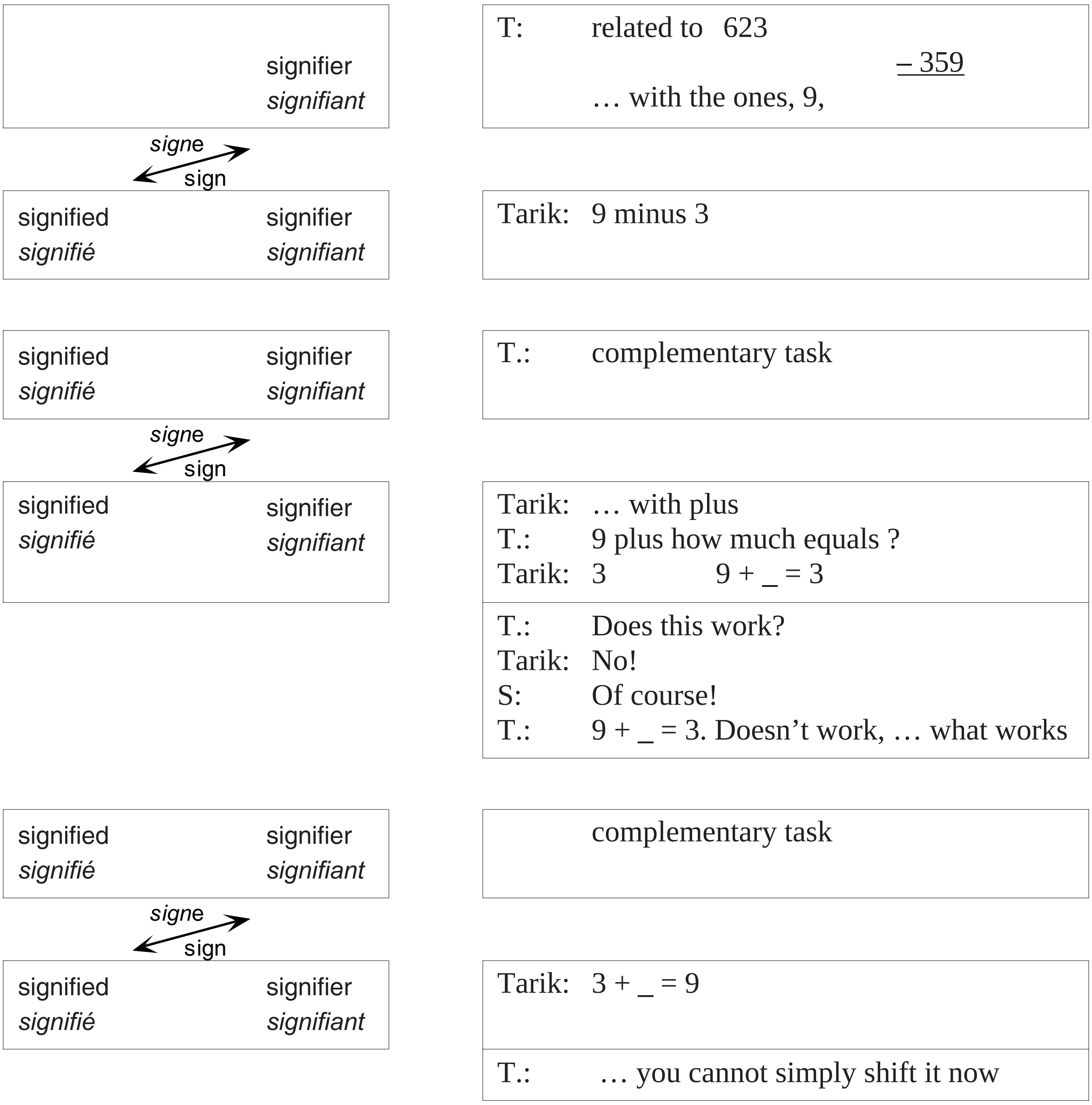
She asks for an alternative to the former signifier “9 + _ = 3": ”… but, what works?" Tarik replaces the former signifier “9 + _ = 3" by the new signified / signifier ”3 plus how much equals 9", which is in relation with the signified “complementary task” (20). In this way the reference: signifier “complementary task” ↔ signified “3 + _= 9" constitutes perhaps an interactively generated sign.

28 Tarik: 3 plus how much equals 9.

The signifier “3 + _= 9" and the possible sign is not accepted; the teacher breaks down the interaction with Tarik.

29 T: Yes, but you cannot simply shift it now, Tarik look from your seat again. …

The chain of conveyed signifiers / signifieds (cf. de Saussure, 1967, p. 137) in the course of interaction with Tarik can be schematically illustrated in the following diagram:



At this point no further communicative connection seems to be possible; finally Tarik has tried to maintain the reference between the signifieds / signifiers: “9 + _ = 3" and ”3 + _ = 9" and the former signifieds / signifiers: “complementary task” (tasks “with plus”). The signified / signifier now looked for cannot be directly taken exclusively from the conveyances offered in the course of this interaction, e.g. by a simple combination of already provided signifieds / signifiers. A new signified / signifier must be introduced, reaching beyond the frame of social interactions.

2.2 Analysis of the 5. phase: The mathematical interaction with Svenja

The teacher calls Svenja; she formulates a new signified / signifier: “9 plus how much equals 13?”(30) in relation to the complex signifier “Interpreting the task of written subtraction 623 – 359 at a single value position as a complementary task”.

29 T.: … Svenja come once more.

30 Svenja: 9 plus how much equals 13

The teacher confirms Svenja’s proposal; additional questions refer to the choice of the signifier “13" within the signifier ”9 + _ = 13".

31 T.: Good, you say 13, why?

32 Svenja: Because 9, the other doesn’t work at all.

33 T.: Yes good, you take what thereto?

Svenja denotes the new signified / signifier with “Ten”; in interaction with the teacher she explains the enlargement of single place values by “one Ten” in the minuend and the subtrahend.

34 Svenja: The ten.

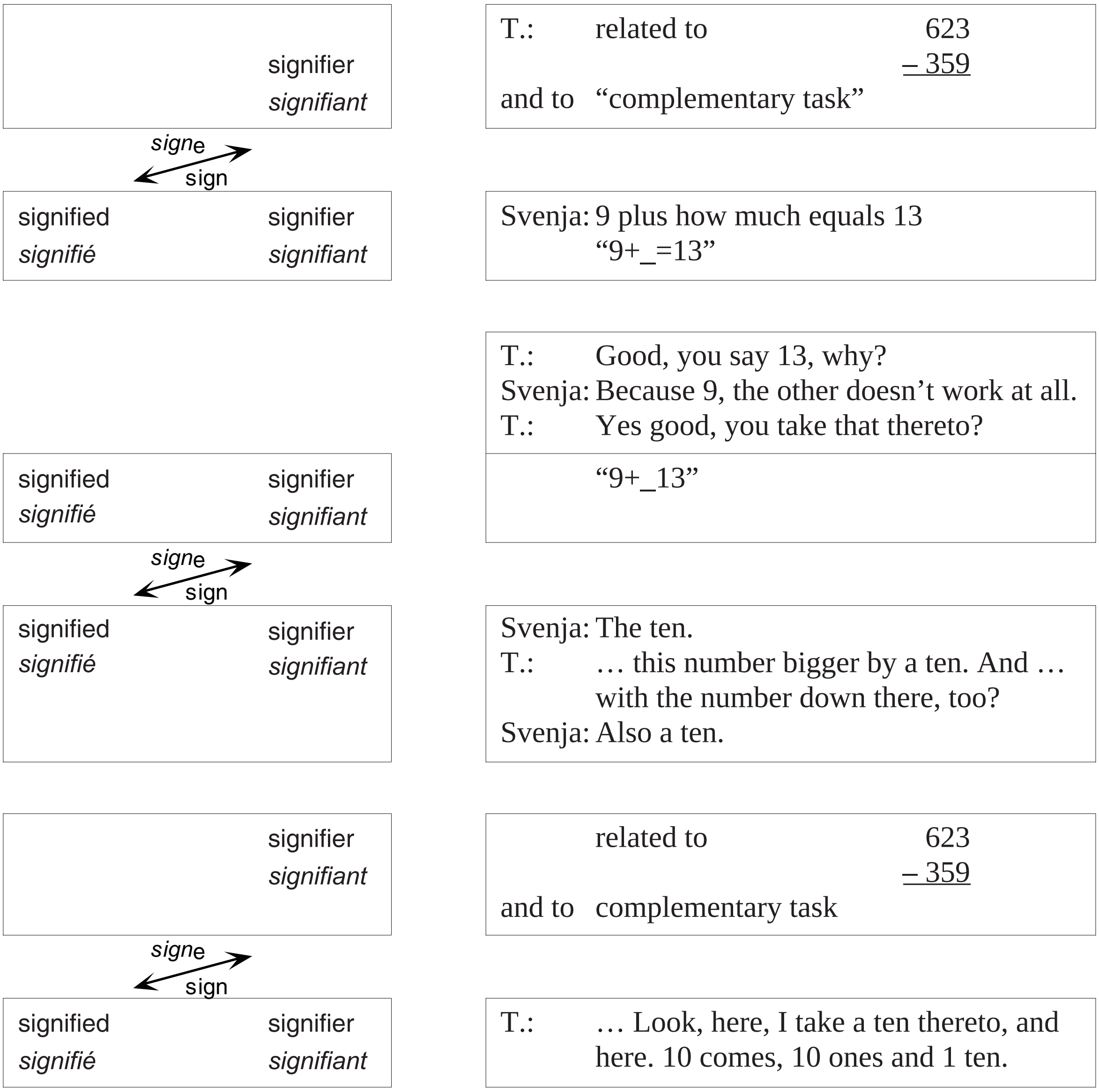
35 T.: ‘A ten; you make this number bigger by a ten. And what you have to make then with the number down there, too?

36 Svenja: Also a ten.

The teacher confirms and she gives a summarizing description of the signified / signifier “Ten” for the subtraction at single positions according to the complementary strategy.

37 T.: Exactly, this gets also a ten before. So, then we write again our task on the board. Look here, I take a ten thereto, and here. 10 come, 10 ones and 1 ten.

The interaction with Svenja looks as follows in the schematic diagram:



During the interaction with Svenja communicative signs are produced which seem to be accepted. Svenja's reformulation of the complementary task "9 + _ = 3" (which doesn't work) into the complementary task "9 + _ = 13" is not simply a product of elements already provided in this communication; it is in principle a new constituted signified / signifier in this interaction. This seems to be the reason for the success of the interaction with Svenja.

3. Comparing the two interactions from an epistemological perspective

In the interaction with Tarik the following leading idea of communication as a self-referential process is observable:

- *combining existing mathematical signs (by way of trial)*

together with the idea: signs are names for objects, subtraction is explained as taking some objects away, multi-digit numbers are put together of single-digit numbers.
During the interaction with the teacher Tarik combines the one-digit numbers at the value positions in a way, that one gets with these numbers subtraction or complementary tasks that indeed can be calculated, e.g. by a concrete action of taking some amount away or by adding some amount these tasks could be solved.
In the interaction with Svenja the following leading idea of communication as a self-referential process is observable:

- *constructing new mathematical connections / relationships*

together with the idea: multi-digit numbers consist of Ones, Tens, Hundreds; there are *relations* between the different place value positions.
The explanation for the transformation of the former complementary task “9 + __ = 3" into the complementary task ” 9 + __ = 13" makes it necessary to utilize the relations between the place values, and not to consider ciphers and numbers at the different positions as isolated entities, but as interrelated elements in a structural network.

The different interactive meanings of mathematical ideas constituted in the communication with Tarik and with Svenja can be characterized with the epistemological triangle (Steinbring, 1997; 1998). In comparison with the semiotic triangle according to the terminological distinction defined by de Saussure with “*signifiant, signifié, signe*” the epistemological triangle uses a different notation: “Sign / Symbol, Reference Context, Concept”, trying to take into account the fact that *mathematical signifiers* are already always *signs* themselves and that in mathematics the *mathematical concept* has its own central epistemological meaning beyond the mathematical “*Sign / Symbol*” (and therefore the *Concept* is put at the position of the sign).

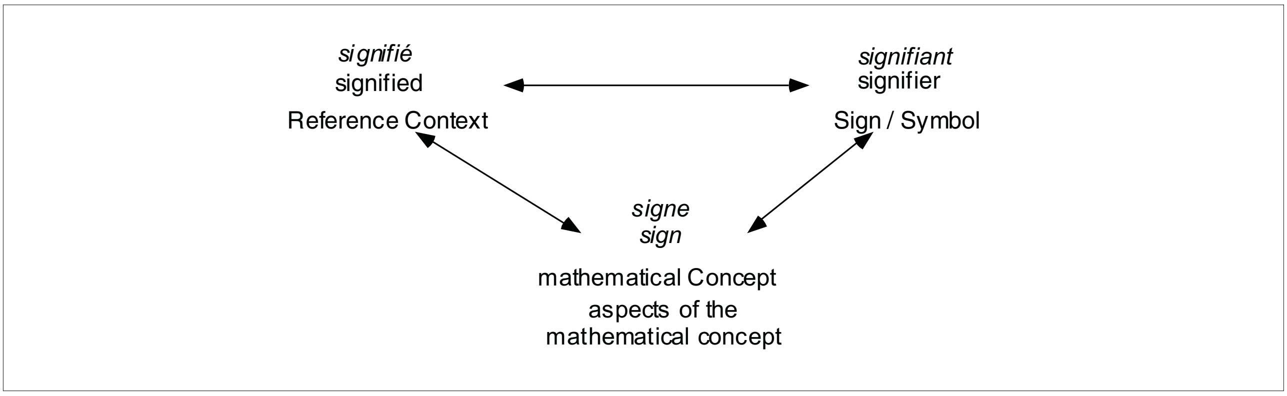


Fig. 1: The semiotic and the epistemological triangle

The interactively constituted mathematical meaning of numbers in the interaction with Tarik (phase 4) according to the epistemological triangle:

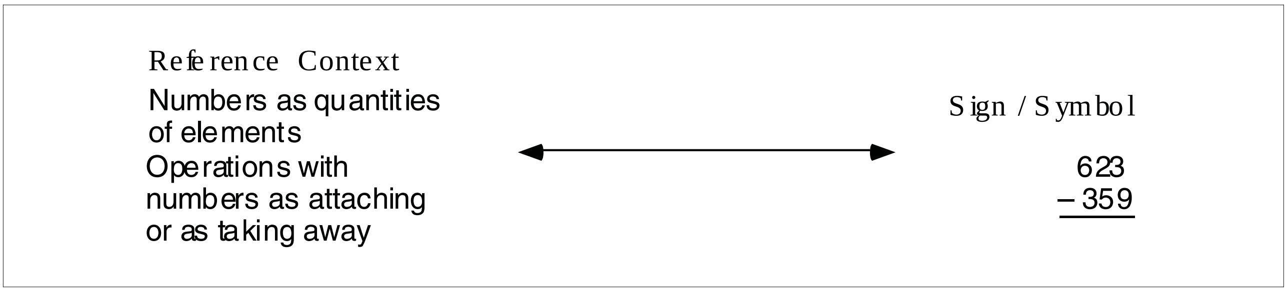


Fig. 2: Analysis of Tariks number use with the epistemological triangle

The interactively constituted mathematical meaning of numbers in the interaction with Svenja (phase 5) according to the epistemological triangle:

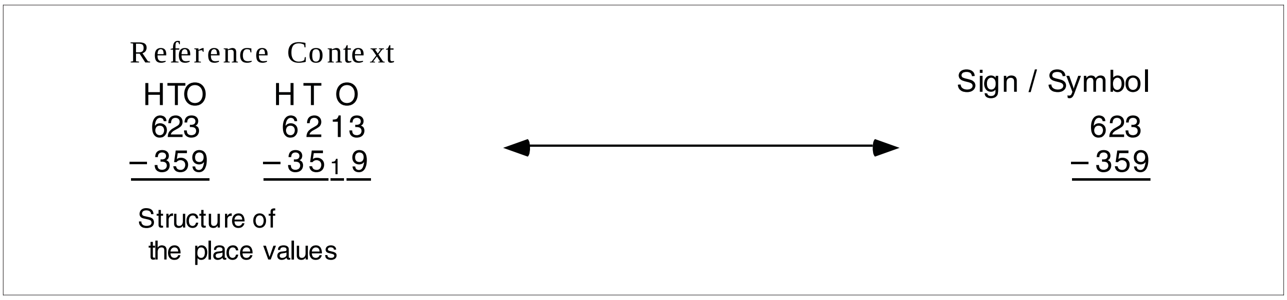


Fig. 3: Analysis of Svenjas number use with the epistemological triangle

On the level of social communication a contribution could either be accepted or rejected; a *justification* for an acceptance or a refutation cannot explicitly be noticed as a contribution in the frame of social communication.

Tarik and Svenja both are able to successfully participate in the *social* interaction; they try to keep going the social interaction with the teacher. The reason for the break down of the communication with Tarik cannot be explained exclusively on the level of the social system of communication; to give an explanation requires the position of an observer together with a “reflective, epistemological theory” about the epistemological nature of mathematical knowleddge generated in interaction.

4. Specific issues of mathematical communication processes

Mathematical interactions are social systems, being at the same time characterized by very specific intentions. They are *educational* communications, and at the same time *mathematical knowledge* is in some special way the object of the communication.

- Interactions between teacher and students are pedagogically intended communications with the aim to mediate knowledge. This implies a superposition in the autopoietic development of the social communication with an “additional sense”, which is the result of the teacher’s intention of *teaching*. When trying to achieve their educational intentions teachers are often unconsciously unaware of the borders between the social and the psychic system and they believe that the meaning conveyed in the communicative process could be transferred instantly and unchanged into the student’s consciousness.
- Mathematical communication treats mathematical knowledge; this implies for the (external) observer (the researcher) to analyze from an epistemological perspective the knowledge that is interactively generated in the course of communication. The analysis of the specific status of school mathematics and its interactively constituted meaning shows that it can be interpreted as a “symbolically generalized communication medium" (Luhmann); in analogy to “scientific truth” (GLU, 1997, p. 190) one can speak here of “school mathematical correctness”.

The conflict between direct pedagogical intentions and epistemological constrains of mathematical knowledge in social communication creates a *paradox of mathematical, educational communication*: The mathematical information (the *intention*) should be transmitted together with the conveyance, what is not directly achievable and is in principle even impossible. Therefore in many cases a "conversion" takes place of a *communication about mathematical knowledge* to a *communication about the teacher's intended information* associated with the conveyance he or she states.

Such "compensating" communicative strategies can also be observed in the present episode. In the course of such communicative patterns students do not explain the given (mathematical) signifiers / signifieds with reference to other *mathematical* signifiers / signifieds and do not produce in this interaction an elementary *epistemological* relationship, but they try to decipher the information (and intentions) the teacher has in mind. Thus it can be observed that the central possibility of the autopoiesis of communication is dodged in ordinary mathematical interaction: The receiver of the conveyance (a student) first of all can ascribe only the given conveyance to the conveyor (the teacher or another student); the possibility to detach the information of the conveyance the conveyor had in mind implies the possibility of the autopoiesis of the social system. In the course of mathematical interactions one can often observe how the students – in another type of communication system and in reciprocal reaction with their teacher – try to tie the information intended by the teacher to his given conveyance or to infer the needed information directly from the teacher's conveyance. The conveyance of the teacher (and of other students) becomes the proper object of communication, it is not just conveyed in communication for relating (and interpreting) it interactively to other *mathematical* signifiers / signifieds.

The interactions on the social level "produce" interactive, epistemological readings of the mathematical signs / symbols; the ascription of meaning to signifiers in social interaction which can be observed and theoretically reflected on the epistemological level are necessary for assessing the reasons for a success or a failure of a mathematical communication (for what is considered as true or not-true in mathematical communication, and which has to be distinguished from simple "mathematical truth" or "falsehood").

5. References

GLU (1997): *Glossar zu Niklas Luhmanns Theorie sozialer Systeme*, von C. Baraldi, G. Corsi & E. Esposito. Frankfurt am Main: Suhrkamp.

Luhmann, N. (1996): Takt und Zensur im Erziehungssystem. In N. Luhmann & K.-E. Schorr (eds.), *Zwischen System und Umwelt. Fragen an die Pädagogik*, pp. 279-294. Frankfurt am Main: Suhrkamp.

Luhmann, N. (1997): *Die Gesellschaft der Gesellschaft*. Frankfurt am Main: Suhrkamp.

Maturana, H. R. & Varela, F. J. (1987): *Der Baum der Erkenntnis*. Bern: Scherz.

Saussure, F. de (1967): *Grundfragen der allgemeinen Sprachwissenschaft*. Berlin: Walter de Gruyter & Co.

Steinbring, H. (1997): Epistemological investigation of classroom interaction in elementary mathematics teaching. *Educational Studies in Mathematics*, 32(1), 49–92.

Steinbring, H. (1998): Elements of epistemological knowledge for teachers. *Journal of Mathematics Teacher Education*, 1(2), 157–189.

ON THE ACTIVATING ROLE OF PROJECTS IN THE CLASSROOM

Marie Tichá[1], Marie Kubínová[2]
[1]Department of Didactics of Mathematics, Mathematical Institute, Academy of Sciences of the Czech Republic, 11567 Praha 1, Czech Republic
ticha@math.cas.cz
[2]Department of Mathematics and Mathematical Education, Faculty of Education, Charles University, 11639 Praha 1, Czech Republic
marie.kubinova@pedf.cuni.cz

Abstract: *In our contribution we will deal with following items:*
to explain what goals we intend to reach when we include the projects in mathematics education,
- *to show how we proceed when working on projects,*
- *to present concrete samples from school practice,*
- *to show the areas in which the work on project was profitable for students (did we achieve our goals?),*
- *to mention main obstacles for including projects in classroom,*
- *to formulate questions that remained open, to show future direction of our work.*

On the other hand we will not aim at theoretical specification of concept of project.
Keywords: *project, grasping situation, social interactions*

1. Introduction

Our experience from teaching process showed that one of fundamental questions of mathematical education is improperly directed activity of both students and teachers in the course of education (lessons). Everything is, at the first sight, often all right. However deeper observation shows that teaching process deals mainly with the receptive education and intensive exercise of certain operations according to models, samples etc. even when solving word problems. Teaching process framed this way is easy and convenient for both teachers and students. It is also easy to evaluate its results. The question „What I know from mathematics?“ is not the substantial one for students but rather the question „For what is it useful?“ (i.e. the question of sense, utilisation).

But students’ answer to these questions could be deformed by the effort (their own or that of their parents and teachers) to gain good assessment (marks). The students (and often also the teachers) do not realise that the result of mathematical education that is reduced to the work according to certain instructions and rules is only formal knowledge. Under „formal knowledge“ we mean a parrot-like knowledge, a knowledge without understanding. It is a knowledge, which is preserved as isolated memory data and it can be applied only in standard task situations. Outside of these such knowledge is useless. Adjective „formal“ in no sense means here „symbolic“.

Therefore we look for the ways how to open up effectively to the students the world of mathematics. For our needs we understand under the „world of mathematics“ the area in which for the grasping of reality the student (with regard to his/her age) uses the mathematical notions and the relations between them and mathematical as well as nonmathematical experience gained until now. We seek how to ensure that pieces of knowledge will be organic parts of the student´s cognitive net, that pieces of knowledge will be connected to student’s experience and that students will be able to apply them in the course of problem solving, especially of those from everyday life. In our opinion, one of solutions is the involving of project into the mathematics education.

Our first projects arose spontaneously as a reaction to certain student’s suggestion or as a means which should help to prevent the gaining only the formal knowledge as a result of the teaching process.

However, gradually we started to assign the projects some specific aims (utilise the projects during mathematics education purposefully). We also devoted our attention to the theoretical questions that concern the particular phases of the work on projects (preparation, realisation, evaluation, etc.). We connected the work of two groups - Koman-Tichá (Koman & Tichá, 1996, 1997, 1998) and Kubínová-Novotná (Kubínová 1996, 1997, Kubínová, Novotná & Littler 1999) - and we utilised the results of their research done hitherto.

2. Backgrounds

When we include the projects in the mathematics education we want to increase the competencies in the *content-cognitive* area (permanent knowledge and skills, etc.) as well as *functional-cognitive* (problem appreciation, ability to think critically, processually, to reason, etc.). A further goal is the improvement of abilities in the area of *self-competence* (self-reflection, self-assurance, initiative, etc.) and *social-competence* (ability of co-operation, skill to act, behave, negotiate, to present, to cope with conflicts, etc.). Very important goal is *the improvement of school climate* (relationships between students and teachers, among students, elimination of stress, etc.) (Grecmanová & Urbanovská 1997).

The ideas of „Project education“ appeared in Czechoslovakia in twenties (Príhoda, Vrána) as a response to the work of J. Dewey and W.H. Kilpatrick (their work is mentioned also in (Ludwig, 1996-1997)). Vrána (1936) specified „project“ as something aiming at a certain goal, which stresses to arrange, to find out, to survey, to carry out, etc. According to Vrána the project is also powerful *motivating factor* - students are ready to work, they are motivated through responsibility in results of their work and therefore they are able to assess its quality. On the other hand Vrána emphasises that it is not possible to use only „project method“, *it is not possible to teach only in the form of projects*.

The fundamental starting point of our approach to projects are ideas of *didactical constructivism* (Hejný & Kurina 1998) which are connected with ideas of social constructivism (Jaworski 1994); its conception was created within the framework of our research „Mathematical education of 5-15-year-old students“. From the point of view of utilisation of projects in mathematics education are important especially the following ideas of didactical constructivism:

- Mathematics is a specific sort of human activity.
- Pieces of knowledge are not transferable (only information is transferable).
- Pieces of knowledge are created on the basis of experience and foregoing knowledge.

- The construction of pieces of knowledge is an individual process, however social interaction in the classroom contributes strongly to its development.
- The basis of mathematical education is the milieu that stimulates creativity, that is created, made, formed, evolved by teacher, by the heterogeneity of mathematics and by the social climate in the class.
- Various forms of communication and various languages of mathematics (graphs, tables, mathematical symbolism, etc.) play an important role.
- The education (teaching) based only on transferring of information or instructions (directions) does not enable its utilisation and development, on the contrary, it leads to superficial knowledge that is easy to forget.
- It is necessary to focus mathematical education to the understanding of mathematics, to mastering of mathematical craft as well as to the applications of mathematics.

3. Work on projects

Our work on projects can be divided into the following three *phases* (in more detail in Kubínová, Novotná & Littler 1998):

- Preparation of project which involves determination of the goal of project, the choice of topic, the formulation of project posing, suggestions of alternatives approaches and procedures, etc. We suppose the involvment of student in this phase, it enables them to direct project according to their interests.
- Project realisation. In this phase are active mainly the students. The teacher stays behind in the role of a co-worker, adviser, judge, stimulator. In the course of project realisation the students often initiate enlargement of the original goals of the work on project.
- Evaluation and assessment. The teacher and the students review the course of project preparation and realisation from the point of view of general goals of including projects into the education and of the goal of concrete project. This phase includes also self-reflection.

A substantial part of „project realisation" is the process of *grasping situations* which penetrates both problem solving and the problem posing (Koman & Tichá 1996, 1997, 1998). („Situation" means for us the area which we want to investigate and in which we want to find out subjects for activities and problems.) According to our experience it is possible to itemise the mechanism of grasping situations into several phases. In similar manner Hejný (1995) itemises the grasping of word problems. Some of following points concern mainly grasping of real situation arising from everyday life:

- Perception of situation as a section of real life (imagining the milieu; reviving former experience, skills, knowledge, capabilities; putting oneself into the position of the acting person).
- Discovering the key objects, phenomena and relations. Judgement of relevance of the data, consideration about possibilities of getting the missing data.
- Setting-up certain direction of grasping real situation.
- Awareness of problems growing from the situation and of questions which could be asked. This phase penetrates all others.

The phases given above are directed to the investigation and understanding of the situation and their aim is the *problem posing* (divergent and convergent problems). The views from within (i.e. views of interested person) change the views from without (we step from the position of acting person and observe the situation as a whole).

In the further phases of grasping of situations predominates the problem solving.

- Searching for answers to the questions, formulation of results.
- Interpretation and evaluation of results and answers and their judgement from the point of view of acting (concerned) person (or persons).
- Identification of a new situation.

The work on projects covers various activities (not only mathematical, but also non-mathematical) of both students and teachers which relate to various levels of behaviour.

Until now we have prepared projects related either to various school subjects (mathematics, physics, chemistry, etc.) or to one subject (to mathematics itself or to various areas of mathematics). Contextually the projects arise either from everyday life (shopping, travelling, votes, etc.) or from mathematics itself (fractions with denominator equal to one, regular solids, etc.).

4. Two samples from the work on projects

Sample 1

We would like to show one part of solving of project named „*How is our class?*“ which was directed to the elaboration of data. This projects was short-lasting and was elaborated by students in the age 12 years (grade 6).

The tasks were formulated as relatively open (What could we say about ourselves?). The students should decide what data are important for the characterisation of their class (group of students) from their point of view. Students solved the project independently with only graduated teacher’s help.

At first they collected the data from two areas:

a) Data which they were able to indicate quantitatively (height and age of each student, shoe size, number of brothers and/or sisters, etc.) Here appeared a.o. the connection with physics, because it was necessary to consider units of measurement.
b) Data that describe characteristic features of personality (desirable properties etc.)

During the next phase the students assessed and evaluated collected data.

- The students discovered by themselves „arithmetical average“ as a quantitative aid for description of properties of the group of investigated objects; it was one of the purposes of the work on project. They also discussed the sense and utilisation of „arithmetical average“.

- For the visualisation of collected data students used also bar charts (Fig. 1). In the course of visualisation they discovered various character of such distributions and their dependence (height) or independence (age) of time. Originally this was not one of the aims of project, but the students themselves enriched the project during the solution by the propedeutics of dependence.

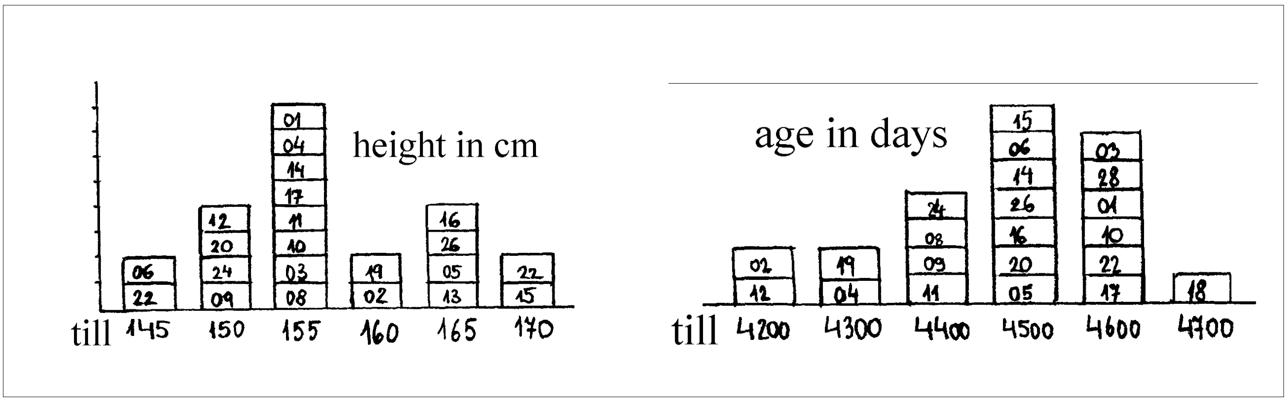


Fig. 1: Students’ bar charts

At the end of the work on the project the students decided to picture the „average student“ in his real life-size (height, hair colour, etc.). Some features the students were able to paint, other characteristic properties illustrated in words, that surrounded the picture of „average student“.

Other suitable theme of projects is *individual transport*, i.e. organisation, calculation of expenses, etc.

When we grasp the situation that concerns individual transport and pose the problems we can vary especially the topology and metric parameters of traffic net and also the number of persons and the rules of sharing expenses. It allows to prepare the clusters and cascades of problems of various degree of difficulty and so each student can solve such problems that are appropriate to his/her capability. In this case appears also the question of fairly sharing of expenses (if this sharing is fair, advantageous for both, etc.). Students should also realise the importance of agreement.

Sample 2 - Two or more go in one car

We commute to work in our own car every day. Our friends who live in the nearby towns and villages and who we work with use their own cars as well.
Simple example (Fig. 2). Imagine that one of you is Adam and the other Bob. You decided to go together with Adam's car. Suggest what amount of money each of you should contribute to the total expenses.

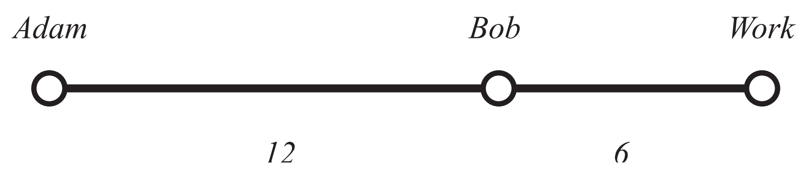


Fig. 2: Simple Example: two or more go in one car

We would like to show how the 10-year-old students (4th grade) discussed the situation and gradually suggested various possibilities of sharing expenses, its advantages and acceptability (they supposed, that numbers at the picture mean the distances and both friends pay 1 Kc per 1 km in case that they go with their own cars).

Peter: I suggest 12 and 6 Kc. Adam will pay the first part of journey, i.e. 12 Kc. The second part, i.e. 6 Kc, will pay Bob.
Honza: This suggestion is not good because Bob will not save any money.
David: I agree with Peter, I will buy it! I will save my car and I do not lose anything.
Marta: I do not agree with you David! You should save your car and also economise money.

Note, that David and Marta speak about themselves, but they probably don't realise it. Great part of students started to discuss among themselves.

Teacher: Can somebody suggest any other possibility?
Honza: I suggest 15 and 3. Adam will pay the first part of journey and they will share expenses along the second part.

Michal: That's nonsense. Adam cannot agree with it, that is clear.

Started strong discussion within students.

Gaby: I also think that Honza's suggestion is unacceptable for Adam because he will spend 3 Kc as well as Bob and that's unfair. Adam should drive and in addition to it he will destroy his car. I suggest 15 and 6.

Jirka: But this way they will pay more than they need.

David: Then Adam can put 3 Kc in his pocket!

Michal: Or they can share these 3 Kc.

He goes to the blackboard and writes 13.50 and 4.50.
Reaction of the class was positive.

Teacher: Do you agree? What about you, Marta? Are there any other objections?

Marta: Adam saves 4.50, Bob saves 1.50. Yes. That's all right.

After that they proceed:

Teacher: I have other question. *What if* Adam does not live on the Adam's route to work? How could it look out?

Tomas drew immediately new picture on the blackboard - triangle with distances and suggested: I can speak about sharing of money with Peter and so it will be better, if we will speak about us (and wrote to the picture T and P instead of A, B).

It was possible to observe that the discussion (social interaction) positively influenced the development of individual points of view and interpretation of situation of students concerned in this discussion. Interesting were also the sense for „fair play" and comparatively high level of economical thinking.

According to our experience from the teaching process, the obstacle for the implementation of the projects into the mathematical education is not the lack of teacher's experience with this activity. The obstacle is the lack of teacher's belief in the usefulness of projects. Our main problem was to overcome the initial barrier of disbelief and to start. (Also to convince the teachers that the success of students in entrance examinations could not be considered as the only measure of evaluation of their work.)

5. Current experience with projects

- During the work on projects increased active participation of students in the teaching/learning process and their interest in work. However, it showed that when the students worked on the same project for too long time they lost the interest. Only some students like to deal with one problem for a long time, to thoroughly investigate it. (Usually they solve one task and then quickly pass to another one.) The same student's reaction was found when projects were posed too often. There are also some students who are more satisfied when practising the techniques.
- Teacher's belief in suitability of including projects into lessons has increased. This took place despite of great time demands of project and exacting character of preparation, realisation and evaluation. When they started to work with projects, they were able to work very creatively.
- The work with projects is also the contribution to the cultivation of teachers.
- The social climate in class has improved (relations between students, etc.)
- It was possible to follow also the development of cognitive competencies of students. The students started to investigate, discover the phenomena and the relations between them, to create hypotheses and to verify them, to seek for links etc.

We consider as one of greatest achievements of our work on projects done until now the saying of one student: „We don't learn mathematics, we do it." Could it be new J. Dewey?

6. Intentions for further research

- Based on metaanalysis to precise the specification of the concept of project from the point of view of needs of mathematical education with regard to ideas of didactical constructivism relating to the creation of student's cognitive structure.
- To work out the methodology of creation of projects supporting the change of receptive approaches to recognition and education and the change of teachers belief.
- To prepare, realise and evaluate in the concrete teaching process the longitudinal research directed to the study of the activating role of projects.
- To study social interactions among students as well as between the students and the teacher in the course of work on the project.
- To prepare investigation of students' and teacher's opinion on including the projects into lessons.

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7. References

Grecmanová, H. & Urbanovská, E. (1997): Projektové vyucování a jeho význam v soucasné škole. (Project education and its role in contemporary school, in Czech). *Pedagogika*, 47 (1), 37-45.

Hejný, M. (1995): Zmocnování se slovní úlohy (Grasping of word problem, in Czech). *Pedagogika*, 45 (4), 386-399.

Hejný, M. & Kurina, F. (1998): Konstruktivní prístupy k vyucování matematice (Constructivist approaches to teaching of mathematics, in Czech). *Matematika-fyzika-informatika*, 7 (7), 385-395.

Jaworski, B. (1994): *Investigating Mathematics Teaching: A Constructivist Enquiry*. The Falmer Press.

Koman, M. & Tichá, M. (1997): Grasping of Situations and the Development of Activity and Cognitive Abilities. In M. Hejný, J. Novotná (eds.), *Proceedings ERCME 97*, Prometheus, Praha, 94-97.

Koman, M. & Tichá, M. (1997): Wie die 9-15 jährige Schüller die Alltagssituationen untersuchen und begreifen. In K.P. Müller (ed.), *Beiträge zum Mathematikunterricht*. Voträge auf der 31. Tagung für Didaktik der Mathematik, Franzbecker, 490-493.

Koman, M. & Tichá, M. (1997): Alltagssituationen im Mathematikunterricht. Folge 1: Eine Situation aus dem Handel. Folge 2: Eine Situation aus dem Verkehrswesen. *Mathematik in der Schule*, Berlin, 35 (10, 11), 527-535, 590-596.

Koman, M. & Tichá, M. (1998): On travelling together and sharing expenses. *Teaching Mathematics and its Applications*, (accepted to print).

Kubínová, M. (1996): Does experimental work belong to the mathematical education of 9-16-year old students? In *Proceedings VSP Rzeszów*.

Kubínová, M. (1997): Students' projects from the point of view of diagnostics. In M. Hejný, J. Novotná (eds.), *Proceedings SEMT 97*, Prometheus, Praha, 88-91.

Kubínová, M., Novotná, J. & Littler, G. (1999): Projects and Mathematical Puzzles - a Tool for Development of Mathematical Thinking. In I. Schwank (ed.), *European Research in Mathematics Ecucation*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Ludwig, M. (1996-1997): Projektorientierter Mathematikunterricht. Folge 1.-7. *Mathematik in der Schule*, Pädagogische Zeitschriftenverlag, vol. 34-35.

Tichá, M. & Koman, M. (1996): Handels- und Beförderungssituatinen als Thema für Unterrichtseinheiten. In K. P. Müller (ed.), *Beiträge zum Mathematikunterricht.* Vorträge auf der 30. Tagung für Didaktik der Mathematik, Franzbecker, 425-428.

Vrána, S. (1936): *Ucebné metody* (Teaching/Learning Methods, in Czech). Brno - Praha.

AUTHORS: VOL. 1

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