A REAL OPTION MODEL WITH BAYESIAN LEARNING AND ITS APPLICATION TO ENVIRONMENTAL PRESERVATION

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Abstract : Uncertainty on effective stocks or damages is the core of most irreversible economic decisions involving natural resources and the environment. Although there is an extensive literature dealing with the theoretical aspects of this problem, economists still lack an operational decision rule. The matter is that the real option theory, now widely used to deal with Markovian uncertainty, i.e. independent exogenous repeated chocks affecting the expected return from an irreversible project as time goes, is not adequate to analyze problems characterized by Bayesian uncertainty, i.e. problems where uncertainty about the effective value of parameters is resolved with time by a learning process. This paper is aimed at remedying at these shortcomings. The real option theory is first adapted to an explicit Bayesian learning process in continuous time. In the context of a simple model of environmental preservation, the following question is then addressed: how to define and characterize the optimal tradeoff between the "look before you leap" principle on the one hand and the precautionary principle on the other hand?

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1. INTRODUCTION

Does the state of scientific knowledge justify to take immediate and costly measures to protect the environment? Has exploration of a new oilfield provided a sufficient knowledge about its quality and extent to invest in extraction? Economists dealing with irreversible decisions in the face of uncertainty are concerned with numerous questions of this kind.

Since the seminal articles by Arrow and Fisher [1974] and Henry [1974] and the clarification made by Hanemann [1989], economists generally invoke the "irreversibility effect" to explain that the more uncertain we are about future returns from an irreversible project, the more the postponement of the project is relevant. This is also known as the intuitive “look before you leap” principle: rather than taking a decision immediately, it is better to wait for more information and then act according to the information received. The “look before you leap” principle has received much attention due to the ability to implement it to irreversible investment choices thanks to the real option theory synthesized in the now well known articles and textbook of Pindyck [1991], Dixit [1992], and Dixit and Pindyck [1994]. Although the real option theory is originally build on a comparison with financial options (see, for instance, the pioneering works of McDonald and Siegel [1986] or Brennan and Schwartz [1985]), its relationship with the concept of options and the irreversibility effect originally developed and clarified by Arrow Fisher Henry and Hanemann (AFHH) has been outlined, among others, by Lund [1991] and Fisher [2000].

Our own lecture suggests that some important differences still remain between the two approaches. As noted by Ulph and Ulph [1997], assumptions about stochastic variables made in the real option theory are not appropriate for a wide range of environmental problems implying both irreversibility and uncertainty. The reason for this is that the real option theory focuses on problems where independent exogenous and repeated chocks affect the evolution of a key variable in the decision of whether to develop or postpone an irreversible project. This approach, thereafter called Markovian approach, is relevant to deal with the uncertainty on future prices but it is not suitable to deal with the uncertainty about an unknown parameter, the true value of which is invariant with time. The latter kind of problems entails the use of Bayesian uncertainty. The matter is that, in its primary version, the concept of option introduced by Arrow and Fisher [1974] is sufficiently large to be thought of as a problem with either Markovian uncertainty or Bayesian uncertainty while the reference to a learning process and Bayesian uncertainty is made more explicit in the subsequent literature. Fisher and Hanemann [1985] and Hanemann [1987], for instance, explicitly refer to Bayesian uncertainty when focusing on uncertainty about biological and engineering parameters or uncertainty as to whether the offshore structures contain oil in commercial quantities.

Furthermore, the “look before you leap” principle often need to be balanced with the precautionary principle which emerged in the mid 1980’s as a clause in international treaties such as the Conference of Rio on Environment and Development. For instance, Article 15 of the Rio declaration states that “where there are threats of serious and irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost effective measures to prevent environmental degradation”. Similar definitions have been enacted in other national or international laws (see Gollier [2001]). The question of whether the precautionary principle is compatible with the irreversibility effect then arises. Ulph and Ulph [1997] refuted the initial idea of Chilchilnisky and Heal [1993] that the precautionary principle could be systematically assimilated with the irreversibility effect. In the more specific context of greenhouse gas accumulation due to economic activity, Gollier Jullien and Treich [2000] stressed the role of the attitude toward risk to justify the precautionary principle while Kolstad [1996a and b] outlined that two contradictory irreversibility effects should be considered since "one may wish to under-emit today to avoid potential environmental irreversibilities" on the one hand and "one may wish to under-invest in pollution control capital, avoiding investments in sunk capital that turn out to be wasted" on the other hand.

In this framework, the present paper is aimed at providing the reader with a coherent and explicit Bayesian approach of the real option theory and to contribute to the debate on the "look before you leap" principle versus the precautionary principle. For this purpose, a suitable Bayesian process is defined in section 2 in order to model how knowledge about unknown parameters evolves in response to an arrival of new information. This is first done in discrete time and then extended to continuous time. Section 3 details how to use such a Bayesian stochastic process together with the concept and mathematical tools of the real option theory to define an optimal decision rule for irreversible projects in the face of uncertainty about an unknown but fixed parameter. Two cases are examined: the first one deals with passive learning, where information is costless; the second one with active learning, where the receipt of new information requires a costly action. More specifically, the case of active learning allows to determine whether the precautionary principle or the "look before you leap" principle should prevail in the context of the simple Bayesian real option model proposed in the paper, without any reference to the attitude toward risk but at the cost of a crucial dependence on initial beliefs.

2. MODELING THE EVOLUTION OF KNOWLEDGE

Intuition suggests that in a world where information comes as time goes, it may be of interest to delay some project in return for less uncertainty. The real option theory of investment under uncertainty, we should rather say under risk, widely confirms this intuition in the context of a world submitted to repeated, independent and exogenous chocks that follow some known distribution of probability. Unfortunately, it seems somewhat abusive to directly implement the models developed by the real option theory in the context of a lack of information about the true value of some key parameters in the decision problem. The matter is thus to construct an adequate mathematical representation of the learning process when facing uncertainty about an unknown but invariant parameter. With this aim in view, we first discuss how important it is to distinguish between the concepts of Markovian and Bayesian processes, the latter being more relevant in some fields of economics, environmental and resource economics in particular. We then turn to the construction of a stochastic process suitable both for depicting Bayesian learning and implementing the real option theory. This is firstly made in a discrete time context and it is secondly extended to the more convenient continuous time context.

2.1. General considerations

Since the seminal article of Arrow and Fisher [1974], the prospect of an arrival of new information is the core of the analysis of irreversible economic decisions. New information takes either the form of the observation of exogenous random chocks (described as Markov processes) affecting the evolution of some key variables or the form of messages used to revise the beliefs of the economic decider. The real option theory typically focuses on the first representation of information while the work of AFHH and, more especially, the subsequent literature is merely influenced by the second representation. This second representation constitutes the basis for modeling the evolution of knowledge about the effective value of an unknown parameter affecting the decision of whether to develop an irreversible project or not. It is generally assumed that there is a finite set of possible values of the unknown parameter and that there exists an *a priori*, or subjective, discrete probability distribution on this set representing the beliefs on the true value of the unknown parameter. Messages arriving with time are used to revise the probability distribution. Some basic or intuitive properties of the revision process are expected.

First of all, uncertainty is reduced in the long term. This means that the probability for one of the possible values tends to increase with time while the probability assigned to other values tend to decrease. Some authors rule out the eventuality of contradictory messages, which allows to represent the evolution of knowledge as an information structure which becomes finest as time goes[1]. As a result, the evolution of probabilities assigned to the different possible values of the unknown parameter is monotonous. If the eventuality of contradictory or noisy messages arriving at different dates is not ruled out, the evolution of probabilities is not monotonous: it may be the case that a probability increases between two dates and then decreases between the last date of these two dates and the following date. Kolstad [1996a] for instance takes account of such noisy messages. Therefore, the evolution of probabilities may be viewed as a stochastic process with an attractor corresponding to a vector made of zeros except one of its components equal to unity. Another important property of the evolution of beliefs is that the revision of probabilities may be consistent with the probability theory, more especially with Bayes' theorem. This point has not always received attention in the literature dealing with irreversible decision when facing uncertainty. The reason for this is that, following Arrow and Fisher [1974], these works are often

[1] This is typically the case in Freixas and Laffont [1984] or Kolstad [1996b] who merely follow the primary model of Henry [1974].

interested in a two period problem with a perfect acquisition of information at the second period. A noticeable exception is the article by Kelly and Kolstadt [1999]. It seems that the description of the evolution of knowledge in this article is too complex to construct a decision rule similar to that obtained with the real option theory in the case of Markov' processes. Nonetheless, it has set the intuition for our model.

2.2. The discrete time approach

We consider a decision problem where the unknown parameter V, the discounted sum of expected net benefits generated by an irreversible project for instance, may have two possible values, $V^{\sup}$ and $V^{\inf}$ with $V^{\sup} > V^{\inf}$. The restriction to two possible scenarii, the optimistic scenario corresponding to $V^{\sup}$ and the pessimistic scenario associated with $V^{\inf}$, is intended to simplify the representation of how beliefs change with time. At time t indeed, the beliefs on the true value of V are described by the probabilities X_t and $1-X_t$ assigned respectively to $V^{\sup}$ and $V^{\inf}$. The study of how beliefs evolve thus only requires to describe the evolution of a single variable, X_t. Similarly, it is assumed that only two kinds of messages, denoted by M^a and M^b, are received between two consecutive dates. The conditional probabilities of these two messages are given by the following matrix:

$$\begin{pmatrix} \Pr\left[M^a/V^{\sup}\right] & \Pr\left[M^a/V^{\inf}\right] \\ \Pr\left[M^b/V^{\sup}\right] & \Pr\left[M^b/V^{\inf}\right] \end{pmatrix} \equiv \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \tag{1}$$

where $\Pr\left[M^i/V^j\right]$ denotes the probability of receiving a message M^i ($i = a$ or b) when the true value of V is V^j ($j =$ sup or inf). We assume that, when V is equal to $V^{\sup}$, the likelihood of a message M^a is higher than the likelihood of a message M^b and conversely when V is equal to $V^{\inf}$, so that $p > 1/2$. Therefore, the receipt of a message M^a means that a good news is received while messages M^b assimilate to bad news. The symmetry in (1) implies that, when the actual scenario is the pessimistic one, the learning process goes as fast as when the actual scenario is the optimistic one.

The learning process consists in revising the probability X_t as a representation of beliefs according to the arrival of new messages. This is done by using Bayes' theorem, which yields the probabilities $\Pr\left[V^j/M^i\right]$ that the true value of V is equal to V^j ($j =$ sup or inf) when a message M^i ($i = a$ or b) is received between t and the following date $t+\Delta t$:

$$\begin{pmatrix} \Pr\left[V^{\sup}/M^a\right] & \Pr\left[V^{\sup}/M^b\right] \\ \Pr\left[V^{\inf}/M^a\right] & \Pr\left[V^{\inf}/M^b\right] \end{pmatrix} \equiv \begin{pmatrix} \dfrac{p\,X_t}{A} & \dfrac{(1-p)\,X_t}{B} \\ \dfrac{(1-p)(1-X_t)}{A} & \dfrac{p(1-X_t)}{B} \end{pmatrix} \tag{2.a}$$

with

$$A \equiv p\,X_t + (1-p)(1-X_t) \tag{2.b}$$
$$B \equiv (1-p)\,X_t + p(1-X_t) \tag{2.c}$$

The first column of the matrix given in (2) is nothing else that the new probabilities $X_{t+\Delta t}$ and $1-X_{t+\Delta t}$ that characterize the beliefs at $t+\Delta t$ if a message M^a is received between t and $t+\Delta t$. Conversely, the second column yields the new probabilities $X_{t+\Delta t}$ and $1-X_{t+\Delta t}$ that characterize the beliefs in $t+\Delta t$ if a message M^b is received between t and $t+\Delta t$. One easily checks that the difference $X_{t+\Delta t}-X_t$ is positive and increasing with p if a message M^a is received between t and $t+\Delta t$ while it is negative and decreasing with p if a message M^b is received between t and $t+\Delta t$: the higher p, the more informative the messages are.

We are interested in obtaining a more convenient way to represent the evolution of beliefs than the description given by matrix (2). With this aim in view, we now turn to the examination of how the ratio between X and $1-X$ changes. A careful examination of (2) shows that

$$\frac{X_{t+\Delta t}}{1-X_{t+\Delta t}} = \begin{cases} \dfrac{p}{1-p}\dfrac{X_t}{1-X_t} & \text{if} \quad M^a \quad \text{is received} \\ \dfrac{1-p}{p}\dfrac{X_t}{1-X_t} & \text{if} \quad M^b \quad \text{is received} \end{cases} \tag{3}$$

Then, it is worthwhile examining the evolution of the natural logarithm Y_t of the ratio between the probabilities assigned to the two possible values of V:

$$\Delta Y = \ln\left(\frac{X_{t+\Delta t}}{1-X_{t+\Delta t}}\right) - \ln\left(\frac{X_t}{1-X_t}\right) \tag{4}$$

We already known that, if the true value of V is $V^{\sup}$, then M^a and M^b are respectively received with probabilities p and $1-p$. Conversely, if the true value of V is $V^{\inf}$, then M^a and M^b are respectively received with probabilities $1-p$ and p. As a result, we obtain Proposition 1.

<u>Proposition 1:</u> the variation, on a time interval of length Δt separating two consecutive dates, of the natural logarithm Y_t of the ratio between the probabilities X_t and $1-X_t$ is the random variable

$$\Delta Y = \begin{cases} \ln\left(\dfrac{p}{1-p}\right) \text{ with probability } p \\ -\ln\left(\dfrac{p}{1-p}\right) \text{ with probability } 1-p \end{cases} \qquad \text{when } V = V^{\sup}$$

or

$$\Delta Y = \begin{cases} \ln\left(\dfrac{p}{1-p}\right) \text{ with probability } 1-p \\ -\ln\left(\dfrac{p}{1-p}\right) \text{ with probability } p \end{cases} \qquad \text{when } V = V^{\inf}$$

According to Proposition 1, Y_t is a stochastic process which satisfies all of Markov's properties. A careful examination of the distribution for future values of Y_t yields the results expressed in Proposition 2:

<u>Proposition 2:</u> since the successive steps ΔY are independent, the cumulated change $Y_T - Y_0$ between dates $t=0$ and $t=T$ separated by $N = T/\Delta t$ time intervals of identical length Δt is a binomial variable with mean

$$E[Y_T - Y_0] = \begin{cases} \dfrac{T}{\Delta t}\ln\left(\dfrac{p}{1-p}\right)(2p-1) \text{ when } V = V^{\sup} \\ -\dfrac{T}{\Delta t}\ln\left(\dfrac{p}{1-p}\right)(2p-1) \text{ when } V = V^{\inf} \end{cases}$$

and variance

$$Var[Y_T - Y_0] = \frac{T}{\Delta t}\left(\ln\frac{p}{1-p}\right)^2 4p(1-p) \text{ whatever the value of } V$$

The discrete time stochastic process described in the two previous proposition has some intuitive and important features. For instance, from $p > 1/2$ it clearly appears that the stochastic process Y_t has a positive drift if the true value of V is $V^{\sup}$ and a negative drift if the true value of V is

$V^{\inf}$. However, rather than to go further on the study of the discrete time approach, we now examine the continuous time approach as a limit case, for infinitesimal time intervals Δt, of the discrete time approach.

2.3. The continuous time approach

Continuous time stochastic processes can be derived as the continuous limit of discrete-time processes. Dixit [1993], for instance, proposes an explanation of how to derive the Wiener process as the continuous limit of a random walk. Although quite similar, our problem is more complicated. Indeed, the discrete time processes defined in Propositions 1 and 2 differ from a standard random walk because of the magnitude and the probability of upward and downward moves that are linked to each other via the probability p. The key point is thus to express p as a function of the time interval Δt so that letting Δt approach zero yields a known continuous time process, namely a Ito process. In order to obtain such a result, we consider the following expression of p:

$$p = \frac{e^{\sigma\sqrt{\Delta t}}}{1+e^{\sigma\sqrt{\Delta t}}} \tag{5}$$

By construction, expression (5) lies in the interval $[0\ ,1]$, which is consistent with the fact that p is a probability. Behind expression (5), there is the intuition that if we let the length Δt of the time interval separating two consecutive dates decreases, more messages will be received but these messages should be less informative in order not to affect the speed at which knowledge evolves. With this intuition in mind note that, *ceteris paribus*, p as expressed in (5) tends to $1/2$ as $\sqrt{\Delta t}$ goes to zero so that the probabilities of bad news and good news become closer. Moreover, since this expression of p is increasing with respect to σ, this parameter may be thought of as indicating the degree of "informativeness" of messages.

We now assume that the length Δt of time intervals approaches zero, or equivalently that the number $N = T/\Delta t$ of time intervals of identical length Δt separating $t = 0$ and any finite date $t = T$ goes to infinity. The binomial distribution of $Y_T - Y_0$ described in Proposition 2 then converges to a normal distribution with mean

$$\lim_{\Delta t \to 0} E[Y_T - Y_0] = \begin{cases} T\dfrac{\sigma^2}{2} & \text{when } V = V^{\sup} \\ -T\dfrac{\sigma^2}{2} & \text{when } V = V^{\inf} \end{cases} \tag{6}$$

and variance

$$\lim_{\Delta t \to 0} Var\left[Y_T - Y_0\right] = T\,\sigma^2 \text{ whatever the value of } V \qquad (7)$$

Moreover, according to Proposition 1, the variation rate $\Delta Y / \Delta t$ of Y_t on each time interval of length Δt goes to $\pm\infty$ as Δt approaches zero, so that Y_t is not differentiable and dY/dt does not exist. Thus, as Proposition 3 states, the continuous limit of Y_t satisfies all the properties characterizing a Brownian motion[2].

> <u>PROPOSITION 3:</u> *the continuous limit of the discrete time process defined in proposition 1 is an Ito process, more precisely a Brownian motion, the evolution of which is described by the following differential equation:*
>
> $$dY = \begin{cases} \dfrac{\sigma^2}{2}\,dt + \sigma\, d\omega \text{ when } V = V^{\sup} \\ -\dfrac{\sigma^2}{2}\,dt + \sigma\, d\omega \text{ when } V = V^{\inf} \end{cases}$$
>
> *where* $d\omega$ *is the increment of a Wiener process,* $\mathrm{E}\left[dY\right]/dt = \sigma^2/2$ *is the instantaneous drift rate and* $V\left[dY\right]/dt = \sigma^2$ *is the instantaneous variance rate.*

By Ito's lemma, we directly derive Proposition 4 from Proposition 3:

> <u>PROPOSITION 4:</u> *the probability assigned to the optimistic scenario,* X_t, *follows a Ito process, the evolution of which is defined by the differential equation*
>
> $$dX = \begin{cases} \sigma^2 X\left(1 - X\right)^2 dt + \sigma\, X\left(1 - X\right) d\omega \text{ when } V = V^{\sup} \\ -\sigma^2 X^2\left(1 - X\right) dt + \sigma\, X\left(1 - X\right) d\omega \text{ when } V = V^{\inf} \end{cases}$$

In spite of its apparent complexity, the Ito process X_t defined in Proposition 4 has some interesting features. Among those, the most important one is undoubtedly that the process admits two absorbing point at $X = 0$ and $X = 1$. Thus, starting with any initial value X_0 in $\left[0\,,1\right]$, X_t will never go outside this interval, which is consistent with the fact that the process X_t describes the evolution of a probability. As shown in Figure 1, the instantaneous variance rate $V\left[dX\right]/dt = \sigma^2 X^2\left(1 - X\right)^2$ admits a maximum at $X = 1/2$, is null for the two absorbing

[2] For an introduction to stochastic processes and stochastic calculus see, among others, Harrison [1985] or Karatzas ans Shreve [1996].

points $X = 0$ and $X = 1$ and has symmetric values for X and $1 - X$. Furthermore, when the true value of V is $V^{\sup}$, the instantaneous drift rate $\mathrm{E}[dX]/dt = \sigma^2 X (1-X)^2 dt$ is positive for any $X \in \,]0,1[$ while it takes the negative value $\mathrm{E}[dX]/dt = -\sigma^2 X^2 (1-X) dt$ for any $X \in \,]0,1[$ when the true value of V is $V^{\inf}$. We conclude that $X = 1$ is an attractor when $V = V^{\sup}$, in the sense that X_t tends to approach this value with a decreasing variance as time goes. Conversely, $X = 0$ is an attractor when $V = V^{\inf}$. Consequently, the continuous version of the process X_t seems to be adequate to describe how the arrival of information improves the knowledge on the true value of the V.

Insert Figure 1

The use of the process X_t in the theory of decision, more especially in optimal stopping time problems, is simplified by a particular feature of this stochastic process. To see this, imagine the following optimal problem. Consider an irreversible project that generates a payoff $G(X)$ if realized while a flow cost $C(X)$ is incurred as long as the project is not realized. The discount rate is r. The problem is to determine the optimal trade off between delaying the realization of the project to benefit from an expected better knowledge in spite of the flow cost $C(X)$ or realizing immediately the project. In formal terms, we have to solve the program

$$F(X_0) = \underset{\tau}{Max}\, \mathrm{E}_0 \left[\int_0^{\tau} -C(X_t) e^{-rt} dt + e^{-r\tau} G(X_\tau) \right] \qquad (8.a)$$

where E_0 stands for mathematical expectation conditional on the initial value X_0 of X_t at $t = 0$. The first stopping time is

$$\tau = Inf\{t \geq 0\,;\, X_t \notin \Omega\} \qquad (8.b)$$

where

$$\Omega = \{X \in [0,1]\,;\, F(X) > G(X)\} \qquad (8.c)$$

is the continuation or waiting region. Program (8) is typically solved by dynamic programming[3]. The matter is that, as stressed by Proposition 4, in Program (8) we have to take account of the fact that the evolution of X_t is described by two different stochastic processes according to the true

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[3] Actually, program (8) belongs to the class of optimal stopping time problems. Shiryayev [1978] constitutes a good and complete presentation of this specific class of dynamic programming problems. Krylov [1980] also contains a shorter introduction. A survey of applications in resources economics is made by Clarke and Reed [1990].

value of V. It is shown in appendix A that, in the interior of Ω, the value function $F(X)$ satisfies the following Bellman equation:

$$\frac{\sigma^2}{2} X^2 (1-X)^2 F_{XX} - r\, F(X) - C(X) = 0 \quad \forall X \in \Omega \qquad (9)$$

The main point to be outlined is that (9) may be directly obtained by assuming that X_t follows an unique stochastic process whatever the true value of V, the evolution of which is described by a differential equation obtained by deleting the deterministic components of dX in Proposition 4. We thus conclude this section with Proposition 5:

> <u>PROPOSITION 5:</u> *we can work "as if" the evolution of the probability X assigned to the value $V^{\sup}$ of V was described by the differential equation*
>
> $$dX = \sigma\, X(1-X)\, d\omega$$
>
> *whatever the true value of V is.*

Proposition 5 means that the modeling of the evolution of knowledge when uncertainty concerns a parameter with two possible values can be summed up as the use of the simple purely stochastic process (there is no deterministic component) described in the proposition. This process admit $X = 0$ and $X = 1$ as absorbing points so that, starting within $[0, 1]$, the process will never quit this interval. The instantaneous variance rate is identical to that shown in Figure 1. Proposition 5 proves to be useful when studying irreversible decisions under uncertainty with learning.

3. ACTING, LEARNING OR ABANDONING?

With the mathematical preliminaries behind us, we can now turn to the analysis of irreversible decisions with the prospect of learning more about some uncertain parameters. In a first stage, we examine how to adapt the canonical model of real option as presented in Dixit and Pindyck [1994], and derived itself from the model originally developed by McDonald and Siegel [1986]. For this purpose, it is assumed that learning is passive in the sense that no specific and costly decision has to be made in order to acquire more information. In a second stage, the more realistic case of active learning where it is costly to acquire more information is considered. The trade off between the "look before you leap" principle and the precautionary principle is more specifically discussed in this last case.

3.1. The case of passive learning

We first consider the problem of whether to postpone the realization of an irreversible project in return for a better knowledge about one of the key parameters involved in the decision problem or to immediately concretize the project at the risk that it appears *ex post* that the true value of the initially unknown parameter does not justify the realization of the project. Uncertainty is supposed to affect the expected sum of net benefits resulting from the realization of the project which amounts to $V^{\sup} > 0$ if the optimistic scenario is the correct one, or $V^{\inf} < 0$ if the pessimistic scenario is the correct one. Although it is not restricted to the field of environmental and resource economics, this kind of problem is quite similar to the standard problem of environmental preservation examined by Arrow and Fisher [1974]: the project considered may be the construction of a dam in an area, say a nice valley in mountains, with entertainment as an alternative and incompatible use. Irreversibility is then due to the fact that, once the dam is built, it is impossible to restore the original wilderness of the area and the resulting amenities for entertainment are definitely lost. Uncertainty typically concerns the true money value of these amenities rather than the gain from the production of electricity. V is thus the difference between the gain from the production of electricity and the loss of amenities from wilderness.

The modeling of how beliefs on which one of the pessimistic or the optimistic scenarii is the correct one change as time goes, follows the same lines as in the previous section. To make things simple, economic agents are assumed to be risk neutral. Accordingly, the net payoff from realizing at time t the project is valued at its expected value:

$$\begin{aligned} G(X_t) &= X_t V^{\sup} + (1 - X_t) V^{\inf} \\ &= X_t \left(V^{\sup} - V^{\inf}\right) + V^{\inf} \end{aligned} \tag{10}$$

where X_t still denotes the probability assigned to the optimistic scenario at time t. The optimal trade off between postponing or immediately realizing the project corresponds to the solution of the following optimal stopping problem:

$$F(X_0) = \underset{\tau}{Max}\, \mathrm{E}_0\left[e^{-r\tau} G(X_\tau)\right] \tag{11}$$

where r is the instantaneous discount rate and E_0 stands for mathematical expectation conditional on the initial value X_0 of the probability assigned to the optimistic scenario at time $t = 0$. The maximization in (11) is subject to the equations given in Proposition 4, or equivalently in Proposition 5, for the evolution of X_t. The optimal stopping time τ and the associated waiting region Ω are identical to those defined in (8.b) and (8.c).

Program (11) is quite similar to one of the first and simplest real option problems, that of McDonald and Siegel [1986]. It clearly appears in (10) that the negative net payoff $V^{\inf}$ received if the project is realized when the pessimistic scenario is correct acts as the sunk investment cost of most real option models of irreversible investment. The presence of the positive coefficient $V^{\sup} - V^{\inf}$ implies that the expected net payoff $G(X)$ is linearly increasing with respect to the probability X assigned to the optimistic scenario, which replaces the project gross value of irreversible investment problems as the state variable. The only difference in program (11) compared with the model of McDonald and Siegel [1986] and most of the models developed in the subsequent literature, like those detailed in Dixit and Pindyck [1994], is that the state variable X_t does not follow a geometric Brownian motion but the new, and unused until the present work to our knowledge, diffusion process described in Propositions 4 and 5.

At any time t, postponement of the project is optimal for, at least, all value of X_t such that the net payoff given in (10) is negative. Consequently, Ω necessarily includes the waiting region associated with the net present value criteria, $\Omega_{VAN} = \{X \in [0\,,1]\,;\, G(X) < 0\}$ that is $\left[0\,, -V^{\inf}/\left(V^{\sup} - V^{\inf}\right)\right]$. Hence, we can guess that Ω takes the form $\left[0\,, X^*\right[$ where $X^* > -V^{\inf}/\left(V^{\sup} - V^{\inf}\right)$ is the optimal but unknown upper boundary of Ω. The value function $F(X)$ and X^* solve the following standard system of equations:

$$\frac{\sigma^2}{2} X^2 (1-X)^2 F_{XX} - r\,F(X) = 0 \quad \forall X \in \left[0\,, X^*\right[\tag{12.a}$$

$$F(0) = 0 \tag{12.b}$$

$$F\left(X^*\right) = X^*\left(V^{\sup} - V^{\inf}\right) - V^{\inf} \tag{12.c}$$

$$F_X\left(X^*\right) = V^{\sup} - V^{\inf} \tag{12.d}$$

Equation (12.a) is identical to the Bellman equation (9) except that there is no flow cost incurred when the project is postponed. Equation (12.b) is a constraint associated with the lower boundary of Ω, it results from the fact that $X = 0$ is an absorbing point of the stochastic process X_t: once the probability assigned to the optimistic scenario reaches the null value, there is no more uncertainty so that we are sure a definitive abandonment of the project is optimal. In the terminology introduced by Dumas [1991], equation (12.c) is the standard value matching condition of optimal stopping problem while equation (12.d) is the smooth pasting condition. Equation (12.a) differs from standard Bellman equations characterizing most of real option problems. Fortunately, there exists an analytical solution to this Bellman equation, the expression of which is

$$F(X) = A_1 \sqrt{X(1-X)} \left(\frac{X}{1-X}\right)^{\beta_1} + A_2 \sqrt{X(1-X)} \left(\frac{X}{1-X}\right)^{\beta_2} \tag{13}$$

where A_1 and A_2 are two constants to be determined according to the boundary conditions associated with the Bellman equation. We see by substitution that (13) satisfies (12.a) provided that β_1 and β_2 are given by

$$\beta_1 = \frac{\sqrt{8\,r + \sigma^2}}{2\,\sigma} > 0 \text{ and } \beta_2 = -\frac{\sqrt{8\,r + \sigma^2}}{2\,\sigma} < 0 \tag{14}$$

Since $\beta_2 < 0$, the boundary condition (12.b) requires that $A_2 = 0$. Then, we obtain A_1 and X^* as solutions of the system formed by the value matching condition (12.c) and the smooth pasting condition (12.d). After some algebraic manipulations, we obtain the value of X^* given in Proposition 6:

<u>PROPOSITION 6:</u> *It is optimal to postpone the project and learn more about the true value of V as long as the probability assigned to the optimistic scenario is lower than the optimal threshold value*

$$X^* = \frac{V^{\inf}\left(1 + 2\,\beta_1\right)}{\left(V^{\sup} - V^{\inf}\right)\left(1 - 2\,\beta_1\right) + 2V^{\inf}}$$

with $0 < X_{NPV} \leq X^* \leq 1$ *if* $V^{\sup} > 0$ *and* $V^{\inf} < 0$ *and where* $X_{NPV} = -V^{\inf} / \left(V^{\sup} - V^{\inf}\right)$ *is the critical probability associated with the net present value criteria. The expected value of the project then amounts to*

$$F(X) = \begin{cases} A_1 \sqrt{X\left(1 - X\right)} \left(\dfrac{X}{1 - X}\right)^{\beta_1} & \text{if } X \leq X^* \\ X \left(V^{\sup} - V^{\inf}\right) + V^{\inf} & \text{otherwise} \end{cases}$$

where

$$A_1 = \frac{V^{\sup} - V^{\inf}}{2\,\beta_1 + 1 - 2\,X^*} \frac{2\sqrt{X^*\left(1 - X^*\right)}}{\left(\dfrac{X^*}{1 - X^*}\right)^{\beta_1}}$$

Figure 2 illustrates Proposition 6 for $V^{\sup} = 20$, $V^{\inf} = -25$, $\sigma = 0.5$ and $r = 0.03$, it outlines the classical analogy with an American financial call option on an asset which is worth $X\left(V^{\sup} - V^{\inf}\right)$, with an infinite expiration date and an exercise price equal to $-V^{\inf}$. The value function is drawn as a continuous line while the terminal payoff $X\left(V^{\sup} - V^{\inf}\right) + V^{\inf}$ corresponds to the dashed line. For these values of the parameters, X^* amounts to 0.882353 and X_{NPV} is equal to 0.555556.

Insert Figure 2

Some results of comparative statics are given in Table 1, they follow on from the fact that β_1 increases with r, decreases with σ and from the limits $\lim_{r \to 0} \beta_1 = 1/2$, $\lim_{r \to \infty} \beta_1 = \infty$, $\lim_{\sigma \to 0} \beta_1 = \infty$ and $\lim_{\sigma \to \infty} \beta_1 = 1/2$. A high instantaneous discount rate r or a low speed of learning σ lessen the interest of learning more since they reduce the gap between the optimal threshold X^* obtained with the real option criteria and the optimal threshold $X_{NPV} = -V^{\inf} / \left(V^{\sup} - V^{\inf}\right)$ characterizing the net present value criteria. Conversely, a low instantaneous discount rate or a high speed of learning strengthen the interest of acquiring more information and, at the extreme, justify to postpone the project until there is no more doubt that the true value of V is $V^{\sup}$, i.e. until X attains and remains at its higher value 1 which is an absorbing point.

Insert Table 1

One of the more striking features of the optimal decision rule detailed in Proposition 6 is that abandonment of the project never succeeds postponement. To say it in an other way, postponement is only aimed at making it sufficiently sure that the optimistic scenario is the correct one to justify the realization of the project. It is not aimed at making it sufficiently sure that the pessimistic scenario is the correct one to definitely abandon it. The reason for this result is that learning induces no particular costs, which means that no specific effort is required to acquire more information. This is designated as passive learning. The case of active learning where information is costly is now examined.

3.2. The case of active learning

So far, postponement of the project was optimal as long as the probability of the optimistic scenario remained below an optimal threshold value. We can guess that the existence of learning costs resulting from an active learning process may invalidate such a decision rule.

Indeed, intuition suggests that the realization of the project should generate a sufficiently high expected net payoff in return for the cost of active learning to justify postponement rather than a definitive abandonment of the project with no learning. To make this idea more concrete, we now introduce a constant learning flow cost c incurred as long as the economic decider is engaged in the learning process. The flow cost c results from the need to do scientific observation and experiences to evaluate the consequences of the destruction of the natural area. It may also results from the need to explore the area in order to learn more about its geological characteristics and the corresponding costs and benefits of building and exploiting the dam if we consider the standard model of environmental preservation introduced by Arrow and Fisher [1974]. The problem is thus now to decide whether to postponed the project and learn more, or to realize the project immediately, or to definitely abandon it. Postponement keeps the options to realize or abandon the project alive while its realization is definitive due to its irreversibility and its abandonment is definitive also due to the absence of any new acquisition of information. The optimal stopping problem to be solved may thus be written as

$$F(X_0) = \underset{\tau}{Max}\, E_0\left[-\int_0^{\tau} c\, e^{-r\,t} + e^{-r\,\tau} G(X_\tau)\right] \tag{15.a}$$

with

$$G(X) = Max\begin{cases} X\left(V^{\sup} - V^{\inf}\right) + V^{\inf} \\ 0 \end{cases} \tag{15.b}$$

and subject to the equations given in Proposition 4, or equivalently in Proposition 5, for the evolution of X_t. The first term in the terminal payoff function (15.b) corresponds to the terminal payoff in case of an immediate realization of the project while the second term corresponds to abandonment of the project. The optimal stopping time and the continuation or waiting region are identical to those defined in (8.b) and (8.c).

We already know that, like in the case of passive learning, the optimal waiting region Ω necessarily includes all the values of X such that abandonment of the project is preferred to immediate realization according to the net present value criteria. Moreover, since it is an absorbing point, if $X = 0$ there is no more expected change in the probability assigned to the optimistic scenario and abandonment is preferred to postponement in order to avoid learning costs. We can guess that for values sufficiently close to $X = 0$, abandonment is preferred to postponement also. Therefore we conclude that postponement is more interesting for intermediate values of X than for values approaching either zero or unity, that abandonment is optimal for low values of X and that immediate realization is preferred for high values of X. We thus have $\Omega = \left]X^{**}, X^{*}\right[$

where $X^{**} \geq 0$ is the lower bound of the waiting region behind which abandonment is optimal and $X_{NPV} \leq X^{*} \leq 1$ is the upper bound of the boundary region above which an immediate realization of the project is optimal. Then, going along the same lines as in the case of passive learning, we find that the value function $F(X)$ and the two optimal thresholds for the probability assigned to the optimistic scenario solve the Bellman equation

$$\frac{\sigma^2}{2} X^2 (1-X)^2 F_{XX} - r\,F(X) - c = 0 \quad \forall X \in \left[X^{**}, X^{*}\right] \tag{16.a}$$

subject to the boundary conditions

$$F\left(X^{**}\right) = 0 \tag{16.b}$$

$$F_X\left(X^{**}\right) = 0 \tag{16.c}$$

$$F\left(X^{*}\right) = X^{*}\left(V^{\sup} - V^{\inf}\right) - V^{\inf} \tag{16.d}$$

$$F_X\left(X^{*}\right) = V^{\sup} - V^{\inf} \tag{16.e}$$

Conditions (16.b) and (16.c) are respectively the value matching and smooth pasting conditions associated with the lower bound X^{**} of the waiting region; they replace the condition (12.b) in the program (12) characterizing the case of passive learning. Conditions (16.d) and (16.e) are identical to the value matching and smooth pasting conditions (12.c) and (12.d) associated to the upper bound X^{*} of the waiting region. The general expression of $F(X)$ is still given by (13), augmented by the term c/r to take account of the existence of learning costs. Substituting this expression in conditions (16.b), (16.c), (16.d) and (16.e) yields a system of four equations to be solved in X^{**}, X^{*}, A_1 and A_2. Unfortunately, we are unable to find an analytical solution and numerical computations are required to solve the problem. Figure 3 illustrates the solution for $V^{\sup} = 20$, $V^{\inf} = -25$, $\sigma = 0.5$, $r = 0.03$, and $c = 0.1$.

Insert Figure 3

The lower bound X^{**} of the optimal waiting region amounts to 0.0796468 and the optimal value 0.856003 of the upper bound X^{*} of the waiting is slightly lower than the optimal threshold value of X above which an immediate realization of the project is optimal in the case of passive learning. In order to get more insights into the comparative statics of the model, we proceed with a sensitivity analysis illustrated by Figures 4.a and 4.b. It clearly appears that the upper bound X^{*} reacts to changes in the value of the degree of "informativeness" σ of messages and in the value of the discount rate r in the same direction as the optimal threshold value of the probability in the case of passive learning does. Conversely, the way the lower bound

X^{**} reacts to changes in the values of the same parameters is just opposite. Therefore, the optimal waiting region tends to be wider as the discount rate r decreases or as σ increases and, thus, as messages are more informative. Since the waiting region is the set of values of the probability assigned to the optimistic scenario such that postponement is preferred to both an immediate realization of the project or a definitive abandonment, it typically indicates the state of knowledge for which the "look before you leap" principle prevails. Above the upper bound of the waiting region, there are sufficiently strong beliefs that the optimistic scenario is the correct one to justify an immediate realization of the project. Conversely, below the lower bound of the waiting region, there are sufficiently weak beliefs that the pessimistic scenario is the true one not only to abandon the project, but the learning process too. Otherwise stated, below the lower bound of the waiting region, the precautionary principle prevails.

Insert Figures 4.a and 4.b

Note that, for values of σ around 0.2, the lower bound X^{**} of the waiting region approaches relatively high values (about 0.35) and is highly sensitive to changes in the degree of "informativeness" of messages. This result confirms the intuition that the conjunction of a costly learning process and noisy messages weakens the interest of postponing the project to learn more about the unknown parameter. It justifies the application of the precautionary principal in the sense that the prospect of acquiring more information should not serve as an argument to postpone the irreversible project rather than to abandon it and preserve the environment, if learning costs are sufficiently high and if we are sufficiently sure that the real scenario is not the optimistic one but the pessimistic one (that is, if the probability X assigned to the optimistic scenario to represent the beliefs on the true value of V is lower than X^{**}). In the context of the simple Bayesian real option model developed in this paper, it thus appears that the precautionary principle may follow on from the existence of learning costs and crucially depends on initial beliefs. The strong dependence on initial beliefs may highlight why different countries adopt different decisions while facing an apparently identical irreversible project. For instance, former negative experiences such as the so-called "mad cow" disease may explain that the authorities in charge of the health and sanitary policy in the European Community are more conservative than their American homologues when considering the introduction of Genetically Modified Organisms in agricultural practices.

4. CONCLUSION

The simple Bayesian real option model developed in this paper constitutes a first step in the attempt to unify the real option theory and the Bayesian approach of decision problems involving both irreversibility and uncertainty. A reexamination of some former work dealing with such decision problems outlines the need for an unified approach. For instance, in their application of option valuation to the case of offshore petroleum leases, Paddock Siegel and Smith [1988] pointed that "the primary uncertainty surrounding the exploration stage is the quantity of hydrocarbons". However, they assumed that the exploration costs are sunk costs incurred at the date of exploration and thus ignored the importance of the time to learn. The article by Pindyck [2000] may also be viewed as an example of a real option model dealing with a problem involving Bayesian uncertainty but avoiding to do so. Indeed, the author is interested in uncertainty about future damages caused by anthropogenic Green House Gases accumulating in the atmosphere but prefer to consider that this uncertainty is due to changes in tastes or technology whereas Kolstad [1996a] or Kelly and Kolstad [1999], among others, stressed the role of scientific uncertainty about the effective value of parameters linking the concentration of Green House Gases and money valued damages. Applications of the simple model proposed here to environmental and resource economics are numerous and include all problems where economic deciders are uncertain about the existence and magnitude of externalities or about the quality and quantity of a natural resource. Moreover, the model seems to be adequate to analyze the role of learning costs, noisy messages and primary beliefs to justify (a version of) the precautionary principle and discuss the conditions that make it more relevant than the traditional "look before you leap" principle supported by the real option theory.

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APPENDIX A: DETERMINATION OF THE BELLMAN EQUATION

Consider a value of X_0 in the interior of Ω. Then, there always exists a time interval Δt such that the probability $X_{\Delta t}$ lies in the interior of Ω also. Thus, we can write

$$F(X_0) = \mathrm{E}_0\left[\int_0^{\Delta t} - C(X_t)e^{-r\,t}\,dt + e^{-r\,\Delta t}F(X_{\Delta t})\right] \tag{A.1}$$

Following Kamien and Schwartz [1991] or Dixit [1993], by the mean value theorem and the linear approximation of the exponential function in the neighborhood of zero, (A.1) also reads

$$F(X_0) = -C(X_0)\Delta t + \mathrm{E}_0[F(X_{\Delta t})]/(1 + r\,\Delta t) \tag{A.2}$$

The numerator in the last term of (A.2) is

$$\mathrm{E}_0[F(X_{\Delta t})] = X_0\,\mathrm{E}_0^{\sup}[F(X_{\Delta t})] + (1 - X_0)\mathrm{E}_0^{\inf}[F(X_{\Delta t})] \tag{A.3}$$

where $\mathrm{E}_0^{\sup}$ (respectively $\mathrm{E}_0^{\inf}$) denotes mathematical expectation conditional on the initial value X_0 of X and on the fact that the true value of V is $V^{\sup}$ (respectively $V^{\inf}$). After some rearrangements, (A.2) then becomes

$$0 = X_0\frac{\mathrm{E}_0^{\sup}[F(X_{\Delta t}) - F(X_0)]}{\Delta t} + (1 - X_0)\frac{\mathrm{E}_0^{\inf}[F(X_{\Delta t}) - F(X_0)]}{\Delta t} - C(X_0) - r\,F(X_0) \tag{A.4}$$

Ultimately, we are interested in the limit when Δt goes to zero. Then, Ito's lemma gives the two expected terms in (A.4), the Dynkins of $F(X)$ for the two processes described in Proposition 4:

$$\lim_{\Delta t \to 0}\frac{\mathrm{E}_0^{\sup}[F(X_{\Delta t}) - F(X_0)]}{\Delta t} = \frac{\sigma^2}{2}X_0^2(1 - X_0)^2 F_{XX} + \sigma^2 X_0(1 - X_0)^2 F_X \tag{A.5.a}$$

$$\lim_{\Delta t \to 0}\frac{\mathrm{E}_0^{\inf}[F(X_{\Delta t}) - F(X_0)]}{\Delta t} = \frac{\sigma^2}{2}X_0^2(1 - X_0)^2 F_{XX} - \sigma^2 X_0^2(1 - X_0)F_X \tag{A.5.b}$$

where F_X and F_{XX} respectively stand for the first and the second derivatives of F. Substitute in (A.4) and simplify to finally obtain the Bellman equation

$$\frac{\sigma^2}{2}X_0^2(1 - X_0)^2 F_{XX} - r\,F(X_0) - C(X_0) = 0 \quad \forall X_0 \in \Omega \tag{A.6}$$

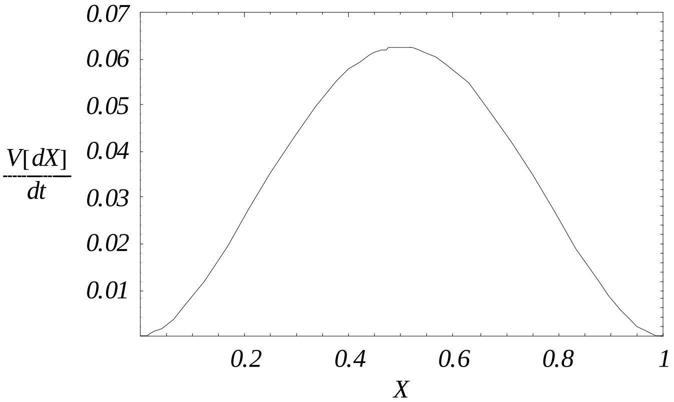
Table 1: comparative statics for the optimal threshold X^* in the context of passive learning

$\partial X^* / \partial r$:	-
$\lim_{r \to 0} X^*$:	1
$\lim_{r \to \infty} X^*$:	X_{NPV}
$\partial X^* / \partial \sigma$:	+
$\lim_{\sigma \to 0} X^*$:	X_{NPV}
$\lim_{\sigma \to \infty} X^*$:	1

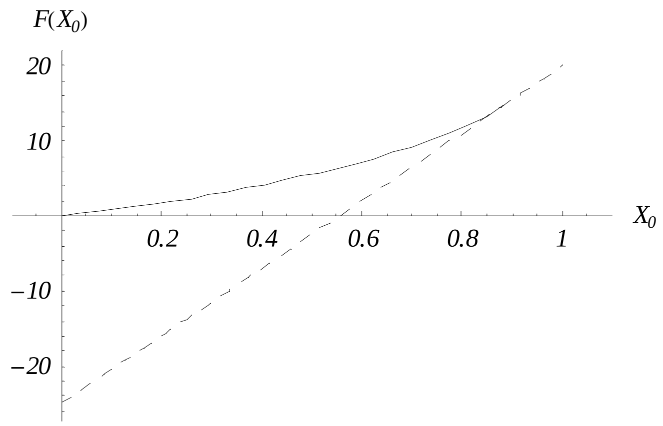
Figure 1:

$V[dX]/dt = \sigma^2 X^2 (1-X)^2$ *as a function of* X

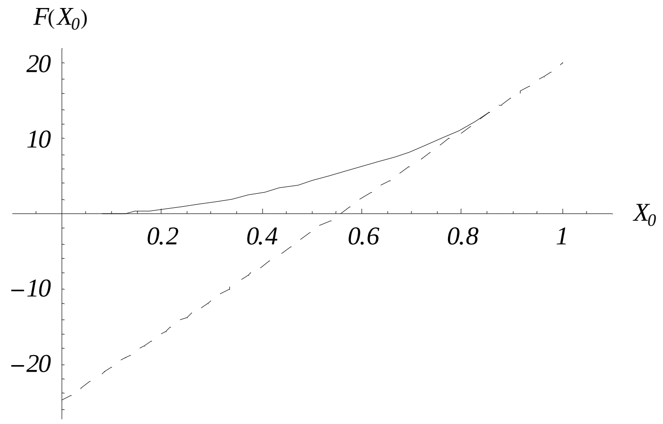
for $\sigma = 1$



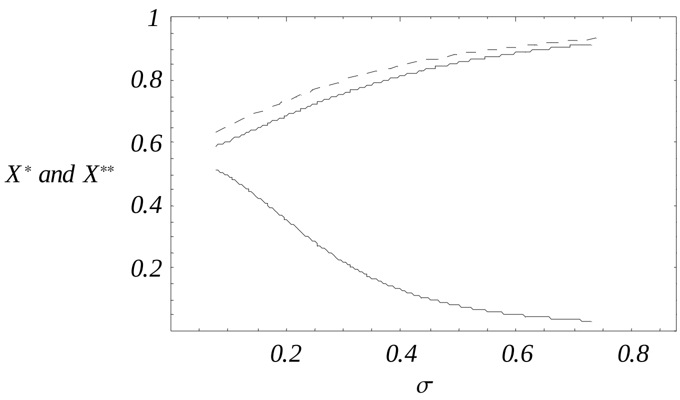
<u>*Figure 2:*</u>
The value of the project as a function of X
In the case of passive learning



<u>*Figure 3:*</u>
The value of the project as a function of X
In the case of active learning



<u>Figure 4.a:</u>
The upper and lower bound of the waiting region as functions of σ
(dashed curve: optimal threshold value of X in the case of passive learning)



<u>Figure 4.b:</u>
The upper and lower bound of the waiting region as functions of r
(dashed curve: optimal threshold value of X in the case of passive learning)