Simulation and Numerical Methods in Real Options Valuation

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Abstract

This paper provides an overview of numerical methods and its applicability for solving real option problems. It discusses alternative approaches and shows that both forward and backward induction procedures have a place in real options valuation.

A case-project with the option of investing in the future contingent on a stochastic output price is valued using binomial trees, finite differences, and simulation. The Black and Scholes (1973) analytical solution to this problem is used as a benchmark. All four methods provide similar results.

The extension of simulation methods to American-Type options is discussed and a solution to Brennan and Schwartz´s (1985) classic mine valuation problem is presented. The benefits of this approach, with its better handling of complex uncertainty modeling and path-dependent cash flows, are discussed.

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1. - Introduction

Traditional finance literature stresses that the value of an asset is determined by future cash flows. As long as these cash flows are certain, the task of determining asset values amounts to finding adequate inter-temporal discount factors to transform these future flows into equivalent present values.

Uncertainty introduces two additional complications into asset valuation. First, we must find a way to penalize the present value of the risky cash flow so that we can take into consideration not only the time value of money but also risk aversion. This may be done by determining risk premiums that should be added to inter-temporal discount factors or, alternatively, by replacing expected cash flows by their certainty equivalents.

A second and somewhat more difficult issue arises when cash flows are a nonlinear function of a risky state variable. Risk premiums become then much more difficult to be derived, and the contingent claims approach of finding certainty equivalents for cash flows at each state of nature becomes the only practical approach.

It has long been noted that simulation lends itself nicely to valuing assets under uncertainty. As long as cash flows may be determined using past information, a nonlinear cash flow function poses no additional burden. An example of *past-dependent* nonlinear cash flows is the *European* call option. The value of this option is nonlinear by being equivalent to the maximum value between two assets, and *past dependent* in the sense that cash flows depend only on past information. This approach, which values a function by unfolding uncertainty as it evolves from the past, is known as *forward induction* and may be successfully applied whenever present cash flows do not depend on future events.

An additional difficulty arises when nonlinear cash flows are dependent on future information. For example, *American* options may be exercised at any of several dates. Thus, cash flows on a given date depend not only on past information but also on expectations of future events. It has long been known that whenever uncertainty can be described by a markovian process (in the sense that all past information may be embedded in current state variable values) the value of a security may be obtained by some kind of backward induction. This procedure works its way into the present starting from some known value, typically at option expiration.

A number of backward induction procedures have been proposed for valuing assets, from dynamic programming, to binomial and multinomial trees, to finite difference procedures for solving partial differential equations. All these procedures start from some boundary conditions and solve simultaneously for asset value and the optimal exercise policy, determining the shape of the cash flow function in such a way as to maximize asset value.

Both forward and backward induction procedures have a role in asset valuation. While forward induction handles in an easier way complex uncertain processes, including path dependent cash flows that may arise because of technical or tax reasons, backward induction is specially appropriate for handling American-type options very common in real option problems. Until recently simulation procedures were only recommended for European-type options in forward induction implementations. In section 3 we present the basic intuition of how to use simulation for American-type options and provide an actual application to a well-known real option problem.

2. - Standard Simulation and Other Numerical Approaches

2.1 European Real Option Valuation by Numerical Procedures

Some contingent claims problems may be solved through closed-form analytical expressions, but most can not. If there is no analytical solution numerical methods must be used. To illustrate our discussion on alternative numerical approaches and compare them to a standard simulation, we define a simple real-option valuation problem and solve it using different methods.

Let's assume we want to value a project with cash flows contingent on a stochastic output price, S. The project requires no initial investment at time $T_0 = 0$ but requires an investment I_1 at time T_1 which, if undertaken, generates cash flows with a present value at T_1 of V(S). Depending on the value of V(S) the manager may decide not to invest at T_1, in which case the project must be abandoned at zero liquidation value. The value of this project may be modeled as a call option written on V(S) with exercise price I_1 and time to maturity T_1.

We may be able to solve for the value of this project using the well-known Black and Scholes (1973) formula for a call option, provided some assumptions are made. Among them are that S is a tradable asset, that markets are sufficiently complete as to allow the hedging of output price risk and that output price returns follow a Brownian motion with constant interest rates and volatility. If these assumptions apply then there is no need to resort to numerical procedures.

Most real option models, however, do not have closed-form analytical solutions. There are many reasons why this may be the case, including a more complex uncertainty model or project-flexibility specification. In these cases, to value the contingent claim numerical solution procedures must be used.

There are many numerical methods that may be used to value a European-type contingent claim. In order to place standard Monte Carlo simulation procedures into perspective and to discuss its comparative strengths and weaknesses, we solve the above problem using three alternative numerical procedures: binomial trees, finite differences, and simulation. We assume Black and Scholes (1973) holds so we have an analytical solution that can be used as a benchmark for our approximate numerical methods.

For concreteness we assume risk-adjusted prices follow a geometric brownian motion with a discretized specification as follows:

$$\Delta S = rS\Delta t + \sigma S\sqrt{\Delta t}Z \tag{1}$$

with

$r = 0.10$
$\sigma = 0.2$
Δt = time interval
Z = random variable with a standardized Normal distribution.

Also the investment project described earlier has the following specification:

$V(S) = aS + b$
$a = 10$
$b = 0$
$I_1 = 10$
$T_0 = 0$
$T_1 = 1$

As stated earlier, the value of this real option can be directly obtained using Black and Scholes (1973), which gives the value of the project for an initial output price of 1 is \$1.30. We now solve this same valuation problem using the three alternative numerical approaches. We implement all three numerical solutions using standard Excel spreadsheets.

2.2 Binomial Trees.

Discrete tree representations of stochastic processes and their use in option valuation have been proposed by Cox et al (1979), Rendleman and Barter (1979), and Sharpe (1978). We explain the binomial tree, which is the simplest one. Binomial trees assume that uncertainty at any state can be represented by two alternative states. These two states are defined such that the implied price distribution matches as closely as possible the probability distribution of the underlying continuous state variable. Given our stochastic process for output price, we must restrict the values and probabilities of the two states in such a way that the expected price return over the next time interval is equal to $r\Delta t$ and that its volatility is $\sigma\Delta t^{0.5}$. A solution is to define the two values for the state variable as Su and Sd, with probabilities p and $(1-p)$, with:

$$u = e^{\sigma\sqrt{\Delta t}} \tag{2}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \qquad d = 1/u \tag{3}$$

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{4}$$

Depending on the required accuracy we determine the size of the time interval Δt, or equivalently, the number of subintervals in which we partition the total time to maturity. In our case the time to maturity is $T_1 = 1$, and we arbitrarily divide it into 10 subintervals, setting $\Delta t = 0.1$. Once the binomial tree that represents the underlying price distribution is obtained, it is possible to value the derivative asset (in our case the investment project) as a contingent claim on output price.

In Figure 2.1 a binomial tree solution to our investment valuation problem is presented. A two-dimensional vector is specified at each binomial node: output price and option value. To obtain option values, we compute the cash flows at maturity (T_1) as:

$Max(V(S) - I_1; 0)$.

These values are presented bolded at the extreme right of the figure.

The next step is to compute the option value for each of the preceding nodes. As an example, on the top-right corner of Figure 2.1 a box with three nodes is presented. The option value of **7.77** is computed as the expected option value at the two following nodes using the risk neutral probabilities:

$$OptionValue = \left[p * 8.82 + (1 - p)6.59\right]e^{r\Delta t} = 7.77$$

This procedure is repeated column by column from right to left. The last node represents the current option value of **1.31** for an initial output price of **1.** This option is very similar to our analytical solution value using Black and Scholes.

Time to Maturity										
1.00	**0.90**	**0.80**	**0.70**	**0.60**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.00**
										1.88
										8.82
	Information in each node:									
									1.77	
			Output Price S						**7.77**	
			Real Option Value							
								1.66		1.66
								6.78		**6.59**
							1.56		1.56	
							5.86		5.67	
						1.46		1.46		1.46
						5.01		4.81		**4.62**
					1.37		1.37		1.37	
					4.21		4.01		3.82	
				1.29		1.29		1.29		1.29
				3.47		3.27		3.08		**2.88**
			1.21		1.21		1.21		1.21	
			2.80		2.59		2.38		2.19	
		1.13		1.13		1.13		1.13		1.13
		2.22		2.01		1.78		1.55		**1.35**
	1.07		1.07		1.07		1.07		1.07	
	1.72		1.51		1.29		1.04		0.75	
1.00		1.00		1.00		1.00		1.00		1.00
1.31		1.11		0.91		0.68		0.42		**0.00**
	0.94		0.94		0.94		0.94		0.94	
	0.80		0.63		0.44		0.23		0.00	
		0.88		0.88		0.88		0.88		0.88
		0.42		0.28		0.13		0.00		**0.00**
			0.83		0.83		0.83		0.83	
			0.17		0.07		0.00		0.00	
				0.78		0.78		0.78		0.78
				0.04		0.00		0.00		**0.00**
					0.73		0.73		0.73	
					0.00		0.00		0.00	
						0.68		0.68		0.68
						0.00		0.00		**0.00**
							0.64		0.64	
							0.00		0.00	
								0.60		0.60
								0.00		**0.00**
									0.57	
									0.00	
										0.53
										0.00

Figure 2.1 Binomial tree solution to a European Real Option Investment

2.3 Finite Differences.

An alternative to binomial trees is to use finite differences for solving the valuation equation. In this case we can use standard no-arbitrage conditions to derive a partial differential equation for the value of the contingent claim. For our real option problem, the standard Black and Scholes differential equation for the value of the real option H(S,t) is:

$$\frac{1}{2}H_{SS}S^2\sigma^2 + rSH_S - rH + H_t = 0 \tag{5}$$

with the following boundary condition at maturity

$$H(S, t = T_1) = Max[V(S)-I_1;0] \tag{6}$$

Also S is an absorption state, thus:

$$H(0,t) = 0 \tag{7}$$

Schwartz (1977) proposed the finite difference procedure of discretizing all state variables, setting the value of the contingent claim at the boundary conditions, replacing first and second derivatives by a finite difference approximation, and solving backwards using a discretized version of the partial differential equation that represents the valuation equation. There are two basic finite difference approaches: the implicit and the explicit method. Even though the former is more robust, we implement the latter for expositional reasons.

In our problem, the value of the project is a function of two state variables: output price, S, and time to maturity, T. Time is discretized into M intervals, and price S into N intervals. The Black and Sholes differential equation is then replaced by the following difference approximations:

$$\Delta S = Smax/N \tag{8}$$

$$\Delta T = T_1/M \tag{9}$$

$$H_S = \frac{H_{i+1,j+1} - H_{i+1,j-1}}{2\Delta S} \tag{10}$$

$$H_{SS} = \frac{H_{i+1,j+1} - 2H_{i+1,j} + H_{i+1,j-1}}{\Delta S^2} \tag{11}$$

$$H_t = \frac{H_{i+1,j} - H_{i,j}}{\Delta t} \tag{12}$$

Once these approximations are substituted into the differential equation, we obtain:

$$a_j H_{i+1,j-1} + b_j H_{i+1,j} + c_j H_{i+1,j+1} = H_{i,j} \tag{13}$$

with

$$a_j = \frac{1}{1 + r\Delta t}\left(-\frac{1}{2} rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right) \tag{14}$$

$$b_j = \frac{1}{1 + r\Delta t}\left(1 - \sigma^2 j^2 \Delta t\right) \tag{15}$$

$$c_j = \frac{1}{1 + r\Delta t}\left(\frac{1}{2} rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right) \tag{16}$$

Thus, knowing the values of the contingent claim at i + 1 we can obtain the values at i. Given that we have a boundary condition that gives initial values at i = M, it is possible to work backwards from i = M to i = 0.

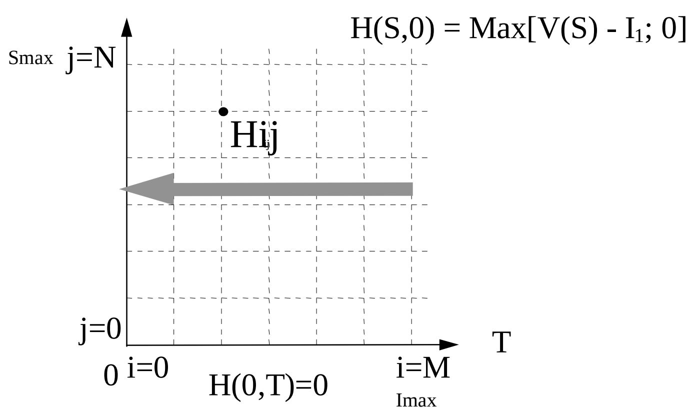

Figure 2.2 State Space Discretization

Figure 2.3 presents an explicit finite difference solution to the investment valuation problem. The three right columns compute the constants necessary for the calculations. The next column to the left values the real option at the boundary:

$$Option\ Value\ at\ Maturity = Max(V(S) - I_1; 0).$$

Using the above equations, the preceding columns are computed. Finally, project value for each initial price, when time to maturity is 1, is presented. It can be noted that for an initial output price of **1**, the computed option value is **1.30**, again very similar to our previous results.

					Real Option Value										
		Time to Maturity(i)													
j	**Price**	**1.0**	**0.9**	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**	**0.0**	**aj**	**bj**	**cj**
20	**2.0**	10	10	10	10	10	10	10	10	10	10	**10**	0.6931	-0.5941	0.8911
19	**1.9**	9.2	9.3	9.2	9.3	9.2	9.2	9.2	9.2	9.1	9.1	**9**	0.6208	-0.4396	0.8089
18	**1.8**	8.6	8.4	8.5	8.4	8.4	8.3	8.3	8.2	8.2	8.1	**8**	0.5525	-0.2931	0.7307
17	**1.7**	7.6	7.7	7.6	7.6	7.5	7.4	7.4	7.3	7.2	7.1	**7**	0.4881	-0.1545	0.6564
16	**1.6**	6.8	6.7	6.7	6.6	6.6	6.5	6.4	6.3	6.2	6.1	**6**	0.4277	-0.0238	0.5861
15	**1.5**	5.9	5.8	5.7	5.7	5.6	5.5	5.4	5.3	5.2	5.1	**5**	0.3713	0.0990	0.5198
14	**1.4**	4.9	4.8	4.8	4.7	4.6	4.5	4.4	4.3	4.2	4.1	**4**	0.3188	0.2139	0.4574
13	**1.3**	4.0	3.9	3.8	3.7	3.6	3.5	3.4	3.3	3.2	3.1	**3**	0.2703	0.3208	0.3990
12	**1.2**	3.01	2.9	2.8	2.7	2.6	2.5	2.4	2.3	2.2	2.1	**2**	0.2257	0.4198	0.3446
11	**1.1**	2.11	2.0	1.9	1.8	1.7	1.6	1.5	1.3	1.2	1.1	**1**	0.1851	0.5109	0.2941
10	**1.0**	**1.30**	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.4	0.2	**0**	0.1485	0.5941	0.2475
9	**0.9**	0.67	0.6	0.5	0.4	0.4	0.3	0.2	0.1	0.1	0.0	**0**	0.1158	0.6693	0.2050
8	**0.8**	0.26	0.2	0.2	0.1	0.1	0.1	0.0	0.0	0.0	0.0	**0**	0.0871	0.7366	0.1663
7	**0.7**	0.07	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	0.0624	0.7960	0.1317
6	**0.6**	0.01	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	0.0416	0.8475	0.1010
5	**0.5**	0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	0.0248	0.8911	0.0743
4	**0.4**	0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	0.0119	0.9267	0.0515
3	**0.3**	0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	0.0030	0.9545	0.0327
2	**0.2**	0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	-0.0020	0.9743	0.0178
1	**0.1**	0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	-0.0030	0.9861	0.0069
0	**0.0**	0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0**	0.0000	0.9901	0.0000

Figure 2.3 (Explicit) Finite difference solution to a European Real Option Investment

2.4 Standard Simulation.

Boyle (1977) proposed a Monte Carlo simulation approach for European option valuation. The method is based on the idea that simulating price trajectories can approximate probability distributions of terminal asset values. Option cash flows are computed for each simulation run and then averaged. The discounted average cash flow using the risk free interest rate represents a point estimator of the option value.

There are several ways to increase estimation accuracy, the simplest one being to increment the number of simulating paths. Efficiency may also be improved by using variance reduction techniques, including the control-variate and antithetic-variate approaches [Hammersley and Handscomb (1964)]. In what follows we solve our real option problem implementing the latter.

Figure 2.4 shows a spreadsheet with simulation runs to value the real option. Each run starts with an initial output price of **1**, and by using a specific set of random numbers, a price trajectory is computed. For example, our first price trajectory ends (at option maturity) with an output price of **0.88**. The next column to the right shows the option payoff for that specific price. Given that for that price the project would be abandoned, option value is zero.

To implement the antithetic variate variance reduction technique, our next row in Figure 2.4 presents the price trajectory using the same random numbers as before, but with a change in sign. Given that our initial random numbers were such that output price at maturity was low (0.88), by changing random number signs this second row provides a price trajectory with high output price at maturity (1.34). For this new output price the option is now valuable, providing a cash flow of **3.44**. It is easy to see that an average of these two rows provides a lower variance estimate of actual cash flows than using any single one.

Once an adequate number of price trajectories is generated, real option value may be computed by discounting average option cash flows at the risk free rate. It can be seen that in our case with a set of only 30 independent price trajectories (60 rows including their antithetic values) we are able to obtain a very close value for our real option problem (1.30). In most cases it is necessary to make a much higher number of simulation runs to obtain accurate estimates.

	Time to Maturity(i)												
Run	**1.0**	**0.9**	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**	**0.0**	**V(S)**	**NPV**
1	1.00	0.98	0.98	0.91	1.00	0.99	0.95	0.97	0.95	0.91	**0.88**	0.00	0.00
	1.00	1.04	1.06	1.16	1.06	1.09	1.16	1.16	1.20	1.28	1.34	3.44	1.56
2	1.00	0.98	0.99	1.11	1.13	1.11	1.23	1.19	1.26	1.27	1.31	3.12	1.98
	1.00	1.04	1.05	0.95	0.94	0.98	0.89	0.94	0.91	0.91	0.90	0.00	1.49
3	1.00	0.98	1.07	1.09	1.06	1.01	1.06	1.00	1.06	1.08	1.25	2.50	1.30
	1.00	1.04	0.97	0.97	1.01	1.08	1.05	1.13	1.09	1.08	0.94	0.00	1.37
4	1.00	0.93	0.89	0.90	0.88	0.75	0.77	0.82	0.90	0.80	0.72	0.00	1.28
	1.00	1.09	1.16	1.17	1.22	1.43	1.41	1.34	1.24	1.40	1.58	5.81	1.68
5	1.00	0.92	0.88	0.72	0.68	0.64	0.60	0.59	0.60	0.67	0.70	0.00	1.30
	1.00	1.10	1.17	1.41	1.52	1.63	1.77	1.84	1.85	1.65	1.60	6.01	1.89
10	1.00	1.00	1.02	0.99	0.96	1.09	0.99	0.94	0.89	0.90	0.88	0.00	1.29
	1.00	1.02	1.02	1.07	1.12	1.00	1.11	1.18	1.27	1.29	1.33	3.33	1.43
15	1.00	1.07	1.08	1.15	1.09	1.14	1.24	1.43	1.45	1.54	1.61	6.10	1.33
	1.00	0.95	0.96	0.92	0.98	0.96	0.89	0.77	0.78	0.75	0.73	0.00	1.44
20	1.00	0.91	0.93	0.93	0.93	0.94	0.92	0.94	1.01	1.13	1.18	1.75	1.32
	1.00	1.11	1.11	1.13	1.16	1.16	1.22	1.21	1.15	1.03	1.01	0.10	1.37
25	1.00	1.10	1.10	1.06	1.13	1.18	1.23	1.24	1.35	1.35	1.35	3.53	1.31
	1.00	0.92	0.94	0.99	0.95	0.92	0.90	0.91	0.85	0.86	0.88	0.00	1.34
30	1.00	1.05	1.02	1.05	1.09	1.08	1.13	1.07	1.04	1.12	1.18	1.82	1.30
	1.00	0.97	1.01	1.01	0.99	1.02	0.99	1.06	1.11	1.05	1.01	0.15	**1.30**

Figure 2.4 Simulation Solution to a European Real Option Investment

2.5 Comparing Alternative Numerical Procedures.

We have presented simple spreadsheet implementations of three alternative numerical approsches to the same real option problem. Each of them has its own merits and is especially useful for specific types of valuation problems.

Maybe the most important factor in choosing the appropriate numerical approach is the type of option we are trying to value. Standard simulation is a forward induction procedure, and as such presents problems for valuing American-type options. In situations when the optimal strategy is not known in advance, standard simulation procedures are not able to correctly value these options. As discussed later, many real options allow decision-makers to change production or investment levels at different points in time, and are therefore modeled as American options. Finite difference and binomial trees, on the other hand, are backward induction procedures that can determine optimal exercise policies, correctly valuing these American options.

What is a weakness for simulation in handling American-type options becomes a strength when there are path-dependent cash flows. For example, current tax payments normally depend on past profits, presenting a difficulty for all backward induction procedures. It is always possible to circumvent this problem by defining new state variables that represent path dependent information, but this may increase model complexity in a substantial way. Therefore, in the presence of path-dependent cash flows, simulation is a much better procedure than backward induction procedures.

The main characteristic that makes simulation so attractive is its ability to cope with uncertainty in a very simple way. The recent trend in modeling price uncertainty using multi-factor models is much easier to implement using standard simulation than using other numerical approaches. Something similar can be said on the use of complex stochastic processes for modeling the dynamics of these risk factors, which are simpler to implement using simulation.

Finally, the fact that the cost of computing has been going down so dramatically in the past years and that this trend shows no sign of weakening in the near future, presents a favorable prospect for increasing use of simulation. Moreover, its major drawback, the inability to successfully handle American-type options, has been tackled by new research in recent years, as described in the following section. With lower computational costs we can expect handling increasing modeling complexity, and an enhanced use of simulation techniques.

3. - Simulation for American Options

3.1 Introduction

As stated earlier, there have been several recent efforts to extend Monte Carlo simulation techniques for solving American-type options. These include Barraquand and Martineau (1995), Broadie and Glasserman (1997a, 1997b), Broadie, Glasserman and Jain (1997), Raymar and Zwecher (1997), Longstaff and Schwartz (1998), and others. These methods attempt to combine the simplicity of forward induction with the ability of determining the optimal option exercise of backward induction. In this section we give the basic intuition of this new approach and in the next the results of its application to the classical Brennan and Schwartz (1985) model for evaluating natural resource investments.

Longstaff and Schwartz (1998) propose a promising new procedure for solving American options. Their approach basically consists of estimating a conditional expected payoff function for each date for the continuation value of the American option, and comparing this value with its exercise value. Whenever the exercise value is higher than the continuation value the option should be optimally exercised, while in the opposite case it would not. To estimate the conditional expected payoff value they first run several thousand state variable paths, and at a second stage they make a backward induction analysis of when it is optimal to exercise. At any point in time (starting from the end) each path generates one observation on the optimality of exercising or not for that path. Using cross sectional regressions it is possible to estimate when it is optimal to exercise for given date and state variable values, and solve recursively backwards.

Another approach (used in the next section) proposed by Barraquand and Martineau (1995) has been applied to solve some financial options. Its main insight is that in most cases it is possible to discretize and to reduce the dimensionality of the valuation problem, and still get reasonably good approximations. For example, if we assume a multi-factorial process for a state variable S, the procedure calls for making several thousand Monte Carlo simulations on S and grouping the obtained values into a fixed set of "bins", as shown in Figure 3.1. Then, by making successive simulation runs it is possible to empirically determine the transition probabilities between successive bins and finally to solve backwards the valuation process using each bin as a decision unit (see Figure 3.2).

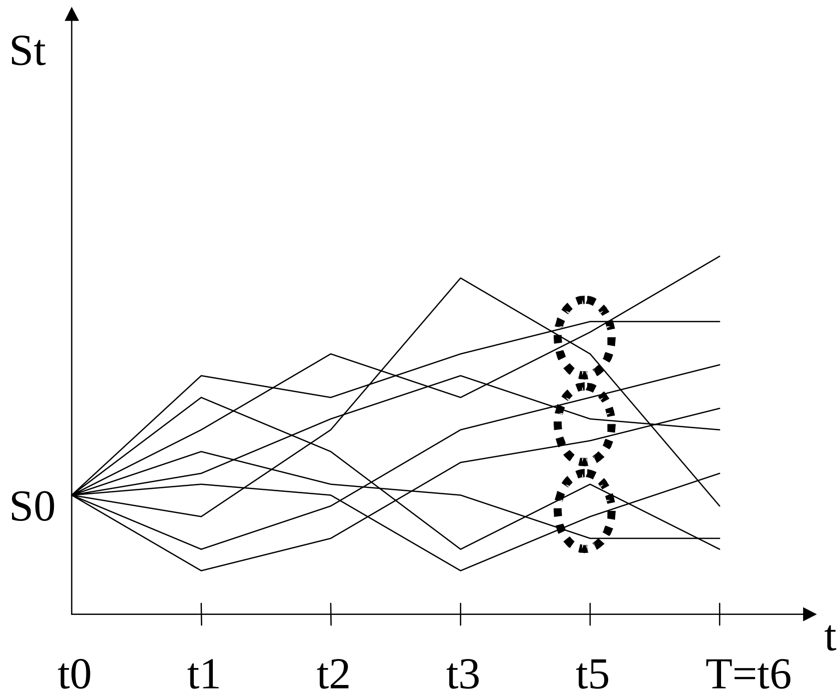

Figure 3.1 Simulation paths for the uncertain state variable and grouping into bins

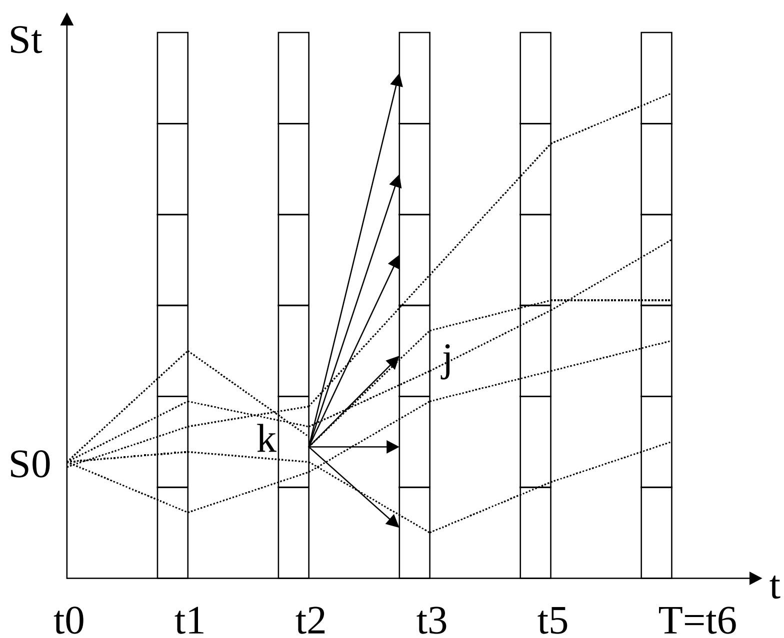

Figure 3.2 Determination of Transition Probabilities between Bins.

One of the crucial success factors in using this methodology is the selection of the one-dimensional state variable that will represent all state variables in the problem. As Broadie and Glasserman (1997b) point out, this procedure does not ensure convergence when, for example, there are disjoint optimal exercise sectors. In this case, further increases in the number of one-dimensional bins are not able to determine the optimal exercise policy, as Figure 3.3 shows.

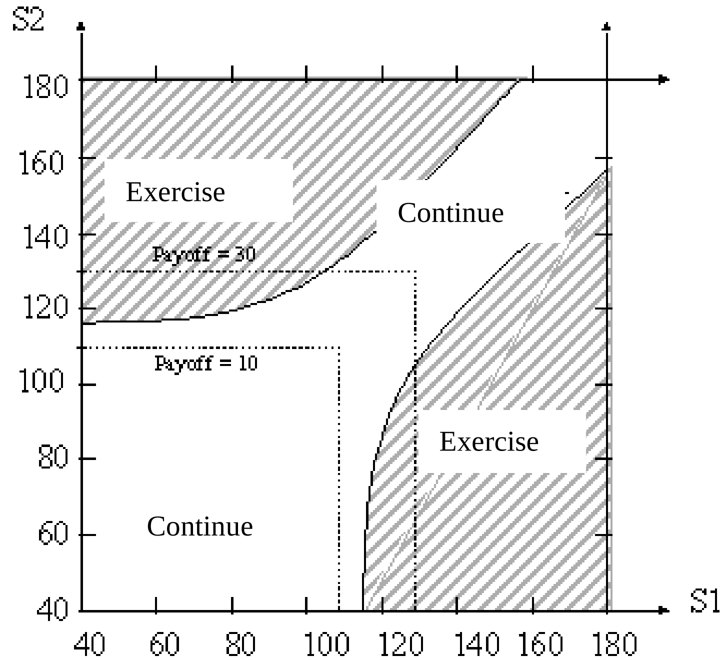

Figure 3.3 Example of Disjoint Optimal Exercise Regions

To reduce this problem, Raymar and Zwecher (1997) recommend adding a second dimension to the bin grouping process, so the state of the economy can be represented by more than a one-dimensional vector.

Most of the literature for solving American options by simulation has concentrated on valuing financial options. In the next section we illustrate the use of the Barraquand and Martineau (1995) procedure for the more complex optimal operation of a copper mine, initially modeled in Brennan and Schwartz (1985). Other applications of simulation to real options include Cortazar and Schwartz (1998), who solve for the optimal timing of a development investment in an oil reserve using the Barraquand and Martineau (1995) procedure, and Cortazar, Acosta and Osorio (1999) who compare different simulation alternatives for valuing real options and extend some of the results presented in the next section.

3.2 Evaluating Natural Resource Investments: Solving the Brennan and Schwartz (1985) Mine Problem Using Monte Carlo Simulation.

3.2.1 The Model and Solution Procedure.

Most real investments have embedded American-type options in their cash flow function. Whenever investments may be delayed, capacity expanded at different dates, and/or production suspended or resumed, the optimal timing of this decision is crucial for asset value maximization.

In this section we use the Barraquand and Martineau (1995) simulation procedure for solving the Brennan and Schwartz (1985)[2] model for valuing a copper mine which involves several American-type options. In this model, mine value depends on the discounted cash flows of future production, with copper prices following a stochastic process. The mine has finite copper reserves (Q), a constant-returns-to-scale technology with average unit cost of production A(Q), a flexible tax structure, a diversifiable expropriation risk, and several American-type options including the flexibility to temporally stop or resume production (with associated costs) or to abandon the mine. The mine has a maximum production rate and will optimally either produce at the maximum rate or close down.

Brennan and Schwartz (1985) solve their model by backward induction using finite difference numerical approximations to the partial differential equations that describe the value of the mine. They start the valuation process at known boundary conditions, including setting the value of the mine to zero when resources are exhausted or if output price becomes zero.

The key elements to solve such a real option problem by simulation are: 1) the definition of a proper valuation grid, with each node representing a markovian state of the mine susceptible to be solved by backward induction, and 2) the discretization of the risk-neutral price process to simulate possible price paths.

First, to define the valuation grid, the value of the mine at any moment can be seen as depending on copper price, on remaining reserves, and on whether the mine is open or closed. It is convenient to discretize total reserves into units of production, q. Each unit of production represents total copper output that will be produced after a decision in that regard has been made and before a new review on whether or not to continue producing is made.

Depending on the current state of the mine, the feasible next stages can be determined. For example, if the mine is open and has n units of copper reserves (each one of q pounds of copper) the mine manager might decide to maintain the mine open (reducing next-stage reserves to $n - 1$ units), to close the mine (with next-stage reserves

[2] See chapter **XX** of this book for a full presentation of this model.

remaining the same), or to definitely abandon the mine. Each decision is associated with different cash flows. In the first case the cash flow is equal to the revenues minus the costs of production, in the second to the closing and maintenance costs, and in the third to zero.

Similarly, if the mine is currently closed and has *n* units of copper reserves, the manager might decide to keep it closed, to open it (reducing next-stage reserves to *n* -1 units), or to abandon the mine. The cash flows in this case are equal to the maintenance costs for the first alternative, to the revenues minus the costs of producing a unit minus the cost of opening the closed mine for the second case, and to zero in the third alternative.

To solve for the optimal decision for each state of the mine we compute the expected value of each alternative decision, which, in turn, depends on the transition probabilities for the changes in copper prices and on the optimal continuation value once any of the three decisions is made.

The second element in solving by simulation such an evaluation problem is the definition of the stochastic process that models uncertainty. To obtain the price change probabilities we must specify the price stochastic process. Brennan and Schwartz (1985) assume a random walk process for the risk-adjusted commodity price returns. We discretize this process using the following equation:

$$\Delta S = (r - \kappa)S\Delta t + \sigma S\sqrt{\Delta t}Z \qquad (17)$$

where S is the commodity price, r is the (real) risk-free interest rate, κ the convenience yield, σ the volatility of price returns, Δt the time-increment, and *Z* a standardized Normal random variable. Even though a mean-reverting process for commodity prices might be better, we use this random walk process for comparison with the finite-difference method used in the original paper. In the next section we discuss possible model extensions that take this issue into account.

Following Barraquand and Martineau (1995), to obtain a discrete number of price states that adequately represent this stochastic process, a first set of simulation runs is performed. For each time interval, ΔT, all price paths are sorted and grouped into 200 bins, each one with the same number of observations. The average, maximum and minimum price in each bin is computed. Successive simulation runs are then performed and used to compute the transition probabilities between bins at successive time intervals.

Once transition probabilities are obtained, backward induction is used on the discrete state space that includes prices and mine states. Figure 3.4 shows a graphical representation of the grid that must be solved by backward induction. Assuming the mine is open and has 2 units of production, the arrows indicate the feasible states that could be reached with some non-zero probability. We solve the grid starting from the end and work our way backwards determining the optimal operating policy for each state vector value.

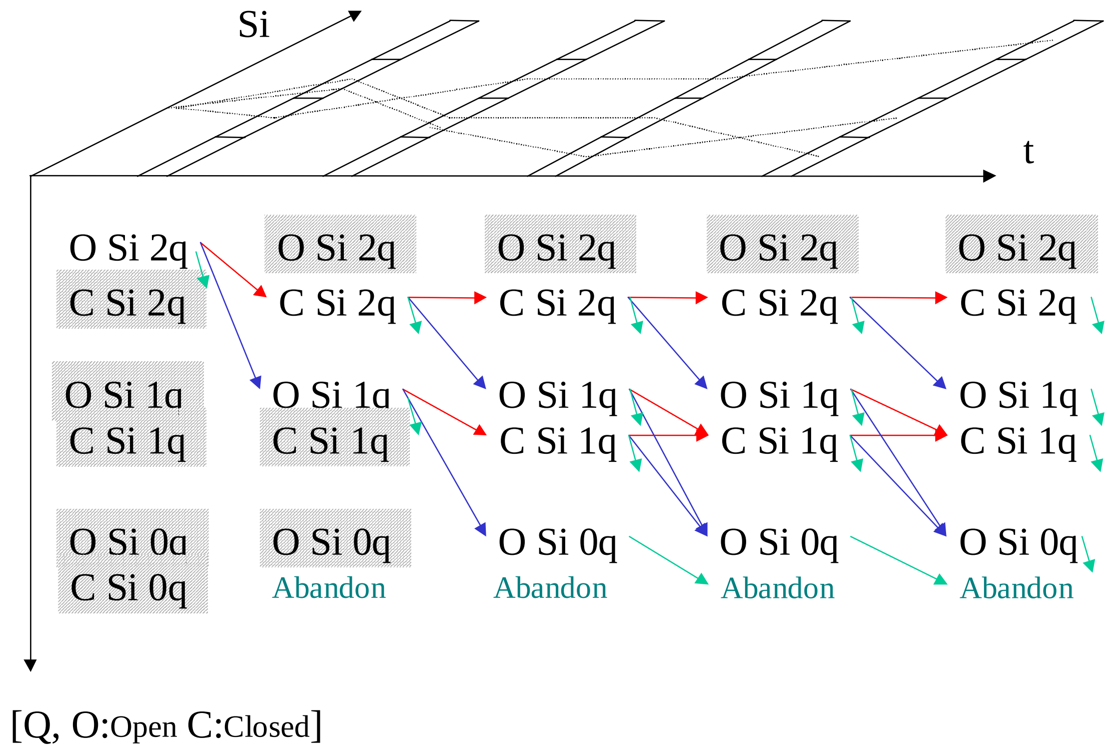

Figure 3.4 Feasible States that can be Reached depending on Managerial Decisions
Each State is defined by the commodity price (Si), the level of reserves (2q, 1q or 0q) and whether the mine is currently Open (O), Closed (C), or Abandoned

3.2.2 Mine Value and Optimal Policy.

To check the simulation method accuracy we compare our results with those obtained by finite differences as reported in Brennan and Schwartz (1985). Table 3.1 presents a comparison of both solutions. It can be verified that the simulation method provides good approximations.

Copper Price (US$/lb)	Mine Value (MMUS$) Brennan-Schwartz		Mine Value (MMUS$) Barraquand-Martineau		Error (%)	
	Open	Closed	Open	Closed	Open	Closed
0.4	4.15	4.35	4.19	4.39	1.0	0.9
0.5	7.95	8.11	7.97	8.17	0.3	0.7
0.6	12.52	12.49	12.53	12.53	0.1	0.3
0.7	17.56	17.38	17.57	17.37	0.1	-0.1
0.8	22.88	22.68	22.90	22.70	0.1	0.1
0.9	28.38	28.18	28.44	28.24	0.2	0.2
1.0	34.01	33.81	34.10	33.90	0.3	0.3

Table 3.1 Mine Value using Simulation and Finite Differences

The optimal policy obtained by the simulation method is illustrated in Figure 3.5.

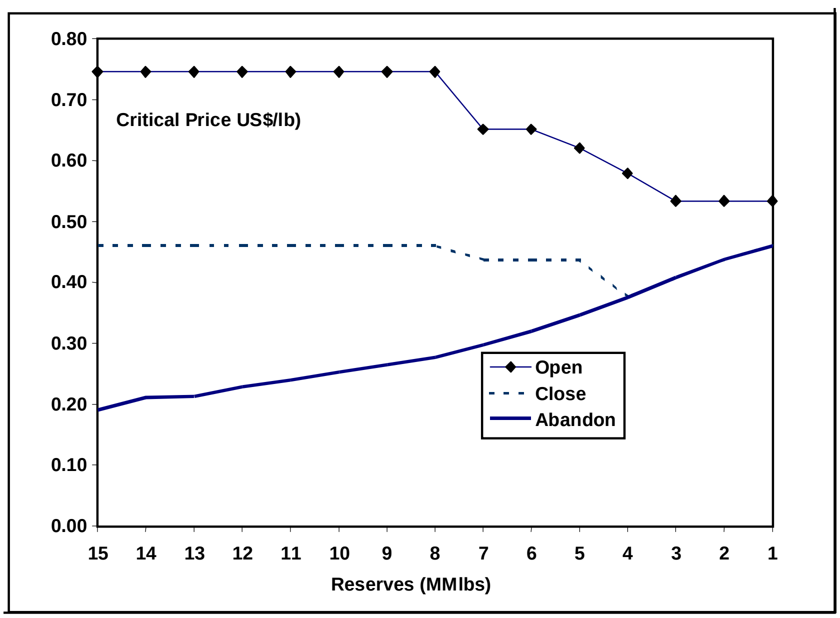

Figure 3.5 Critical Spot Prices for Opening, Closing, and Abandoning the Mine

This result is reasonably close to the finite-differences optimal policy, as can be seen in Figure 3.6, which compares the critical opening prices obtained by finite differences and by simulation.

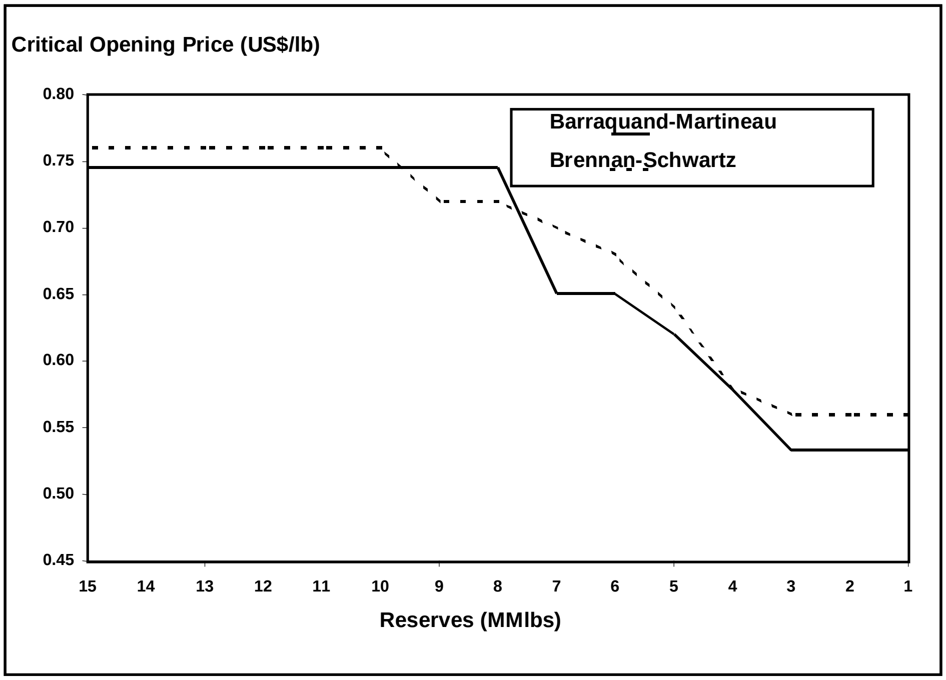

Figure 3.6 Critical Opening Price using Simulation and Finite Differences

3.2.3 Model Extensions.

The main contribution of simulation procedures for solving real option problems is its ability to handle models of increased complexity. In the following we provide some examples of model extensions that can easily be accommodated using these procedures.

The first type of possible extension deals with modeling uncertainty. There is an extensive bibliography on the appropriate number and specification of the risk factors to be used for the stochastic process that defines uncertainty [Schwartz (1997)]. The simple random walk process in the last section was simply used to match the Brennan and Schwartz (1985) model, and is a very strong assumption given state-of-the-art research in commodity prices. Given that simulation is much more efficient to solve problems with multi-factor processes, simulation models can easily be extended to use more complex process specifications.

A second type of possible extension deals with the modeling of the derivative asset. The last section already considered one model extension (over Brennan and Schwartz (1985)) with the use of time as a state variable. This allows for including time-dependent information, like cost inflation or finite-time concessions for mine production. Other extensions of the derivative asset model could easily be considered.

Finally, extensions of the modeling approach itself could be considered. For example, additional dimensions of the state variables could allow for a richer definition of the optimal policy, a source of considerable value in real options investments.

4.- Conclusions

This paper provides an overview of simulation and its applicability for solving real option problems. It discusses alternative numerical approaches to valuing assets and shows that both forward and backward induction procedures have a place in real options valuation.

To highlight the relative merits of the different numerical methods, a case-project with the option of investing in the future contingent on a stochastic output price was valued using binomial trees, finite differences, and simulation. Given that the project can be modeled as a European call option, the Black and Scholes (1973) analytical solution was used as a benchmark. The four methods provided similar results.

Standard simulation methods, even though very powerful for solving European-type options, have traditionally been considered inadequate for solving American-type options, a major drawback for their use in real option valuation. Recent research, however, has proposed extensions of simulation that combine forward and backward procedures for valuing American-type options. We present an application of these extended simulation methods to solve Brennan and Schwartz´s (1985) classic mine valuation problem. The benefits of this approach, with its better handling of complex uncertainty modeling and path-dependent cash flows, are discussed.

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