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TAXATION UNDER UNCERTAINTY – PROBLEMS OF DYNAMIC PROGRAMMING AND CONTINGENT CLAIMS ANALYSIS IN REAL OPTION THEORY

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Abstract

This article deals with the integration of taxes into real option-based investment models under risk neutrality and risk averison. It compares the possible approaches dynamic programming and contingent claims analysis to analyze their effects on the optimal investment rules before and after taxes. It can be shown that despite their different assumptions, dynamic programming and contingent claims analysis yield identical investment thresholds under risk neutrality. In contrast, under risk aversion, there are severe problems in determining an adequate risk-adjusted discount rate. The application of contingent claims analysis is restricted to cases with a dividend rate unaffected by risk. Therefore, only dynamic programming permits an explicit investment threshold without taxation. After taxes, both approaches fail to reach general solutions. Nevertheless, using a sufficient condition, it is possible to derive neutral tax systems under risk aversion as is demonstrated by using dynamic programming.

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1 Introduction

In recent years, literature on capital budgeting has been largely enriched by numerous publications on real options [e.g. Myers (1977), Brennan and Schwartz (1985), McDonald and Siegel (1986), Majd and Pindyck (1987), Dixit (1989), Pindyck (1991), Dixit and Pindyck (1994), Trigeorgis (1996), Grenadier and Weiss (1997), Moretto (2000), Lensink and Sterken (2001)]. Real option theory - the application of option pricing theory on non-financial investment - takes account of uncertainty and irreversibility of investment decisions. Recently, public economists have extended real option theory by integrating taxation [e.g. Harchaoui and Lasserre (1996), Alvarez and Kanniainen (1997), Niemann (1999)].

A real option-based investment model requires the decision between two approaches: either dynamic programming (DP), similar to decision-tree analysis, or contingent claims analysis (CCA), an approach derived from the pricing of financial options. While both methods yield equivalent results in the tax-free case [cf. for a mathematically rigorous analysis of the relationship of DP and CCA in tax-free models Knudsen, Meister and Zervos (1999)], integrating taxes reveals some interesting differences, especially under risk aversion. This raises the question if there is a superior approach and if there are limits to the integration of taxes in capital budgeting.

Our article deals with this agenda beginning with a simplifying assumption widely used in public economics, but often rejected in financial economics: risk neutrality. Our aim is to demonstrate the basic principles of the DP- and the CCA-approach in real option theory. In chapter 2, we present the general assumptions underlying both approaches and demonstrate in chapter 3 that there are similarities in the pre-tax and post-tax case. In chapter 4, the risk neutrality assumption is relaxed and the DP- and the CCA-method are discussed in the more general setting of risk aversion, including a proof of neutral tax systems. In chapter 5, the properties of both approaches are compared before and after taxes, under risk neutrality and under risk aversion. Chapter 6 summarizes with some concluding remarks.

Traditional models of investment under uncertainty neglect an aspect crucial to investment decisions in reality: flexibility when decisions are irreversible. Under conditions of irreversibility, abandoning a project already in place becomes impossible or at least expensive. A supposedly beneficial project can prove disadvantageous when market conditions turn sufficiently unfavorable. Therefore, the possibility to defer a decision and wait for more information can help to avoid harmful decisions. Thus, flexibility constitutes an economic value of its own. Deriving a rule for optimal investment under uncertainty and irreversibility requires the valuation of managerial flexibility. Since the 1970s, an important branch of financial economics deals with this matter: option pricing theory. Therefore, flexibilities in the context of non-financial (real) investment are called real options.

Although real option literature characterizes many other types [for an overview, cf. Trigeorgis (1996)], real options - like their financial counterparts - can be roughly divided into calls and puts. A "real" call is an option to invest. It provides the opportunity to acquire a project with stochastic cash flow and thus uncertain present value. A "real" put is an option to abandon a project already in place. Further real options can be modeled as multi-stage combinations of calls and puts.

The main fields of application are commodity-related investment projects like oil exploration, mining and forestry [cf. Brennan and Schwartz (1985), Morck, Schwartz and Stangeland, (1989)]. Real option-based models are also being applied to real estate development and strategic acquisitions [cf. Quigg (1995), Smith and Triantis (1995)]. Mathematically, these investment decisions are characterized by an optimal stopping problem. Since real options seem to be widely accepted for assessing investment projects in financial theory as well as in business practice, it suggests itself to integrate taxation into real option-based models. By doing so, it is possible to derive an investment rule considering taxational effects and to identify tax systems that are neutral with respect to investment decisions. For risk neutral investors, such tax systems have already been proved [cf. Niemann (1999)].

As under the classical net present value rule, the investment rule in real option-based models is simply that a project's benefits must outweigh its costs. Here, the benefits are defined by the expected present value of the investment project plus the additional real options generated by realizing the project. The costs, in contrast, consist of the strike price, i.e., the project's initial outlay and the value of the real options exercised by carrying out the investment. All components of the costs and benefits can be functions of one or more stochastic variables. The decision rule gives the realization of stochastic variables that is necessary and sufficient to stop waiting and abandon flexibility.

2 General assumptions

The following section provides a summary of the assumptions that are common for all models discussed in this paper. We will consider different models including an option to invest. Its owner permanently faces the decision on either exercising the option, i.e., stopping, carrying out the project and collecting the resulting cash flow or continuing waiting and sacrificing cash flows but keeping the option to avoid unexpectedly low cash flows. It is assumed that the initial outlay I_0 is deterministic and constant and that all uncertainty is summarized in a single continuous-time stochastic process, P, following a geometric Brownian motion

$$\frac{\mathrm{d}P}{P} = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}z \tag{1}$$

with constant drift α and constant volatility σ, where $\mathrm{d}z$ denotes the increment of a standard Wiener process.

Exercising the option to invest is assumed totally irreversible, i.e., it is impossible to abandon a project during its economic life ending at time T. T may be finite or infinite. For reasons of simplicity, we will mostly assume an infinitely-lived project and a perpetual option to invest. The project does not include any additional flexibility, so its only benefit is the expected cash flow. Generally, the project's cash flow π is a function of the stochastic variable P and time t: $\pi \equiv \pi(P, t)$.

Integrating taxation, some simplifying assumptions concerning the tax base and the tax rate are necessary. The tax base equals cash flow π less depreciation allowances d that may be deterministic or stochastic[1]. The tax rate τ is assumed deterministic and constant, so under an immediate loss-offset, the post-tax cash flow π_τ is defined as:

$$\pi_\tau = \pi - \tau\;(\pi - d) = (1 - \tau)\;\pi + \tau\,d. \tag{2}$$

As long as the option to invest is not exercised, available funds yield the risk-free capital market rate r that is assumed constant. The debit and credit rates are identical. A fraction γ of credit or debit interest payment is liable to tax or tax-deductible, respectively. The risk-free after-tax interest rate r_τ can be written as $r_\tau = (1 - \gamma\,\tau)\;r$.

[1] Depreciation allowances might also include immediate write-offs.

3 Real options under risk neutrality

3.1 Tax-free case

One method to derive the optimal investment rule under uncertainty and to assess the value of the option to invest is a technique widely used in economics: dynamic programming [cf. Bellman (1957)]. An alternative method has emerged from option pricing theory: contingent claims analysis. Since the optimality principles are well known in real option theory, we will only give a brief introduction in order to focus on post-tax problems.

It is known from real option literature that assuming an output price P following a geometric Brownian motion and the investment decision being completely irreversible, yields the general objective function

$$\max_u \left\{\max \left\{V\left(P\right) - I_0, F\left(P\right)\right\}\right\} \tag{3}$$

$$\text{with } F\left(P\right) = \max_{T>t} \left\{\mathrm{E}\left[\left(V_T\left(P\right) - I_0\right) e^{-r(T-t)}\right], 0\right\} \tag{4}$$

$\mathrm{E}\left[\cdot\right]$:	expectations operator
$F(P)$:	value of the option to invest
u:	action variable
$V(P)$:	expected present value of the investment project,

which is independent of applying either the DP or the CCA approach.

To derive a rule for optimal investment, at first we have to assess the value of the underlying asset, the investment project. If the project is in place, it does not involve any flexibility, so its economic value consists solely of its future cash flows. For reasons of simplicity, the cash flow $\pi(P,t)$ is set equal to the geometric Brownian motion P: $\pi(P,t) = P$. Assuming risk neutrality, the project value V is determined by its expected present value computed with the risk-free rate r:

$$V \equiv V(P) = \mathrm{E}\left[\int_t^\infty P\left(\xi\right)\, e^{-r(\xi-t)}\mathrm{d}\xi\right] = \frac{P}{r-\alpha}; \quad r > \alpha. \tag{5}$$

3.1.1 Dynamic programming

Given the value of the underlying asset (eq. 5), the value of the option to invest can be determined. Since the owner of the option can only decide between waiting and exercising, the decision variable is binary, so it is possible to check optimality by complete enumeration. We will start with the continuation region in which the option is kept alive. The optimal transition to the stopping or exercise region will be modeled by boundary conditions.

The option does not carry any cash flows, so its possible payoff only consists of its expected appreciation in value. The continuous-time Hamilton-Jacobi-Bellman equation for determining the value of the call can be written as:

$$r\,F \overset{!}{=} \mathrm{E}\left[\mathrm{d}F\right]. \tag{6}$$

Its economic interpretation is that the owner of the option expects an instantaneous return that in equilibrium equals the risk-free rate. Application of Itô's lemma to the stochastic differential $\mathrm{d}F$ and further transformation yields the partial differential equation:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2\,P^2\,\frac{\partial^2 F}{\partial P^2} + \alpha\,P\,\frac{\partial F}{\partial P} - r\,F = 0. \tag{7}$$

Assuming a perpetual real option, the time derivative $\frac{\partial F}{\partial t}$ vanishes and eq. (7) reduces to the ordinary differential equation:

$$\frac{1}{2}\sigma^2\,P^2\,\frac{\mathrm{d}^2 F}{\mathrm{d}P^2} + \alpha\,P\,\frac{\mathrm{d}F}{\mathrm{d}P} - r\,F = 0 \tag{8}$$

with the general solution

$$F\left(P\right) = A\,P^{\lambda}, \quad A > 0, \quad \lambda = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2\,r}{\sigma^2}} > 1, \tag{9}$$

where A is a constant to be determined. The boundary conditions are:

$$F\left(0\right) = 0 \tag{10}$$

$$F\left(P^*\right) = V(P^*) - I_0 \tag{11}$$

$$\frac{\mathrm{d}F\left(P^*\right)}{\mathrm{d}P} = \frac{\mathrm{d}V\left(P^*\right)}{\mathrm{d}P}. \tag{12}$$

Eq. (10) implies that a call on a worthless underlying is itself worthless. Eqs. (11) and (12) are free boundary conditions determining the transition from the continuation region to the stopping region at the critical investment threshold P^*. Eq. (11) means that a project's benefits must equal its costs at the point of transition. It is called value-matching condition. Eq. (12) is a so-called smooth-pasting or high contact condition requiring the identity of marginal benefits and marginal costs at the critical threshold. Further transformation yields the critical investment threshold P^*:

$$P^* = \frac{\lambda}{\lambda - 1}\left(r - \alpha\right) I_0. \tag{13}$$

The critical value P^* indicates whether investment should be delayed or not. If the actually observed realization P is higher than the critical value P^*, the investment should be carried out immediately, otherwise it must be postponed until P^* is reached.

3.1.2 Contingent claims analysis

In analogy to the Black-Scholes formula [cf. Black and Scholes (1973)] in CCA, a risk-free portfolio has to be formed to assess the value of the option to invest and herewith to determine the decision rule. The portfolio Φ consists of one option to invest and n units in a short position of the investment project or a perfectly replicating asset:

$$\Phi = F(P) - nP. \tag{14}$$

For $n = \frac{\partial F}{\partial P}$ all stochastic terms vanish, so the portfolio is riskless. In the absence of arbitrage opportunities, the riskless total return from the portfolio equals the return from the real option plus the negative return from the short position during a time interval of length $\mathrm{d}t$. Substituting for n yields

$$r\,\Phi\,\mathrm{d}t = \mathrm{d}F - \frac{\partial F}{\partial P}\mathrm{d}P - \frac{\partial F}{\partial P}\delta P\,\mathrm{d}t, \tag{15}$$

with $\delta = r - \alpha$, the dividend rate. After some transformations, the ordinary differential equation (8) from dynamic programming can be derived. Considering identical boundary conditions a critical threshold, which is equivalent to the one from dynamic programming (eq. 13) can be determined.

A critical assumption of CCA is the replication or spanning property. In reality, there are only very few perfectly replicating assets for real investment, especially in commodity-related projects like oil exploration or mining. In any case, the existence of futures markets often enables to construct a replicating portfolio.

An advantage of CCA is the endogenization of the capital market rate whereas DP is limited to cases in which the discount rate is given exogeneously. In this chapter, the analysis is restricted to a risk neutral scenario, so the problem of finding an adequate discount rate is irrelevant. Both approaches refer to the exogeneously determined risk-free rate. The derivation of an appropriate discount rate will gain importance in a context of risk aversion in chapter 4.

3.2 Integrating taxes

The literature on real options hardly considers taxation [Exceptions are Majd and Myers (1987), Mauer and Ott (1995), Harchaoui and Lasserre (1996), Pennings (2000), Agliardi (2001), e.g.]. Neither investment nor taxes are treated in a general mode. Therefore, these approaches are not applicable to draw general conclusions concerning the influence of taxation on investment behavior.

3.2.1 Dynamic programming

The integration of taxes requires the same steps as in the tax-free case. At first, the value of the underlying asset has to be determined. The project value after taxes V_τ is the expected present value of the future after-tax cash flows π_V^τ:

$$V_\tau \equiv V_\tau(P) = \mathrm{E}\left[\int_t^\infty \pi_V^\tau \, e^{-r_\tau(\xi-t)} \mathrm{d}\xi\right] = \mathrm{E}\left[\int_t^\infty \left[(1-\tau)\, P\,(\xi) + \tau\, d\right] \, e^{-r_\tau(\xi-t)} \mathrm{d}\xi\right]. \quad (16)$$

Separating this term into the tax-reduced cash flow and the tax shield on the depreciation allowances yields:

$$V_\tau(P) = \frac{(1-\tau)\, P}{r_\tau - \alpha} + \tau \int_t^\infty \mathrm{E}\,[d] \; e^{-r_\tau(\xi-t)} \mathrm{d}\xi = \frac{(1-\tau)\, P}{r_\tau - \alpha} + \tau\, \mathrm{E}\,[D]\,; \quad r_\tau > \alpha, \quad (17)$$

with D as the present value of immediate write-offs and current depreciation deductions. Whereas the option to invest might principally be regarded as depreciable[2], we focus on a non-depreciable option to invest. Thus, the option's cash flow generally consisting of the tax shield on depreciation deductions vanishes in the case considered here. Computing the post-tax option value F_τ requires the Hamilton-Jacobi-Bellman equation

$$r_\tau \, F_\tau \overset{!}{=} \mathrm{E}\,[\mathrm{d}F_\tau]\,, \quad (18)$$

that can be transformed to the ordinary differential equation

$$\frac{1}{2}\sigma^2 \, P^2 \, \frac{\mathrm{d}^2 F_\tau}{\mathrm{d}P^2} + \alpha\, P \, \frac{\mathrm{d}F_\tau}{\mathrm{d}P} - r_\tau \, F_\tau = 0 \quad (19)$$

with the general solution

$$F_\tau\,(P) = A_\tau \, P^{\lambda_\tau}, \;\; A_\tau > 0, \;\; \lambda_\tau = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2\, r_\tau}{\sigma^2}} > 1, \quad (20)$$

with A_τ, a constant to be determined. The boundary conditions for the generalized case of stochastic depreciations are similar to the tax-free case:

$$F_\tau\,(0) = 0 \quad (21)$$

$$F_\tau\,(P_\tau^*) = V_\tau(P_\tau^*) - I_0 \quad (22)$$

$$\frac{\mathrm{d}F_\tau\,(P_\tau^*)}{\mathrm{d}P} = \frac{\mathrm{d}V_\tau\,(P_\tau^*)}{\mathrm{d}P}. \quad (23)$$

Generally, it is not possible to compute the critical investment threshold in the model with taxes P_τ^* analytically unless the stochastic nature of depreciation allowances is known, e.g., in the special case of deterministic depreciations:

$$P_\tau^* = \frac{\lambda_\tau}{\lambda_\tau - 1} \, \frac{r_\tau - \alpha}{1 - \tau} \, (I_0 - \tau\, D)\,, \quad (24)$$

[2] Acquired real options are depreciable under most tax regimes.

The optimal investment rules before and after taxes at hand, it is possible to derive neutral tax systems by equating the critical investment thresholds and solving for the present value of depreciation allowances. Since neutral tax systems are well-known under certainty [cf. Brown (1948), Preinreich (1951), Samuelson (1964), Johansson (1969), Brennan and McGuire (1975), Boadway and Bruce (1984)] and neutral taxes have already been derived under risk neutrality in real option literature [e.g. Niemann (1999)], we will not discuss their properties in detail and will leave further derivations aside. It should only be noted that the cash flow tax and the Johansson-Samuelson tax are special cases of such neutral tax systems.

3.2.2 Contingent claims analysis

Referring to the tax system from chapter 2, the riskless portfolio with taxes becomes:

$$\Phi_\tau = F_\tau - \frac{\partial F_\tau}{\partial P} P. \tag{25}$$

For tax purposes, it has to be clarified, whether the portfolio consists of depreciable assets or financial and therefore non-depreciable assets. Excluding acquired options, the real option is assumed non-depreciable and therefore has to be interpreted as a financial asset. The short position has to be regarded as a liability. Since the investor is selling short and therefore not owning the output from investment, he is not supposed to depreciate the short position. The writer of the short position receives a taxable payment of $\frac{\partial F_\tau}{\partial P}\delta P$. From the investor's viewpoint, the portfolio is a financial asset.

For simplification, we will assume that the portfolio's pre-tax value in a world with taxation is identical to the one in the tax-free model. The equilibrium condition is

$$r_\tau \; \Phi_\tau \; \mathrm{d}t = N_\tau \; \mathrm{d}t, \tag{26}$$

where N_τ represents the portfolio's after-tax return and Φ_τ the after-tax value of the portfolio. The right hand side of this equilibrium condition describes the after-tax cash flow the investor receives during a time interval of length dt when liquidating the portfolio. Because the portfolio is non-depreciable, this cash flow amounts to:

$$\begin{aligned} N_\tau \, \mathrm{d}t &= (1-\gamma\tau)\left(\mathrm{d}F - \frac{\partial F}{\partial P}\,\mathrm{d}P - \frac{\partial F}{\partial P}\,\delta\, P\,\mathrm{d}t\right) \\ &= \left(\mathrm{d}F_\tau - \frac{\partial F_\tau}{\partial P}\,\mathrm{d}P - \frac{\partial F_\tau}{\partial P}\,\delta\, P\,\mathrm{d}t\right). \end{aligned} \tag{27}$$

In order to reach an investment decision and to prove tax neutrality, the critical value P^*_τ has to be derived. Substituting for N_τ, we find in equilibrium:

$$r_\tau \left(F_\tau - \frac{\partial F_\tau}{\partial P} P \right) \mathrm{d}t = \mathrm{d}F_\tau - \frac{\partial F_\tau}{\partial P} \mathrm{d}P - \frac{\partial F_\tau}{\partial P} \delta P \, \mathrm{d}t \qquad (28)$$

and further the ordinary differential equation known from eq. (19). In contrast to dynamic programming (eq. 17), the investment project's post-tax present value $\overline{V}_\tau(P)$ in CCA does not include the tax shield on depreciation allowances for notational reasons. Whether the tax shield on depreciation allowances is included in the effective initial outlay I_0^τ or in the project value does not affect the boundary conditions. The effective after-tax initial outlay under deterministic depreciation deductions can be denoted as

$$I_0^\tau = I_0 - \tau D. \qquad (29)$$

As in DP, D in eq. (29) reflects the present value of depreciation allowances for the project's entire economic life. Applying the boundary conditions in the post-tax case:

$$F_\tau (0) = 0 \qquad (30)$$

$$F_\tau (P^*_\tau) = \overline{V}_\tau (P^*_\tau) - I_0^\tau \qquad (31)$$

$$\frac{\mathrm{d}F_\tau (P^*_\tau)}{\mathrm{d}P} = \frac{\mathrm{d}\overline{V}_\tau (P^*_\tau)}{\mathrm{d}P}, \qquad (32)$$

yields the same post-tax critical investment threshold as in eq. (24)

$$P^*_\tau = \frac{\lambda_\tau}{\lambda_\tau - 1} \frac{r_\tau - \alpha}{1 - \tau} (I_0 - \tau D). \qquad (33)$$

Comparing it with the pre-tax model's critical value P^* (eq. 13), it is obvious that taxation might cause a distorting effect by asymmetrical treatment of financial and real investment.

4 Real options under risk aversion

As demonstrated above, optimizing the investor's objective function leads to equivalent results for both methods under risk neutrality. Expanding the analysis with respect to risk aversion, this result cannot be acknowledged as will be seen.

4.1 Tax-free case

4.1.1 Dynamic programming

Using the DP approach the derivation of the investment rule requires either an exogeneously determined risk-adjusted discount rate (RADR) or explicit knowledge of an investor's risk preferences. In the preceeding sections, for reasons of simplicity, we assumed risk neutrality to restrict the computations on expected present values.

In reality, investors are typically risk averse, which is reflected by traditional capital market models where risky assets yield higher expected returns than risk-free assets. There are different ways to deal with risk aversion in models of capital budgeting, e.g. RADRs or martingales. We will not use RADRs in the DP approach because even in the tax-free case, the CAPM's meaningfulness is quite limited with additional difficulties in the integration of taxes [e.g. Bogue and Roll (1974), Fama (1977), Constantinides (1980)]. A quite sophisticated method to model risk aversion are martingale measures [cf. e.g. Harrison and Kreps (1979), Harrison and Pliska (1981)]. Since it is not yet clear if martingales do exist after the integration of taxes, we will leave it as a subject for further research. Instead, investors' risk preferences are endogenized by their utility functions.

Strictly speaking, maximizing an individual's utility under risk aversion requires a global model of all decision variables since the utility function's non-linearity does not permit separation of partial decision models like consumption, saving and investment. Therefore, some simplifying assumptions have to be made in order to concentrate on the investment behavior. The optimal division of a cash flow into saving and consumption will be left aside because it requires information from outside the investment model. For this reason, cash flows instead of consumption expenditures will be used as the utility function's argument. This assumption can be justified by defining the utility from a cash flow being the result of optimal intertemporal allocation of consumption.

Apart from this more restrictive assumption, the utility function $U \equiv U(\pi)$ should be well-behaved, i.e., time-invariant ($\frac{\partial U}{\partial t} = 0$), twice continuously differentiable, with positive and diminishing marginal utility ($\frac{\partial U}{\partial \pi} > 0$, $\frac{\partial^2 U}{\partial \pi^2} < 0$). Utility is assumed additive over time. Intertemporal utility transformation can be realized only by intertemporal transformation of cash flows. The risk-free interest rate r is assumed constant. Without restricting generality, utility is standardized to $U(0) = 0$, $U(1) = 1$. Disutility from an investment project's initial outlay $U(-I_0)$ is treated as an individual constant.

Qualitatively speaking, the situation under risk aversion is the same as under risk neutrality. The investor holds a perpetual option to invest in an infinitely-lived project with deterministic initial outlay I_0 =const. and stochastic cash flow $\pi \equiv \pi(P, t)$. Again, the decision variable is binary: the investor can either continue waiting or exercise the option to invest. Nevertheless, the decision rule has to be adopted to risk aversion. Now, exercising the option is optimal when the investment project's discounted future utility exceeds the initial outlay's utility plus the option's utility. In principle, this rule is valid under risk neutrality, too, because utility is equivalent to expected monetary units in that context. In contrast, under risk aversion, the numéraire is utility rather than money.

Again, the investment project and the option to invest will be valued separately. According to the preceeding sections, the project's cash flow π equals the stochastic variable P which is still the only source of uncertainty and which follows a geometric Brownian motion. A project already in place does not offer any flexibility, so its value simply consists of its expected discounted utility:

$$V(P) = \mathrm{E}\left[\int_t^{\infty} U(P)\, e^{-r\,(\xi - t)}\, \mathrm{d}\xi\right]. \tag{34}$$

In the limiting case $P \to 0$, the realized project has zero value because of the standardization $U(0) = 0$: $V(0) = 0$. It should be emphasized that all assets are valued in utility units. For comparison, monetary units have to be transformed by applying the utility function.

As under risk neutrality, it is necessary to evaluate the option to invest under both settings of the control variable. At the stopping region, the option value equals its intrinsic value Ω which is the maximum of subjective project value plus the disutility resulting from the initial outlay and zero:

$$\Omega \equiv \Omega(P) = \max\{V(P) + U(-I_0); 0\}. \tag{35}$$

In the continuation region, the Hamilton-Jacobi-Bellman equation can be written as:

$$r\,F \overset{!}{=} U(\pi_F) + \frac{\mathrm{E}\,[\mathrm{d}F]}{\mathrm{d}t}, \tag{36}$$

where π_F denotes the possible cash flow from holding the option. The economic interpretation is that the risk-free return (measured in utility units) must equal the instantaneous utility plus the expected change in utility during the next infinitesimal time interval. Since the option comprises no pre-tax cash flow ($\pi_F = 0$), its instantaneous utility is zero: $U(0) = 0$. Transforming equation (36) yields the well-known ordinary differential equation:

$$\frac{1}{2}\,\sigma^2\,P^2\,\frac{\mathrm{d}^2F}{\mathrm{d}P^2} + \alpha\,P\,\frac{\mathrm{d}F}{\mathrm{d}P} - r\,F = 0, \tag{37}$$

which is equivalent to the risk neutral case. Therefore, the option value function is equivalent, too:

$$F(P) = A\, P^{\lambda}. \tag{38}$$

The reason is that the differential equation determining the option value only depends on the underlying stochastic process which by assumption is the same as under risk neutrality. The option value itself is different, of course. The investor's utility function determines the underlying asset and therefore the option coefficient A.

Again, solving the investment problem requires two free boundary conditions. At the optimal point of transition, P^*, the expected utility of the continuation alternative must equal the expected utility of the stopping alternative (value matching):

$$F(P^*) \overset{!}{=} V(P^*) + \underbrace{U(-I_0)}_{=\text{const.}}. \tag{39}$$

The same applies to the marginal utilities of the continuation and the stopping alternatives (smooth pasting):

$$\frac{\mathrm{d}F(P^*)}{\mathrm{d}P} \overset{!}{=} \frac{\mathrm{d}V(P^*)}{\mathrm{d}P}. \tag{40}$$

Without explicit knowledge of the utility function $U(P)$ these expressions cannot be further elaborated.

Nevertheless, the procedure can be characterized in general. Because P is a geometric Brownian motion, a continuous function $U(P)$ follows a diffusion process with drift function α_U and volatility function σ_U.[3] Applying Itô's lemma on the stochastic utility differential and further transformation yields:

$$\begin{aligned} \mathrm{d}U &= \frac{\mathrm{d}U}{\mathrm{d}P}\,\mathrm{d}P + \frac{1}{2}\,\frac{\mathrm{d}^2U}{\mathrm{d}U^2}\,(\mathrm{d}P)^2 \\ &= \frac{\mathrm{d}U}{\mathrm{d}P}(\alpha\, P\,\mathrm{d}t + \sigma\, P\,\mathrm{d}z) + \frac{1}{2}\,\frac{\mathrm{d}^2U}{\mathrm{d}P^2}\,\sigma^2\, P^2\,\mathrm{d}t \\ &= \underbrace{\left(\alpha\, P\,\frac{\mathrm{d}U}{\mathrm{d}P} + \frac{1}{2}\,\sigma^2\, P^2\,\frac{\mathrm{d}^2U}{\mathrm{d}P^2}\right)}_{\alpha_U(U)}\,\mathrm{d}t + \underbrace{\sigma\, P\,\frac{\mathrm{d}U}{\mathrm{d}P}}_{\sigma_U(U)}\,\mathrm{d}z. \end{aligned} \tag{41}$$

P can be eliminated by the inverse function $U^{-1}(U(P)) = P$, so that the stochastic differential equation (41) only contains functions of U. Accordingly, utility and expected

[3] These functions may be more complex than a geometric Brownian motion, so that it is not always possible to compute expected utility.

utility at time t can be written as stochastic and deterministic integrals:

$$\begin{aligned} U(t) &= U(0) + \int_0^t \alpha_U(U(\xi))\,\mathrm{d}\xi + \int_0^t \sigma_U(U(\xi))\,\mathrm{d}z_\xi \\ \mathrm{E}\left[U(t)\right] &= U(0) + \int_0^t \alpha_U(U(\xi))\,\mathrm{d}\xi. \end{aligned} \quad (42)$$

Differentiating (42) with respect to t generates the following first-order ordinary differential equation:

$$\frac{\mathrm{d}\,\mathrm{E}\left[U(t)\right]}{\mathrm{d}t} - \alpha_U(U(t)) = 0. \quad (43)$$

Its solution constitutes expected utility at time t. Since $\alpha_U(U)$ may be nonlinear in U, there is no guarantee for analytical solutions. But even if these do exist, simple examples demonstrate that the expected discounted utility

$$V = \int_t^\infty \mathrm{E}\left[U(\xi)\right]\, e^{-r\,(\xi-t)}\,\mathrm{d}\xi \quad (44)$$

cannot always be computed. For economically useful results, sufficiently well-behaved utility functions are required. One such example is the power utility $U(P) = P^R$ $(0 < R < 1)$. Unfortunately, even widely-used utility functions like logarithmic or exponential utility do not permit analytical solutions because solving the differential equation (37) requires standardizing utility $U(0) = 0$ to avoid infinite nonhomogenous parts of the equation.

4.1.2 Contingent claims analysis

In contrast to DP, CCA requires an endogeneous discount rate. This might be useful if we do not succeed in solving problems concerning individual risk preference functions. Well-known from Black-Scholes analysis, option pricing theory, being the basis for CCA, seems to allow a risk neutral scenario for risk averse investors and herewith apparently offers the possibility to ignore individual risk preferences.

The objective function under risk neutrality (3) has to be adopted to risk aversion introducing a RADR μ:

$$\begin{aligned} & \max_u \left\{\max\left\{V\left(P\right) - I_0, F\left(P\right)\right\}\right\} \quad (45) \\ \text{with} \qquad F\left(P\right) &= \max_{T>t} \left\{\mathrm{E}\left[\left(V_T\left(P\right) - I_0\right) e^{-\mu(T-t)}\right],\, 0\right\}. \end{aligned}$$

In option pricing theory forming a portfolio allows risk neutral valuation of financial options even for risk averse investors. For optimization purposes one may refer to the exogeneous risk-free market rate of return r. Nevertheless, we have to find out, whether the problem of a suitable risk-adjusted discount factor can really be avoided in the context of real investment. The equilibrium condition (15) under risk aversion becomes:

$$r\,\Phi\,\mathrm{d}t = \mathrm{d}F - \frac{\partial F}{\partial P}\,\mathrm{d}P - \frac{\partial F}{\partial P}\delta\,P\,\mathrm{d}t. \tag{46}$$

In CCA, the RADR influences the optimization calculus by the dividend rate δ although a risk-free portfolio and herewith a risk neutral scenario is supposedly constructed. The investor has to pay $\frac{\partial F}{\partial P}\delta\,P\,\mathrm{d}t$, where $\delta = \mu - \alpha$, during an interval of length $\mathrm{d}t$ to the writer of the short position.

In the Black-Scholes-formula expanded for dividend payments [cf. Merton (1973b), p. 171)], $\delta = \mu - \alpha$ appears, too. In contrast to the real options approach, risk aversion does not influence the option value in the option pricing model since δ can be interpreted as an exogeneously given dividend rate. The amount of the dividend payment is either known or it can be estimated on the basis of former dividend payments. This seems permissable since dividend payments in general remain constant over time [cf. Lintner (1956), Cragg (1986), pp. 195-196] and financial options are normally short term investments, which implies that they are unlikely to suffer unexpected changes of the dividend. Therefore, δ may be considered exogeneous.

In contrast to financial option pricing, the dividend rate δ in CCA is endogeneous. This is a result from the relation $\mu = \alpha + \delta$, where μ comprises a risk premium $\eta\,\sigma$ and therefore, $\mu = r + \eta\,\sigma$, with η, the risk aversion coefficient [cf. Dixit and Pindyck (1994), pp. 149-150, Trigeorgis (1996), p. 97].

In order to determine the appropriate RADR, the distribution of the the risk premium $\eta\,\sigma$ among the growth rate α and the dividend rate δ has to be explained by a sophisticated capital market model. Even abstracting from the CAPM inherent problems (e.g. application of a one-period model to a multi-period decision problem etc.) [cf. e.g. Merton (1973a), p. 885, Bogue and Roll (1974), pp. 604-606, Roll (1977), Kazemi (1991), p. 224, Kulatilaka and Marcus (1992), p. 94, Sick (1995), pp. 634-636 and 639-643], the real option approach fails to solve this problem.

Dixit and Pindyck assume δ to be independent of σ and only α to include the risk premium [cf. Dixit and Pindyck (1994), pp. 149-150]. Accepting this assumption, analyzing tax systems and applying CCA, there is no need to know μ and the risk premium $\eta\sigma$. Every α comprising term included in the equilibrium condition vanishes after application of Itô's lemma. Consequently, the risk premium has no impact on the critical value P^*. For this scenario, the results under risk neutrality can be transferred unamended to the case of risk aversion. Thus, the critical threshold is given by

$$P^* = \frac{\lambda}{\lambda - 1}\, \delta\, I_0. \tag{47}$$

Whenever the risk premium is supposed to be included in α as well as in δ, a sophisticated capital market equilibrium model is needed, which, among other aspects, allows to deduce a rule for dividing up the risk premium between the two components. Consequently, λ and herewith the critical threshold P^* will be influenced by risk aversion. It is not possible to generate pseudo risk neutrality by forming a hedge portfolio any more.

4.2 Integrating taxes

Under risk neutrality, our focus was the presentation of the DP and the CCA approach as a basis for their comparison in chapter 5. We referred to investment rules and neutral tax systems only incidentically because these aspects are already known in real options literature. In contrast, under risk aversion, no neutral tax systems have been proved until now, so it should be discussed whether an optimal investment rule after taxes exists and derive it to prove neutral tax systems.

4.2.1 Dynamic programming

Using DP, we will refer to the exogeneously determined risk-free rate r. Therefore, the risk-free post-tax discount rate r_τ is the same as under risk neutrality: $r_\tau = (1 - \gamma\tau)r$. The value of the realized project is the intertemporal utility from its after-tax cash flows:

$$\begin{aligned} V_\tau \equiv V_\tau(P) &= \mathrm{E}\left[\int_t^\infty U(\pi_\tau)\, e^{-r_\tau(\xi - t)} \mathrm{d}\xi\right] \\ &= \int_t^\infty \mathrm{E}\left[U((1-\tau)\, P + \tau\, d)\right]\, e^{-r_\tau(\xi - t)} \mathrm{d}\xi. \end{aligned} \tag{48}$$

Since the investor's utility function is non-linear, it is not possible to separate the cash flow and depreciation components in discounted utility terms. For this reason, the post-tax critical investment threshold can be computed only in special cases. Even without taking the option to invest into account, it becomes obvious that the derivation of neutral tax systems involves a circular argument: the depreciation function d is already needed to derive the critical investment threshold which in turn is necessary to compute the neutral depreciation schedule. A special case in which the critical threshold can be explicitly computed is the cash flow tax. It is well known that under risk aversion, the cash flow tax is not neutral. This result can be proved in a real options model, too. Compared with the tax-free case, the cash flow tax will typically promote investment because it reduces the initial outlay I_0 as well as current cash flows by the same factor τ, while the marginal utility obtained by the tax shield on the initial outlay exceeds the marginal utility from the foregone cash flow.

Nevertheless, neutral tax systems can be proved in a generalized model with the following additional assumptions:

- The project's economic life may be finite: $T \leq \infty$.
- Cash flows are a function of the geometric Brownian motion P and time t: $\pi \equiv \pi(P, t)$.
- Cash flows equal zero after time T: $\pi(P, T) = 0$.
- A fraction D_i of the initial outlay I_0 might be written off immediately. To separate immediate write-offs from current depreciation deductions, the former are included in the effective initial outlay I_0^{eff} that is reduced by the tax shield on the immediate write-off: $I_0^{eff} = (1 - \tau\, D_i)\ I_0$.
- The option to invest is assumed depreciable. Depreciation deductions associated with the option to invest are denoted by $d_F \in \mathbb{R}$. Accordingly, it might involve a non-zero cash flow $\pi_F = \tau\, d_F$.[4]
- Depreciation allowances may be stochastic as well as deterministic: $d_V \equiv d_V(P, t)$, $d_F \equiv d_F(P, t)$.

In this case, the project's pre-tax value V is given by:

$$V(P, t) = \mathrm{E}\left[\int_t^T U(\pi_V(P, \xi))\, e^{-r(\xi - t)} \mathrm{d}\xi\right] = \int_t^T \mathrm{E}\left[U(\pi_V(P, \xi))\right]\, e^{-r(\xi - t)} \mathrm{d}\xi. \tag{49}$$

with $V(0, t) = V(P, T) = 0$. The Hamilton-Jacobi-Bellman equation for the option value in the continuation region is once again:

$$r\, F \stackrel{!}{=} U(\pi_F) + \frac{\mathrm{E}\left[\mathrm{d}F\right]}{\mathrm{d}t}. \tag{50}$$

[4] Variables concerning the option are denoted with subscript F, variables concerning the investment project with subscript V.

It can be used in the present context of a non-perpetual option as well, but following from the time-dependence the partial derivative $\frac{\partial F}{\partial t}$ does not vanish, so the option value is determined by the partial differential equation[5]:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 F}{\partial P^2} + \alpha P \frac{\partial F}{\partial P} - r\, F = 0, \tag{51}$$

with $F(0,t) = F(P,T) = 0$. Although the partial differential equation is equivalent to the risk neutral case, it has to be mentioned that the option value as well as the project value is measured in utility units rather than monetary units. The free boundary conditions are

$$F(P^*,t) \stackrel{!}{=} V(P^*,t) + U(-I_0) \tag{52}$$

$$\frac{\partial F(P^*,t)}{\partial P} \stackrel{!}{=} \frac{\partial V(P^*,t)}{\partial P}. \tag{53}$$

Since this problem involves a free boundary problem with a partial differential equation, it typically cannot be solved analytically. For this reason, it is not possible to derive a rule for optimal investment. As a consequence, neutral tax systems cannot be derived by simply equating the pre-tax and the post-tax investment thresholds, i.e., it is not possible to refer to the neutrality condition that is necessary as well as sufficient.

Nevertheless, we can fall back on a condition that is only sufficient. In the model presented here, it is admissable to use the pre-tax model as a yardstick for measuring tax effects, because maximization of individual utility in a tax-free scenario leads to the desired Pareto optimum. As a first step towards neutral tax systems, the post-tax investment problem is formulated. The project value is given by:

$$V_\tau = \int_t^T \mathrm{E}\left[U\left[(1-s)\pi_V + s\, d_V\right]\right]\, e^{-r_\tau(\xi - t)}\, \mathrm{d}\xi \tag{54}$$

with the fixed boundary conditions

$$V_\tau(0,t) = \int_t^T \mathrm{E}\left[U(\tau\, d_V(0))\right]\, e^{-r_\tau(\xi-t)}\, \mathrm{d}\xi \tag{55}$$

$$V_\tau(P,T) = 0. \tag{56}$$

The post-tax partial differential equation for the option value reads

$$\frac{\partial F_\tau}{\partial t} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 F_\tau}{\partial P^2} + \alpha P \frac{\partial F_\tau}{\partial P} - r_\tau\, F_\tau + U(\pi_F^\tau) = 0 \tag{57}$$

[5]Notice that $\pi_F = 0$ and $U(0) = 0$.

with the fixed boundary conditions

$$F_\tau(0,t) \;=\; \int_t^T \mathrm{E}\left[U(\tau\, d_F(0))\right]\, e^{-r_\tau(\xi - t)}\, \mathrm{d}\xi \tag{58}$$

$$F_\tau(P,T) \;=\; 0 \tag{59}$$

and the free boundary conditions

$$F_\tau(P_\tau^*, t) \;\stackrel{!}{=}\; V_\tau(P_\tau^*, t) + U(-I_0^{eff}). \tag{60}$$

$$\frac{\partial F_\tau(P_\tau^*, t)}{\partial P} \;\stackrel{!}{=}\; \frac{\partial V_\tau(P_\tau^*, t)}{\partial P}. \tag{61}$$

Although these investment problems are not analytically tractable, it is possible to derive neutral tax systems by comparing the pre-tax and post-tax problems rather than their solutions. Of course, this neutrality condition is only sufficient but not necessary. The technique is used under risk neutrality in Niemann (1999). Identical solutions may be achieved by different problems whereas identical problems necessarily yield identical solutions. The investment problems under consideration are identical if the pre-tax and post-tax partial differential equations and boundary conditions are equivalent.

We will look at the free boundary conditions first. The value matching and smooth pasting conditions before and after taxes are equivalent if all values - measured in utility units - undergo a linear transformation:

$$\begin{aligned} F_\tau &= c\,F \\ V_\tau &= c\,V \\ U(-I_0^{eff}) &= c\,U(-I_0) \\ 0 < c &= \text{const.} \end{aligned} \tag{62}$$

Then:

$$\begin{aligned} F(P^* t) &= V(P^*, t) + U(-I_0) \\ \Leftrightarrow c\,F(P^*, t) &= c\,\left[V(P^*, t) + U(-I_0)\right] \\ \Leftrightarrow F_\tau(P^*, t) &= V_\tau(P^*, t) + U(-I_0^{eff}), \end{aligned} \tag{63}$$

and therefore:

$$\frac{\partial F(P^*, t)}{\partial P} = \frac{\partial V(P^*, t)}{\partial P} \Leftrightarrow \frac{\partial F_\tau(P^*, t)}{\partial P} = \frac{\partial V_\tau(P^*, t)}{\partial P}. \tag{64}$$

We will now analyze in detail which conditions cause a linear transformation in utility. Proportionality of disutility from the initial outlay before taxes $U(-I_0)$ and after taxes

$U(-I_0^{eff})$ can be achieved by granting a utility-dependent immediate write-off. If the constant of proportionality c is given by tax law, the immediate write-off D_i can be computed using the relationship $I_0^{eff} = (1 - \tau\, D_i)\, I_0$:

$$\begin{aligned} U(-I_0^{eff}) &\stackrel{!}{=} c\,U(-I_0) \\ -I_0^{eff} &= U^{-1}\left[c\,U(-I_0)\right] \\ 1 - \tau\, D_i &= \frac{-U^{-1}\left[c\,U(-I_0)\right]}{I_0} \\ D_i &= \frac{1}{\tau} + \frac{U^{-1}\left[c\,U(-I_0)\right]}{\tau\, I_0}. \end{aligned} \tag{65}$$

As far as the proportionality of pre-tax and after-tax project values is concerned, the non-separability of cash flow- and depreciation-related utility components does not allow the computation of neutral depreciation deductions in present value terms. Thus, it is necessary to derive a neutral depreciation schedule. Instead of directly equating $V_\tau(P,t)$ and $c\,V(P,t)$, we will use a condition that is sufficient for proportionality of pre-tax and after-tax project values, i.e., proportionality of their time derivatives given the terminal conditions $V_\tau\,(P,T) = V\,(P,T) = 0$:

$$\begin{aligned} \frac{\mathrm{d}V_\tau(P,t)}{\mathrm{d}t} &\stackrel{!}{=} c\,\frac{\mathrm{d}V(P,t)}{\mathrm{d}t} \quad \forall\ t \\ r_\tau\, V_\tau - U(\pi_V^\tau) &= c\,\left[r\,V - U(\pi_V)\right] \\ U\left[(1-\tau)\pi_V + \tau\, d_V\right] &= c\,U(\pi_V) - \gamma\,\tau\, r\, c\, V \\ (1-\tau)\pi_V + \tau\, d_V &= U^{-1}\left[c\,U(\pi_V) - \gamma\,\tau\, r\, c\, V\right] \\ d_V &= \frac{U^{-1}\left[c\,U(\pi_V) - \gamma\,\tau\, r\, c\, V\right]}{\tau} - \frac{1-\tau}{\tau}\pi_V. \end{aligned} \tag{66}$$

This depreciation schedule for the investment project implies proportionality of pre-tax and post-tax project values. Additionally, option values before and after taxes are proportional if the partial differential equations that determine their values are equivalent. If F solves the homogeneous partial differential equation

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 F}{\partial P^2} + \alpha P \frac{\partial F}{\partial P} - r\,F = 0, \tag{67}$$

so does $F_\tau = c\,F$, i.e.:

$$\frac{\partial F_\tau}{\partial t} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 F_\tau}{\partial P^2} + \alpha P \frac{\partial F_\tau}{\partial P} - r\,F_\tau = 0. \tag{68}$$

A depreciation schedule for the option to invest that implies the equivalence of eqns (57) and (68) causes proportionality of the option values before and after taxes if the boundary conditions required for uniqueness are also equivalent. The partial differential equations match if:

$$\begin{aligned} -r\,F_\tau &= -r_\tau\,F_\tau + U(\pi_F^\tau) \\ (r_\tau - r)\,F_\tau &= U(\pi_F^\tau) \\ -\gamma\,\tau\,r\,F_\tau &= U(\tau\,d_F) \\ d_F &= \frac{1}{\tau}\,U^{-1}\,(-\gamma\,\tau\,r\,F_\tau) = \frac{1}{\tau}\,U^{-1}\,(-\gamma\,\tau\,r\,c\,F)\,. \end{aligned} \tag{69}$$

Here, it becomes obvious that depreciation allowances on the option to invest are necessary to ensure the sufficient neutrality condition can be met - at least when the interest rate is subject to tax. The neutral depreciation schedules for the investment project and the option to invest are preference-dependent. Typically, it is not possible to eliminate the utility function U and its inverse U^{-1} from d_V, d_F and D_i. Since different investors are characterized by different utility functions, neutral taxation cannot refer to an objective tax base. To guarantee certainty of the law, recommendations for tax reforms should not be deduced from these results. Nevertheless, the results may indicate the possible impact of a tax reform on taxpayers' behavior.

4.2.2 Contingent claims analysis

Integrating taxes into CCA principally permits using the model from risk neutrality as long as the RADR's risk premium is included solely in α. The results from risk neutrality can be transferred to risk aversion. In this specific scenario, the critical investment threshold is given by:

$$P^* = \frac{\lambda_\tau}{\lambda_\tau - 1}\frac{\delta}{1-\tau}\,(I_0 - \tau D)\,. \tag{70}$$

Results concerning tax neutrality derived under risk neutrality hold under these assumptions, too. Otherwise, taxes have to be taken into account when deriving the RADR, e.g. by employing a post-tax CAPM. If interest payments are subject to tax, the pre-tax RADR is not simply supposed to be reduced by the tax rate to receive the after-tax RADR. Therefore, the integration of taxes in CCA under risk aversion requires an appropriate capital market equilibrium model which includes taxation, options and their interdependences in a dynamic context.

The CAPM-RADR in a real option approach depends on the stochastic process P and herewith on the present value of the investment object as well as the risk associated with the option to invest. Thus, endogenizing the risk premium, the RADR itself becomes stochastic. The interdependence of the stochastic underlying asset, CAPM-RADR, option value and herewith the dynamic investment rule, call for a sophisticated capital market equilibrium model, which takes account of all these complex dynamic interactions. With respect to taxation, dividend policy might play an important role. Taxation that discriminates between retained earnings and dividend payments influences δ, too. On the one hand, the dividend rate depends directly on the underlying tax system, on the other hand it is influenced indirectly by taxation via the post-tax RADR.

An extended CAPM suitable for CCA under risk aversion in general has to take these aspects into account. Whereas the standard-CAPM has been extended for taxation [cf. e.g. Brennan (1970), Litzenberger and Ramaswamy (1979), pp. 165-173], a modified multi-period-CAPM that is necessary for long-term real investment still has to be developed. As could be shown in the pre-tax model, generating a hedge portfolio does not create a scenario that complies with "pure" risk neutrality in general. Not only the problem of an appropriate post-tax RADR is unsolved, but missing a consistent pre-tax model as a reference system, there is no yardstick for investigations concerning taxational effects on investment behavior. In analogy with section 4.2.1, expanding the analysis on investment projects with a finite economic life and on stochastic and time-dependent depreciation allowances requires the solution of a free-boundary problem with a partial differential equation making analytical solutions unlikely for this scenario.

5 Comparison of the approaches

The main motivational factors for employing real option approaches instead of traditional models of capital budgeting under uncertainty are the introduction of managerial flexibility in light of irreversibility and the possibility to abstract from individual risk preferences. Both approaches succeed in integrating flexibility under irreversibility. With respect to the abstraction from individual risk preference functions unter risk aversion, each method reveals its specific limitations. As could be shown, implementing taxation does not evoke any difficulties under risk neutrality. Both approaches yield identical critical thresholds and herewith equivalent investment rules under the given set of assumptions.

Since DP employs an exogeneously given discount rate, this method necessarily neglects information from market data in case of risk aversion. Therefore, it is not possible to distinguish between unsystematic and systematic risk. In contrast to risk neutrality, diversification is highly relevant under risk aversion because it allows the elimination of unsystematic risk. Evaluation herewith is supposed to focus on systematic risk which is ignored by DP. By assuming an exogeneous RADR, risk averse investors exclude relevant information from capital budgeting. Finally, these limitations do not suffice to reject DP under risk aversion. An alternative is the application of the investor's utility function as shown in chapter 4. Although this procedure requires restrictive assumptions and limits the analysis to specific classes of utility functions, investors are enabled to gain important support for investment decisions since an explicit investment threshold can be derived at least in the tax-free case.

On the other hand, in CCA all coefficients are either known or observable or can be estimated on the basis of market data [cf. Cox and Ross (1976), pp. 145-146]. CCA's RADR has to be derived from capital market equilibrium whenever a risk premium is included in δ. Only the risk-free market rate is assumed exogeneous. Applying the CAPM in CCA comprises severe restrictions as described in chapter 4. A critical assumption of CCA is the spanning property concerning the underlying cash flows that requires complete markets for risky assets. This implies perfect positive correlation of the cash flow from the investment project and its duplicate. Even if the underlying asset is not quoted on exchange markets[6], the existence of futures markets and the use of securities instead of real assets to construct a duplicating portfolio reduce this problem to an acceptable minimum. DP does not rely on such strict requirements concerning capital market properties. Whenever a specific risk is not traded one may refer to individual estimations of risk. Thus, the objective function can be interpreted as a utility function with a constant discount factor.

Applying real option approaches, either DP or CCA, investment neutrality of a cash flow tax and a tax equivalent to the taxation of true economic profits, can be proved under risk neutrality [cf. Niemann (1999)]. Uncertainty and irreversibility do not violate the neutrality property of such tax systems. In contrast, under risk aversion, proving a tax system's neutrality requires further assumptions concerning the project's life, its cash flows and depreciation allowances in case of DP and the distribution of the risk premium among the growth parameter α and the dividend rate δ in CCA.

[6] In cases of commodities the underlying asset is often quoted.

6 Concluding remarks

The real option approach is an important extension of traditional capital budgeting methods. It covers uncertainty and managerial flexibility in case of irreversibility and allows the integration of taxes in calculus. Thus, significant improvements of investment decisions become possible. Both approaches, DP and CCA, lead to equivalent results for a risk neutral investor under the given set of assumptions. Further, it could be shown for risk averse investors that a comparison of the pre-tax and post-tax decision problems is limited to a rather restrictive set of assumptions. DP and CCA face different limitations under taxation and risk aversion and lose their equivalence.

Although investment decisions under risk aversion supported by the real option approach cannot be attributed to a single, all-embracing investment rule, DP and CCA significantly expand the field of application of investment models to real investment compared with traditional models of capital budgeting. In order to find more general solutions, future research has to answer several questions concerning an appropriate RADR and on a suitable reference system for tax effects on investment behavior.

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