Real Option Valuation using NPV

Tom Arnold
The Robins School of Business
Department of Finance
University of Richmond, VA 23173
O: 804-287-6399
F: 804-289-8878
tarnold@richmond.edu

Timothy Falcon Crack
Dept. Finance and Quant. Analysis,
Otago University,
PO Box 56, Dunedin, New Zealand.
Tel: +64(0)3-479-8310
Fax: +64(0)3-479-8193
tcrack@business.otago.ac.nz

November 19, 2004

Preliminary, do not cite without permission

We thank Richard L. Shockley. Any remaining errors are our own.

Real Option Valuation using NPV

Abstract

We show that a careful net present value (NPV) using risk-adjusted discount rates produces a real option valuation identical to that obtained from a risk-neutral option valuation. This general result demonstrates that NPV and risk-neutral option valuation are equivalent. Although equivalent, we argue that in this context the implementation of a traditional risk-adjusted NPV will often be computationally infeasible—for reasons related to sheer volume of disaggregated sample paths. Fortunately, the risk-adjusted option valuation framework of Arnold and Crack (2000) allows this same risk-adjusted NPV to be executed by seamlessly discounting the payoffs to different sample paths using the correct risk-adjusted discount rates. It also allows the analyst to capture physical probability information not available in a risk-neutral valuation.

Introduction

When executed carefully, the net present value (NPV) of a real option project is shown to be identical to the result of a risk-neutral real option valuation. We compare and contrast these two valuation techniques and conclude that although equivalent, a traditional risk-adjusted NPV technique is often going to be computationally infeasible unless executed using the risk-adjusted option valuation framework of Arnold and Crack (2000).

A project's NPV is the value of the project's discounted expected cash inflows less the value of the project's discounted expected cash outflows. Expectations are usually taken with respect to physical probabilities and discounting is conducted using a risk-adjusted rate of return.

For a project that displays optionality, however, valuation is often via an option pricing technique. Most option pricing techniques (either closed form formulae or numerical techniques) use risk-neutral valuation. Thus, most real option valuations also use risk-neutral valuation. Risk-neutral valuation appears to be different to NPV valuation because it typically uses risk-neutral probabilities (as opposed to physical probabilities) and a riskless discount rate (as opposed to a risk-adjusted discount rate). We show that these differences are really just matters of convenience of execution.

We next discuss optionality and leverage. Then we introduce a real option project which we value using three different techniques: first, risk-neutral option pricing with riskless discount rates; second, option valuation using physical probabilities and risk-adjusted discount rates; and third, traditional NPV techniques using risk-adjusted discount rates. The second technique helps to clarify the link between the first and third techniques, and provides a convenient method to implement a risk-adjusted NPV with computational ease. These three techniques lead to identical valuations. We discuss the pros and cons of each technique and conclude with a summary.

Optionality

A project with optionality allows you to opt into good states of the world and opt out of bad states of the world. In our example in the next section, states of the world are tied to sales forecasts in a product market. Early design and development costs in that example give the option to expend production costs later and go into production if sales forecasts are high or to abandon completely if sales forecasts are low. This is analogous to buying a financial option (with premium analogous to the design and development costs in our example) giving you the right to buy an underlying security (with value analogous to PV of future sales in our example) by expending the strike (analogous to production costs in our example).

The ability to opt into good states of the world and out of bad states of the world means that the holder of the option, be it real or financial, has a leveraged claim on the cash flows accruing to the underlying asset. This leverage means that the discount rate for the option is not the same as the discount rate for the underlying. This point is made very clear in the example that follows. In the case of a simple call option, the discount rate of the option is a leveraged up version of the discount rate of the underlying (Cox and Rubinstein [1985, Chapter 5]). The bottom line is that finding the option's discount rate is all-important, but the complexities of this in the option framework mean that most people turn instead to risk-neutral valuation. We present a risk-adjusted framework as a viable alternative.

A Simple Real Option Project

Suppose a firm is considering entering a particularly volatile product market. Once the product hits the market, it must recoup all of its investment in the initial year because the industry is prone to fads where the product must be different from what customers already own in order to be marketable. If the product is introduced today, it will generate (a present value of) $200 million in sales, but at a production cost of (present value) $300 million. There are development and design costs of $50 million to add to the production cost, so entering the market today is certain to lead to financial failure. Let us ignore taxes (or assume that all cash flows are calculated after tax). Assume there are no costs or benefits other than today's development and design costs, the future production costs, and the future PV of sales.

What if the firm develops the product now, but delays going into production until the market has had the opportunity to expand? The downside to immediate development is that the development and design costs must be incurred now; the upside to immediate development is that patents will be obtained ahead of competitors, thus establishing a toehold. The benefit to delaying production is that uncertain dollar values of sales in the future have at least the potential to cover the future production costs.

Suppose the riskless rate is 5% per annum. Suppose the continuously compounded expected growth in the PV of dollar sales available to the firm in this market is 18% per annum, and the estimated standard deviation of this growth rate is 60% per annum. The expected growth rate of sales and its associated standard deviation are the return and volatility of the underlying process in the model (i.e., the process to which the option applies). The product market payoffs are an asset available for future purchase if the appropriate development expenditure is made today. We need to determine the value of establishing a toehold by developing the product now. If this value exceeds the development and design costs, then we should develop the product now, and otherwise not.

Suppose we restrict ourselves to going into production only at a five-year horizon, at which time the costs of going into production are forecast to be $400 million. The current development of the product thus creates a five-year European-style call option with a strike price of $400 million.[1] The option is only exercised (i.e., we spend the cost of production) when the PV at year five of future sales is above the production cost.

[1] A simple option like this can be valued using Black-Scholes. Our binomial framework, however, handles much more complicated options than this example.

The development of this product provides the right, but not the obligation, to take the product to market in the future. The option-like characteristic of being able to wait and see if the market for the product has sufficiently expanded is valuable. Any valuation technique must capture the value of this optionality.

Option Valuation Technique #1: Risk-Neutral Valuation

A simple risk-neutral valuation (Cox, Ross and Rubinstein 1979, "CRR") begins by cutting the life of the underlying process into equal time steps and building a multi-step binomial tree. At each step of the CRR tree the underlying process is modeled as rising by multiplicative factor $u = e^{\sigma\sqrt{\Delta t}}$ or falling by multiplicative factor $d = e^{-\sigma\sqrt{\Delta t}}$, where σ is the annualized standard deviation of the continuously compounded growth rate of the level of the underlying process (60% here) and Δt is the length of the time step in years. We use a five-step tree, with each step of length one year, so $\Delta t = 1$ here. At each node of the tree, the risk-neutral probability[2] of the up step is given by $q = (e^{r(\Delta t)} - d)/(u - d)$, where r is the annualized continuously compounded riskless rate (5% here), and discounting is performed at the riskless rate. Figure 1 displays the CRR tree of potential sales levels and the final risk-neutral probabilities.

To value the real option using risk-neutral techniques we may either find the value at each final node in Figure 1, and then step backwards through the tree finding repeated one-step riskless discounted expected values using the risk-neutral probability *q*, or we may note that because the option is European style, we need only multiply the final nodal values by the final risk-neutral probabilities and discount using the riskless rate for five years—as shown in Figure 2. Either way, we arrive at the risk-neutral option valuation of $73.25 million. This exceeds the initial development/design costs of $50 million by $23.25 million, so we should proceed.

[2] Under these probabilities the expected growth rate of the underlying process is simply the riskless rate. Thus, these probabilities describe behavior in a world in which investors behave as if they are risk neutral.

Figure 1: Potential Sales Revenue in Millions

Current	Year 1	Year 2	Year 3	Year 4	Year 5	Risk Neutral Probability
$200.00	$364.42	$664.02	$1209.93	$2204.64	$4017.11	0.96%
	$109.76	$200.00	$364.42	$664.02	$1209.93	7.34%
		$60.24	$109.76	$200.00	$364.42	22.52%
			$33.06	$60.24	$109.76	34.55%
				$18.14	$33.06	26.50%
					$9.96	8.13%

Notes: the number $1,209.03 in the table, for example, is the PV three years from now of all future sales if the process has experience three "ups." The final risk-neutral probability 7.34%, for example, is calculated as $\binom{N}{h} q^h (1-q)^{N-h}$, where $N = 5$, $h = 4$, $q = (e^{r(\Delta t)} - d)/(u - d)$, $u = e^{\sigma\sqrt{\Delta t}} = 1.822119$, $d = 0.548812$, $r = 0.05$ is the riskless rate, $\sigma = 0.60$, and $\Delta t = 1$ is the time step.

Figure 2: Risk-Neutral Option Valuation

Year 5 Sales	Max(Sales-Prod. Cost, 0)	Probability	Product
$4,017.11	$3,617.11	0.96%	$34.61
$1,209.93	$809.93	7.34%	$59.45
$364.42	$0	22.52%	$0
$109.76	$0	34.55%	$0
$33.06	$0	26.50%	$0
$9.96	$0	8.13%	$0
			Sum = $94.06
			Disc. Sum = $73.25

Year 5 Sales ($ millions) and probability are taken from Figure 1. "Product" is the product of second and third columns. The discounted sum is calculated as $e^{-rT}\, Sum$, where *r*=0.05 is the riskless rate, *T*=5 years, and *Sum*=$94.06 as shown above.

Option Valuation Technique #2: Risk-Adjusted Option Valuation (GEMPOP)

Arnold and Crack (2000) present the generalized multi-period option pricing model (GEMPOP). The GEMPOP model allows the analyst to discount payoffs one step ahead in a binomial tree using risk-adjusted discount rates and physical probabilities. It is thus a risk-adjusted, rather than a risk-neutral, option valuation technique.

Keeping to the continuously-compounded notation used above, the GEMPOP model is written down in Equation (1) to give the value V_t of a derivative at time *t* as a function of the values V_u and V_d one step ahead at time $t + \Delta t$ in a binomial tree (Arnold and Crack [2000]).

$$V_t = e^{-r(\Delta t)}\left[E\left(V_{t+\Delta t}\right) - \left(\frac{V_u - V_d}{u - d}\right)\left(e^{k(\Delta t)} - e^{r(\Delta t)}\right)\right], \tag{1}$$

where *k* is the physical, rather than risk-neutral, continuously-compounded growth rate of the level of the underlying, *r* is the riskless rate, *u* and *d* are as before, Δt is the step length, and $E(V_{t+\Delta t})$ is taken with respect to the physical probability of an up move given by $p = (e^{k(\Delta t)} - d)/(u - d)$. That is, $E(V_{t+\Delta t}) = pV_u + (1 - p)V_d$.

The GEMPOP model does not use risk-neutral valuation: the term $E(V_{t+\Delta t})$ in Equation (1) uses the physical probability *p*, and the riskless discount rate appearing in Equation (1) goes hand-in-hand with a certainty equivalent interpretation of the term that follows the discounting.[3] If the physical probability *p* appearing in Equation (1) is replaced by the risk-neutral probability *q*, the GEMPOP model reduces immediately to the CRR risk-neutral option pricing model.

[3] The adjustment to the expected value of the option is similar to an adjustment suggested later by Hodder, Mello, and Sick (2001).

Implicit within Equation (1) is a risk-adjusted discount rate k_V for the derivative[4] over the time step Δt. It can be inferred by writing down $V_t = e^{-k_V(\Delta t)} E(V_{t+\Delta t})$, and substituting for V_t and $E(V_{t+\Delta t})$, as in Equations (2).

$$
\begin{aligned}
& V_t = e^{-k_V(\Delta t)} E(V_{t+\Delta t}) \\
& \Leftrightarrow k_V = \frac{-1}{\Delta t} \cdot \ln\left[\frac{V_t}{E(V_{t+\Delta t})}\right] \\
& \Leftrightarrow k_V = r - \frac{1}{\Delta t} \ln\left[1 - \frac{V_u - V_d}{(u-d)E(V_{t+\Delta t})}\left(e^{k(\Delta t)} - e^{r(\Delta t)}\right)\right] \\
& \Leftrightarrow k_V = r - \frac{1}{\Delta t} \ln\left[1 - \frac{V_u - V_d}{(u-d)\left[pV_u + (1-p)V_d\right]}\left(e^{k(\Delta t)} - e^{r(\Delta t)}\right)\right],
\end{aligned}
\qquad (2)
$$

where, as before, $p = (e^{k(\Delta t)} - d)/(u-d)$ is the physical probability of an up move, and k is the physical continuously-compounded growth rate of the level of the underlying.[5]

Given all inputs to the final line in Equations (2), the risk-adjusted discount rate k_V on the option may be determined at any node in the binomial tree. Figure 3 shows the value of the real option (calculated using the GEMPOP model Equation (1)) and the one-period discount rate k_V (calculated using the last line of Equations (2)) for the option at each node in the tree for which the option has value. To implement this, we use k=18% (the growth rate in sales).

[4] See Cox and Rubinstein (1985, p323–324) for an exploration of some related issues for European-style financial options.

[5] If this were a financial option, k would be the risk-adjusted discount rate on the underlying security.

Figure 3: Valuation of the Option-Like Project with Associated Discount Rates (All Dollar Values in Millions)

Current	Year 1	Year 2	Year 3	Year 4	Year 5	Physical Probability
$73.25 25.686%	$170.48 24.970%	$388.45 24.004%	$859.73 22.628%	$1824.14 20.511%	$3617.11	3.42%
	$16.08 30.500%	$42.84 30.500%	$114.12 30.500%	$304.02 30.500%	$809.93	16.50%
		$0.00 N/A	$0.00 N/A	$0.00 N/A	$0.00	31.81%
			$0.00 N/A	$0.00 N/A	$0.00	30.65%
				$0.00 N/A	$0.00	14.77%
					$0.00	2.85%

Note: The final physical probability 16.50%, for example, is calculated as $\binom{N}{h} p^{h} (1-p)^{N-h}$, where $N=5$, $h=4$, $p=(e^{k(\Delta t)}-d)/(u-d)$, $u=e^{\sigma\sqrt{\Delta t}}=1.822119$, $d=0.548812$, $k=0.18$ is the physical growth rate, and $\Delta t=1$ is the time step.

As the option becomes more in-the-money in Figure 3, the discount rate falls. This is a general result: the discount rate for a call option tends to fall as the option becomes more in the money and rise as the option becomes more out of the money.[6] That is, the rate k_V falls as the leverage falls and rises as the leverage rises. The discount rate for the option is thus "path dependent"—it depends upon the path followed through the binomial tree by the underlying. This is not a new result—Black and Scholes' original derivation of the option pricing formula uses a path-dependent discount rate calculated via an "instantaneous CAPM" (Black and Scholes [1973, p645–646], Black [1989, p5]).

[6] There are exceptions to this, for example if the underlying has large negative beta. Similarly, if the option is a put, the opposite result holds. Though in each case, the statement holds true with respect to the magnitude of the option's discount rate.

We can see in Figure 3 that the physical probability that the project will be undertaken (i.e., that the real option ends up in the money) is 3.42%+16.50%=19.92% (the sum of the top two ending nodal probabilities). This probabilistic information cannot be inferred from the CRR risk-neutral probabilities in Figure 1.

Note that although the option values in Figure 3 are calculated using the GEMPOP model, they are identical to those values that would be calculated at each step in the tree using the CRR risk-neutral model (Arnold and Crack [2000]).

An immediate consequence of being able to calculate the risk-adjusted discount rate for the option at any point in the binomial tree is that we may now use these numbers to implement the NPV technique for our real option.

Option Valuation Technique #3: Traditional NPV

Table 1 finds the NPV of the real option project by disaggregating the individual sample paths through the binomial tree that lead to non-zero payoffs. Only one path leads to the highest node (with payoff $3,617.11), but five different paths lead to the second highest node (with payoff $809.93). For example, "uuuud" and "uuudu" both lead to the second highest node (where "uuuud" denotes four up steps followed by a down).

This is a traditional NPV analysis because we have simply added together discounted expected payoffs using risk-adjusted discount rates and physical probabilities. Note that if we ask *"What is the discount rate?,"* this is not a well posed question because different payoffs require different discount rates, and even identical payoffs arrived at via different sample paths of the underlying require different discount rates. The PV of $73.25 is of course identical to the two previous valuations.

Table 1: Calculation of NPV (All Dollar Values in Millions)

Panel A: Present Value of Sample Path Payoffs		
Future Payoff	**Discount Factor Calculated by Compounding Along the Sample Path (from Figure 3)**	**Product**
\$3,617.11	EXP(-(25.686%+24.970%+24.004%+22.628%+20.511%))	\$1113.70
\$809.93	EXP(-(25.686%+24.970%+24.004%+22.628%+20.511%))	\$249.38
\$809.93	EXP(-(25.686%+30.500%+30.500%+30.500%+30.500%))	\$184.95
\$809.93	EXP(-(25.686%+24.970%+30.500%+30.500%+30.500%))	\$195.46
\$809.93	EXP(-(25.686%+24.970%+24.004%+30.500%+30.500%))	\$208.58
\$809.93	EXP(-(25.686%+24.970%+24.004%+22.628%+30.500%))	\$225.67

Panel B: Calculation of NPV		
Product from Panel A	**Probability of Sample Path***	**Product**
\$1113.70	3.42427%	\$38.14
\$249.38	3.30014%	\$8.23
\$184.95	3.30014%	\$6.10
\$195.46	3.30014%	\$6.45
\$208.58	3.30014%	\$6.88
\$225.67	3.30014%	\$7.45

Summation of Discounted Payoff times Probability = \$73.25

Net Present Value = \$73.25 - \$50.00 = \$23.25

*We calculate the probability of observing that sample path using $p^h(1-p)^{N-h}$ where *h* is the number of ups, *N-h* is the number of downs, and *p* is the physical probability as before.

There is a redundancy here that may not be immediately obvious to the reader. To find the discount rates for the option using Equations (2), we walk backwards through the tree and keep track of the option values one step ahead (denoted V_u and V_d in Equation (2)). Thus, we need to know the real option values one step ahead within the binomial tree in order to derive the risk-adjusted discount rate for the real option at that node. Thus, we need to value the option at each node using option pricing techniques (GEMPOP) in order to deduce the discount rate that allows us to value the option using NPV techniques. It is not circular, but it is certainly redundant.

Which Valuation Method and Why?

If we do wish to execute a traditional NPV valuation of our real option, then we need to find discount rates associated with each possible sample path through the tree. This information can be harvested from the GEMPOP model by disaggregating the different sample paths. This leads, however, to a problem as follows.

Our five-step tree has, for example, $\binom{5}{4} = 5$ different paths leading to the second-to-highest ending node (each listed in Table 1), and there are a total of only $\sum_{h=1}^{5}\binom{5}{h} = 2^5 = 32$ sample paths through the binomial tree. In practice, however, absent any accelerated convergence techniques, a 100-step binomial tree may be needed to remove discreteness-induced numerical artifacts (e.g., Rubinstein [1999, p269–270]). If we use a 100-step tree, however, there are $\binom{100}{99} = 100$ different sample paths leading to the second-to-highest ending node, and a total of $\sum_{h=1}^{100}\binom{100}{h} = 2^{100}$ paths through the binomial tree—this is more than the number of atoms in the universe.[7] Disaggregating these sample paths and finding the discount rate for each is not just impractical, but it is in fact *impossible* given *any* computing technology. A back-of-the-envelope calculation suggests that if you list 1,000 discount rates per page, you need a stack of paper 12,000,000,000,000,000 miles high just to list the discount rates for every

[7] We would in fact need to consider only the ending nodes in a 100-step tree with positive option value. With our particular numerical example's parameters, this is only the uppermost 48 ending nodes in the 100-step tree and there are only $\sum_{h=53}^{100}\binom{100}{h} = \frac{2^{100}}{3.2399}$ sample paths leading to these nodes—but that is still more than the number of atoms in the universe (details of calculation available upon request).

sample path through a 100-step tree. You need to square that number for a 200-step tree! You simply cannot analyse a data set this huge.

An NPV alternative is to solve for a single path-*independent* risk-adjusted discount rate that equates our final payoffs to our PV. This discount rate, however, over-discounts the deeper in-the-money sample path payoffs and under-discounts the less in-the-money sample path payoffs. The bottom line is that no single risk-adjusted discount rate correctly applies to all sample paths, but calculating a risk-adjusted discount rate for each sample path is typically going to be computationally infeasible.

Fortunately, each of these sample paths, and their appropriate risk-adjusted discounting, are handled by construction within the structure of the GEMPOP model. The GEMPOP model may thus be considered to be a convenient method for implementing a traditional risk-adjusted NPV analysis that would otherwise be computationally intensive.

If valuation is your only goal, then risk-neutral valuation (e.g., CRR) is probably the most convenient technique because the discount rates on the option-like project are path-*independent*. By altering the probabilities and discarding the risk premium, the CRR technique allows you to use the same discount rate (the riskless rate) for all paths. The downside is that you must use risk-neutral probabilities and riskless discounting, and these can take some getting used to.

If you wish to perform a risk-adjusted NPV and to obtain numerical accuracy, then the number of sample paths is so large that you are likely to need the GEMPOP model to implement the NPV. The GEMPOP model involves only marginally more computational complexity than the CRR model, it avoids risk-neutral pricing and riskless discount rates, and it allows the analyst to capture physical probabilities (e.g., probability of the option finishing in the money).

Conclusion

We show that a careful NPV using risk-adjusted world discount rates produces real option valuations identical to those obtained from a risk-neutral option valuation. Thus, NPV techniques and risk-neutral option valuation techniques are equivalent. Although equivalent, practical implementation of a traditional risk-adjusted NPV analysis may be computationally infeasible because of the large number of sample paths leading to payoffs. For convenience, we recommend either the risk-adjusted GEMPOP option valuation framework of Arnold and Crack (2000) (when probabilistic information is required) or a risk-neutral option valuation framework (when valuation is the only goal).

References:

Arnold, T. and T. Crack. "Option Pricing in the Real World: A Generalized Binomial Model with Applications to Real Options." (2000)

Black, Fischer, "How We Came Up With the Option Formula." *Journal of Portfolio Management*, 15(2) (Winter 1989), 4–8.

Black, Fisher and Myron Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81 (May/June 1973), 637–659.

Cox, J. and M. Rubinstein. *Options Markets*. Englewood Cliffs, NJ: Prentice-Hall, Inc. (1985)

Cox, J.; S. Ross; and M. Rubinstein. "Option Pricing: A Simplified Approach." *Journal of Financial Economics,* 7 (Sept. 1979), 229 – 263.

Hodder J. A. Mello, and G. Sick. "Valuing Real options: Can Risk-Adjusted Duscounting Be Made to Work?" *Journal of Applied Corporate Finance,* (Summer 2001), 90 - 101.

Rubinstein, Mark, *Rubinstein on Derivatives,* Chicago, IL: Risk Books, 1999.

Smith, J. and R. Nau. "Valuing Risky Projects: Option Pricing Theory and Decision Analysis" *Management Science,* (May 1995), 795 – 816.