To Buy Or Not To Buy?

Using Real Options in Real Estate Investment Decisions

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1 Introduction

In this article we consider real estate investment decisions from a real option point of view. More precisely, the situation is as follows. Suppose that an investor is determined to buy the house that he is living in within a fixed period of time. We ask for the optimal point in time to perform the investment, i.e. to change from renting to owning the house. Thereby, we define the optimal point in time to be the one that maximizes the net present value of all cash flows from the point of view of the investor. The cash flows under consideration are the following. Up to the time of the investment, the investor has frequent rent payments constituing negative cash flows. After the investment, he has frequent credit and maintenance payments, both constituing negative cash flows as well. We assume that the investment is fully financed by a credit, and that the period of time for the credit payments is fixed. Finally, we assume that after the original period of time the house is sold, thereby constituing a positive cash flow.

Now, the decision to exercise (i.e. to buy the house) is considered as real option. Consequently, the above problem becomes an optimal stopping problem, where we seek for a stopping time (the exercise time) that maximizes the expected utility (the net present value of the cash flows). For comparison, we also consider the utility for renting the house without ever buying it. Of course, in this case only the frequent rent payments over the full period of time contribute to the utility function. The risk factors taken into account are the price of the house on the one hand, and the risk free rate on the other. Both risk factors are modeled by binomial processes. For the first process we assume a simple geometric random walk whereas for the second we assume a geometric random walk with mean reversion. Moreover, we assume that the house price is in proportion to the general house price index, such that we can extract the model parameters (drift and volatility) from historical data for the general house price index. Finally, we assume that the risk free rate coincides with the overnight rate, such that we can extract the model parameters (mean, drift and volatility) from historical data for the overnight rate. In particular we do not consider term structures in our model. We note that for the case of the geometric random walk, drift and volatility can be computed from the first two moments of the returns given by a historical time series. On the contrary, for the case of the geometric random walk with mean reversion the model parameters have to be determined by fitting them to a historical time series.

2 The Model

We consider a series of $N+1$ equidistant time steps s, where $N \in \mathbb{N}$. We assume that the house price index I_s follows a geometric Random Walk,

$$I_{s+1} - I_s = I_s(\mu + \sigma X_s) \qquad (s = 0, ..., N-1), \tag{2.1}$$

where $\mu > 0$ denotes the drift, $\sigma > 0$ denotes the standard deviation, and the X_s denote i.i.d. random variables such that

$$\mathbb{P}[X_s = 1] = \frac{1}{2} = \mathbb{P}[X_s = -1], \tag{2.2}$$

where $\mathbb{P}$ denotes the probability measure. Moreover, we assume that the risk free rate R_s follows a mean reversion Random Walk,

$$R_{s+1} - R_s = R_s \left(\alpha \, (\overline{R} - R_s) + \beta \, Y_s \right) \qquad (s = 0, ..., N-1), \tag{2.3}$$

where $\overline{R} > 0$ denotes the mean, $\alpha > 0$ denotes the drift, $\beta > 0$ denotes the standard deviation, and Y_s denote i.i.d. random variables such that

$$\mathbb{P}[Y_s = 1] = \frac{1}{2} = \mathbb{P}[Y_s = -1]. \tag{2.4}$$

Finally, we assume that X_r and Y_s are independent for $r \neq s$, and that X_s and Y_s have a joint distribution independent of s. More precisely, we assume that for some $0 < p < 1$ the following equations hold,[1]

$$\mathbb{P}[X_s = 1, Y_s = 1] = \frac{p}{2} = \mathbb{P}[X_s = -1, Y_s = -1], \tag{2.5a}$$

$$\mathbb{P}[X_s = 1, Y_s = -1] = \frac{1-p}{2} = \mathbb{P}[X_s = -1, Y_s = 1]. \tag{2.5b}$$

Throughout the text the expectation of a random variable Z w.r.t. $\mathbb{P}$ will be denoted by $\mathbb{E}[Z]$. Moreover, the expectation of Z taken at time step s will be denoted by $\mathbb{E}_s[Z]$. More precisely, $\mathbb{E}_s[Z]$ denotes the conditional expectation of Z w.r.t. the sigma field generated by $X_0, \ldots, X_{s-1}$ and $Y_0, \ldots, Y_{s-1}$.

Next, we define the utility function for an investor living in a house during the time period $[0, N]$ who is determined to buy the house within that period.

1. First, we consider renting an average house during the time period $[0, n]$. We assume that due to maintenance the value of the house throughout the period is given by the house price index. For simplicity, we assume rent payments of amount $P_s(\text{rent})$ at the time steps $s = 0, ..., n-1$. Moreover, we assume that $P_s(\text{rent})$ is in fixed proportion to the house price index, i.e.

$$P_s(\text{rent}) = \gamma \, I_s, \tag{2.6}$$

   where $\gamma > 0$ denotes a given constant. We define the net present value (NPV) of the rent payments at time step $s = 0$ as the discounted NPV of the rent payments at time step $s = n$. The NPV of the rent payments at time step $s = n$ is defined as the sum of the compounded rent payments where the compounding is performed due to a short term investment strategy. Altogether, the net present value of the rent payments is given by

$$\begin{aligned} \text{NPV}_n(\text{rent}) &= \frac{1}{(1+R_0)^n} \sum_{s=0}^{n-1} \left(P_s(\text{rent}) \prod_{k=s}^{n-1} (1 + R_k) \right) \\ &= \frac{\gamma}{(1+R_0)^n} \sum_{s=0}^{n-1} \left(I_s \prod_{k=s}^{n-1} (1 + R_k) \right). \end{aligned} \tag{2.7}$$

[1] We note that our assumptions on the distributions of X_s and Y_s already determine the joint distribution up to the parameter p. To see this, assume that $\mathbb{P}[X_s = 1, Y_s = 1] = \frac{p}{2}$. Since $\mathbb{P}[X_s = 1] = \frac{1}{2}$ and $\mathbb{P}[Y_s = 1] = \frac{1}{2}$, this implies $\mathbb{P}[X_s = 1, Y_s = -1] = \frac{1-p}{2}$ and $\mathbb{P}[X_s = -1, Y_s = 1] = \frac{1-p}{2}$. Finally, since the joint probabilities have to sum up to 1, we also have $\mathbb{P}[X_s = -1, Y_s = -1] = \frac{p}{2}$.

2. Next, we consider buying an average house at time step $s = n$ at price I_n. We assume that the full price is payed by a credit taken at time step $s = n$. For simplicity, we assume constant repayments of amount $P_n(\text{credit})$ at each time step $s = n+1, \dots, n+K$, where $K \in \mathbb{N}$ denotes the given number of repayments. The residual debt $D_K(k)$ after k payments satisfies the following recursion equation,

$$D_K(0) = I_n, \qquad D_K(k+1) = (1 + R_n + \overline{S})\, D_K(k) - P_n(\text{credit}), \tag{2.8}$$

where $\overline{S} \geq 0$ denotes the given credit spread. The solution is given by

$$\begin{aligned} D_K(k) &= (1 + R_n + \overline{S})^k\, I_n - \sum_{i=0}^{k-1} (1 + R_n + \overline{S})^i\, P_n(\text{credit}) \\ &= (1 + R_n + \overline{S})^k\, I_n - \frac{(1 + R_n + \overline{S})^k - 1}{R_n + \overline{S}}\, P_n(\text{credit}). \end{aligned} \tag{2.9}$$

By assumption, the credit is amortised after K payments. Consequently,

$$D_K(K) = 0 \qquad \Longrightarrow \qquad P_n(\text{credit}) = \frac{(R_n + \overline{S})\,(1 + R_n + \overline{S})^K}{(1 + R_n + \overline{S})^K - 1}\, I_n. \tag{2.10}$$

We define the NPV of the credit payments at time step $s = 0$ to be the discounted NPV of the credit payments at time step $s = n$. The NPV of the credit payments at time step $s = n$ is defined as the sum of the discounted credit payments. Altogether, the net present value of the credit payments is given by

$$\begin{aligned} \text{NPV}_n(\text{credit}) &= \frac{1}{(1 + R_0)^n} \sum_{k=1}^{K} \frac{P_n(\text{credit})}{(1 + R_n)^k} \\ &= \frac{1}{(1 + R_0)^n} \cdot \frac{(1 + R_n)^K - 1}{R_n\,(1 + R_n)} \cdot \frac{(R_n + \overline{S})\,(1 + R_n + \overline{S})^K}{(1 + R_n + \overline{S})^K - 1}\, I_n. \end{aligned} \tag{2.11}$$

3. Next, we consider maintaining the house during the time period $[n, N]$. We assume that due to maintenance the value of the house throughout the period is given by the house price index. For simplicity, we assume maintenance payments of amount $P_s(\text{maint})$ at time steps $s = n, \dots, N-1$. Moreover, we assume that $P_s(\text{maint})$ is in fixed proportion to the house price index, i.e.

$$P_s(\text{maint}) = \delta\, I_s, \tag{2.12}$$

where $\delta > 0$ denotes a given constant. We define the NPV of the maintenance payments at time step $s = 0$ to be the discounted NPV of the maintenance payments at time step $s = n$. The NPV of the maintenance payments at time step $s = n$ is defined as the sum of the discounted and projected future maintenance payments. Altogether, the net present value of the maintenance payments is given by

$$\begin{aligned} \text{NPV}_n(\text{maint}) &= \frac{1}{(1 + R_0)^n} \sum_{s=n}^{N-1} \frac{\mathbb{E}_n[P_s(\text{maint})]}{(1 + R_n)^{s-n}} \\ &= \frac{\delta\, I_n}{(1 + R_0)^n} \sum_{s=n}^{N-1} \left(\frac{1 + \mu}{1 + R_n} \right)^{s-n}. \end{aligned} \tag{2.13}$$

4. Finally, we consider selling the house at time step $s = N$ at price I_N. In accordance with our approach in points 1, 2, 3, we define the NPV of the sale payment as the discounted NPV of the sale payment at time step $s = n$. The NPV of the sale payment at time step $s = n$ is defined as the discounted and projected sale payment. Altogether, the net present value of the sale payment is given by

$$\mathrm{NPV}_n(\mathrm{sale}) = \frac{1}{(1+R_0)^n} \cdot \frac{\mathbb{E}_n[I_N]}{(1+R_n)^{N-n}} = \frac{I_n}{(1+R_0)^n} \left(\frac{1+\mu}{1+R_n}\right)^{N-n}. \tag{2.14}$$

We define the utility function U_n for the above investor by

$$U_n = -\,\mathrm{NPV}_n(\mathrm{rent}) - \mathrm{NPV}_n(\mathrm{credit}) - \mathrm{NPV}_n(\mathrm{main}) + \mathrm{NPV}_n(\mathrm{sale}). \tag{2.15}$$

Now, the optimal time τ for the investor to buy the house can be written as the following optimal stopping problem:

Find a stopping time $\tau \leq N$ such that

$$\mathbb{E}[U_\tau] \longrightarrow \textit{maximal}, \tag{2.16}$$

From the general mathematical theory we know that one solution τ to the above problem is obtained by way of constructing the Snell envelope $\tilde{U}_n$ corresponding to our utility function U_n,

$$\tilde{U}_N = U_N, \qquad \tilde{U}_n = \max\{U_n, \mathbb{E}_n[\tilde{U}_{n+1}]\} \qquad (n = N-1, ..., 0). \tag{2.17}$$

From a numerical point of view the above equations allow to compute $\tilde{U}_n$ recursively from U_n backwards for $n = N, ..., 0$. Finally, having U_n and $\tilde{U}_n$ at hand one solution τ to the above optimal stopping problem is given by

$$\tau = \min\left\{ n \in \{0, ..., N\} \,\middle|\, \tilde{U}_n = U_n \right\}. \tag{2.18}$$

Since one of our goals was to compare buying the house to renting it exclusively, we next define the utility function for an investor living in a house during the time period $[0, N]$ without ever buying it. Given that the investor is determined from the start never ever to buy the house, we define the utility function to be the negative of the NPV of the rent payments. In this case, the NPV of the rent payments is simply the sum of the discounted and projected rent payments. Altogether, the utility function is given by

$$\overline{V} = -\sum_{s=0}^{N-1} \frac{\mathbb{E}[P_s(\mathrm{rent})]}{(1+R_0)^s} = -\gamma\, I_0 \sum_{s=0}^{N-1} \left(\frac{1+\mu}{1+R_0}\right)^s. \tag{2.19}$$

However, given that the investor in indecisive up to time step $s = n$ whether or not to buy the house and only then makes his decision never ever to buy it, we define the utility function to be the negative of the discounted NPV of the rent payments at time step $s = n$. According to points 1 and 3 above the utility function is given by

$$V_n = -\frac{\gamma}{(1+R_0)^n} \sum_{s=0}^{n-1} \left(I_s \prod_{k=s}^{n-1} (1+R_k) \right) - \frac{\gamma\, I_n}{(1+R_0)^n} \sum_{s=n}^{N-1} \left(\frac{1+\mu}{1+R_n}\right)^{s-n}. \tag{2.20}$$

3 Numerical Results

In our model the risk factors under consideration are the price of the house on the one hand, and the risk free rate on the other. In order to be able to extract the model parameters from historical time series, we assume that the house under consideration is an average house rather than a particular one, such that the corresponding price is represented by a general house price index. Since there is no such index, we define it to be the average of the inflation index [2] and the property price index [3]. This is based on the assumptions that the price of the house and the price of the property make up for one half of the total amount each. Moreover, we assume that the risk free rate in our model is represented by the overnight rate. For both, the general house price index and the overnight rate, we have time series at hand that cover a time period of 30 years. In particular, the N time steps in our model precisely correspond to a time period of $T = 30$ years. Since we have assumed that the house price index follows a geometrical random walk, the corresponding drift μ and the corresponding volatility σ can be computed as the arithmetical mean of the first and the second moment of the returns for the given time series, respectively. However, for the risk free rate the situation is more complicated. Since we have assumed that it follows a mean reversion random walk, we have to determine the mean $\overline{R}$, the drift α, and the standard deviation β with the help of a best fit. Therefore, we split our original time series into a number of different time series, where each new time series still covers a range of 30 years. This way we obtain a historical process R_n^{TS} for the risk free rate, where the new time series constitute the paths of the process. Having the process R_n^{TS} at hand we take $(\alpha, \beta, \overline{R})$ to be the minimizer of the following error function,

$$\Phi(\alpha, \beta, \overline{R}) = \frac{1}{2N} \left(\sum_{n=1}^{N} \left(\mathbb{E}[R_n] - \mathbb{E}\left[R_n^{TS}\right] \right)^2 \right)^{\frac{1}{2}} + \frac{1}{2N} \left(\sum_{n=1}^{N} \left| \mathbb{V}\mathrm{ar}[R_n] - \mathbb{V}\mathrm{ar}\left[R_n^{TS}\right] \right| \right). \tag{3.1}$$

We note that our original time series for the risk free rate shows oscillations with comparatively long periods, such that our new time series are strongly correlated (see figure 1). Consequently, our method systematically underestimates the standard deviation β.

We have obtained our numerical results with the help of a C programm. Due to efficient usage of memory it will run even on old machines. However, it takes about 5000 seconds on a 1.4 GHz Pentium machine to calculate the 4^N paths of a tree with $N = 15$ such that less than 2 MB RAM occupied.

In what follows, we investigate the influence of several model parameters on the expected values for the exercise time and the utilities. From figure 2 we can see that the credit spread $\overline{S}$ only has a minor influence on the expected exercise time. Since an increase in $\overline{S}$ makes buying the house more expensive, the expected utility for buying the house decreases with increasing $\overline{S}$. Figure 3 shows that the expected exercise time increases with increasing initial interest rate R_0. In particular, for $R_0 \ll \overline{R}$ we have $\tau = 0$ whereas for $R_0 \gg \overline{R}$ we have $\tau = N$. Of course, this is an artefact from modelling the risk free rate as a mean reversion process. For $R_0 \ll \overline{R}$ we know for sure that the interest rate will rise and thus buying the house becomes more expensive. Consequently, buying the

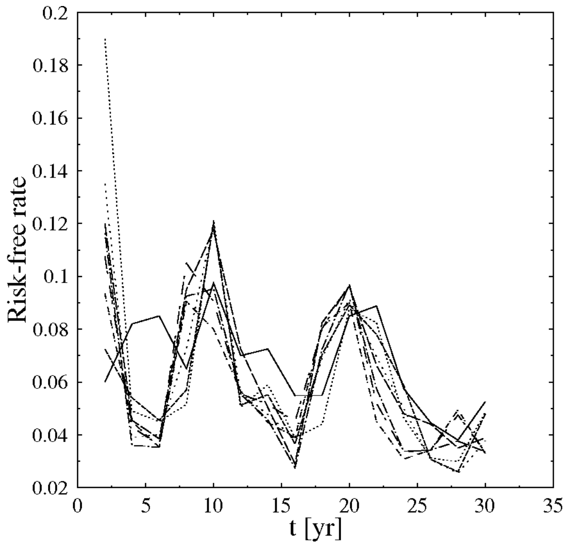

Figure 1: Time Series for the risk free rate. The data have been extracted from the overnight rates [1].

house immediately is an optimal exercise strategy. On the contrary, for $R_0 \gg \overline{R}$ the interest rate will fall and thus buying the house becomes less expensive. Consequently, buying the house as late as possible is an optimal exercise strategy. From the expression for $\overline{V}$ we immediately see that the utility for renting the house exclusively increases with increasing R_0. Interestingly, the graph of the expected utility for buying the house has a global minimum at some $R_0 > 0$. Moreover, it intersects the graph of the expected utility for renting the house exclusively twice. When R_0 is in the range between the two intersection points, renting the house exclusively is more attractive than buying it, whereas the contrary is true otherwise. Figure 4 shows that the parameter p that determines the correlation between the house price index and the risk free rate only has a minor influence on the expected exercise time as well as on the expected utilities. Figure 5 shows that the expected exercise time decreases with increasing parameter γ. Bearing in mind the definition of γ, it is perfectly intuitive that an increasing rent makes buying the house more attractive. Of course, the expected utilities decrease with increasing parameter γ, too. Moreover, we see that the graph of the expected utility for buying the house intersects the graph of the expected utility for renting the house exclusively. When γ is in the range to the left of the intersection point, renting the house exclusively is more attractive than buying it, whereas the contrary is true otherwise. Contrasting the dependence on γ, figure 6 shows that the expected exercise time increases with increasing δ. Bearing in mind the definition of δ, it is also perfectly intuitive that increasing maintenance costs make buying the house less attractive. Moreover, we see that the graph of the expected utility for buying the house intersects the graph of the expected utilitiy for renting the house exclusively. When δ is in the range to the left of the intersection point, buying the house is more attractive than renting it exclusively, whereas the contrary is true otherwise. Figure 7 shows that the expected exercise time decreases with increasing standard deviation β. Moreover, we see that the graph of the expected utility for buying the house intersects the graph of the expected utilitiy for renting the house exclusively. When β is in the range to the left of the intersection point, renting the house exclusively is more attractive than buying it, whereas the contrary is true otherwise. Finally, figure 8 shows that the expected exercise time increases with an increasing number of credit payments K. Due to the credit spread $\overline{S}$, the expected utility for buying the house decreases with increasing K. This is clear since in our model an increase in K corresponds to an extension of the time period in which the credit debt is repayed, and consequently enforces the effect of the credit spread. However, a decrease in K increases the height of the frequent credit payments, such that in reality the financial situation of the investor may give a bound on K. Moreover, we see that the graph of the expected utility for buying the house intersects the graph of the expected utilitiy for renting the house exclusively. When K is in the range to the left of the intersection point, buying the house is more attractive than renting it exclusively, whereas the contrary is true otherwise.

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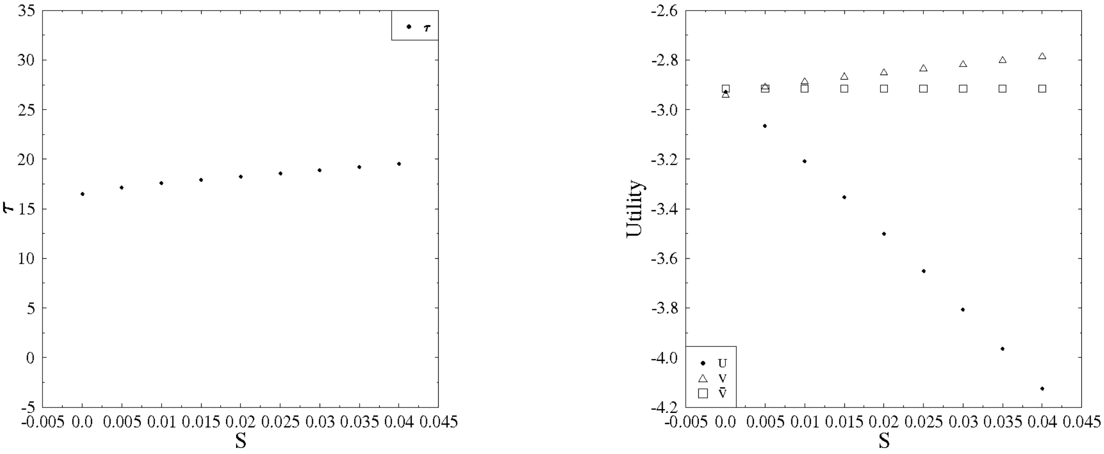

Figure 2: Variation of $\overline{S}$, where $T = 30\,\mathrm{yr}$, $N = 15$, $K = 15$, $I_0 = 1$, $\mu = 0.04775\,\mathrm{yr}^{-1}$, $\sigma = 0.06435\mathrm{yr}^{-1}$, $R_0 = 0.05\,\mathrm{yr}^{-1}$, $\overline{R} = 0.025\,\mathrm{yr}^{-1}$, $\alpha = 2.6$, $\beta = 0.06\,\mathrm{yr}^{-1}$, $\overline{S} = 0.02\,\mathrm{yr}^{-1}$, $p = 0.25$, $\delta = 0.05\,\mathrm{yr}^{-1}$, and $\gamma = 0.10\,\mathrm{yr}^{-1}$.

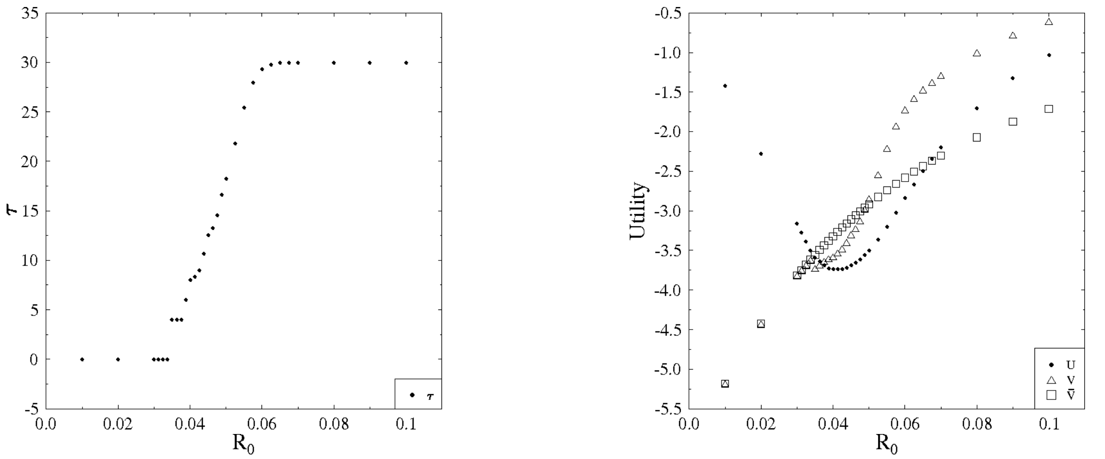

Figure 3: Variation of R_0, where $T = 30\,\mathrm{yr}$, $N = 15$, $K = 15$, $I_0 = 1$, $\mu = 0.04775\,\mathrm{yr}^{-1}$, $\sigma = 0.06435\mathrm{yr}^{-1}$, $\overline{R} = 0.025\,\mathrm{yr}^{-1}$, $\alpha = 2.6$, $\beta = 0.06\,\mathrm{yr}^{-1}$, $\overline{S} = 0.02\,\mathrm{yr}^{-1}$, $p = 0.25$, $\delta = 0.05\,\mathrm{yr}^{-1}$, and $\gamma = 0.10\,\mathrm{yr}^{-1}$.

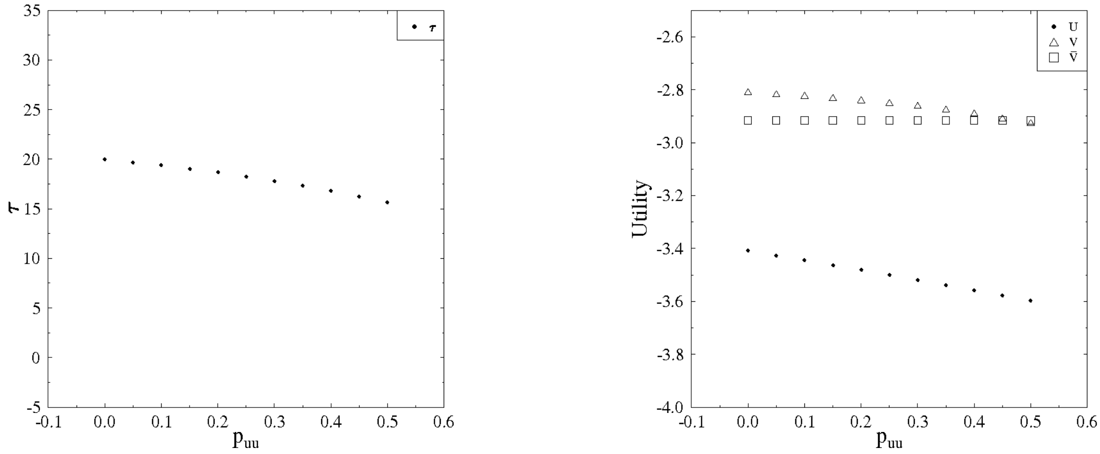

Figure 4: Variation of p, where $T = 30\,\mathrm{yr}$, $N = 15$, $K = 15$, $I_0 = 1$, $\mu = 0.04775\,\mathrm{yr}^{-1}$, $\sigma = 0.06435\mathrm{yr}^{-1}$, $R_0 = 0.05\,\mathrm{yr}^{-1}$, $\overline{R} = 0.025\,\mathrm{yr}^{-1}$, $\alpha = 2.6$, $\beta = 0.06\,\mathrm{yr}^{-1}$, $\overline{S} = 0.02\,\mathrm{yr}^{-1}$, $\delta = 0.05\,\mathrm{yr}^{-1}$, and $\gamma = 0.10\,\mathrm{yr}^{-1}$.

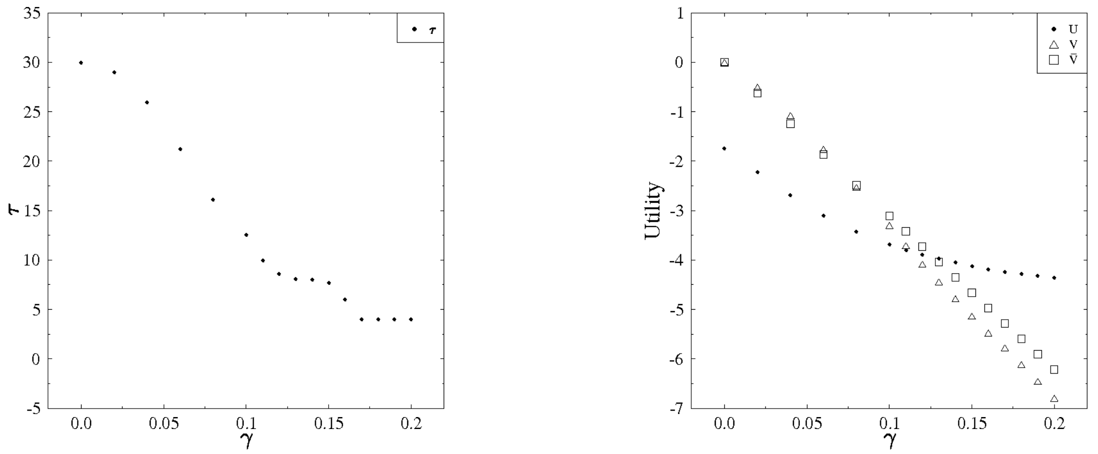

Figure 5: Variation of γ, where $T = 30\,\mathrm{yr}$, $N = 15$, $K = 15$, $I_0 = 1$, $\mu = 0.04775\,\mathrm{yr}^{-1}$, $\sigma = 0.06435\mathrm{yr}^{-1}$, $R_0 = 0.045\,\mathrm{yr}^{-1}$, $\overline{R} = 0.025\,\mathrm{yr}^{-1}$, $\alpha = 2.6$, $\beta = 0.06\,\mathrm{yr}^{-1}$, $\overline{S} = 0.02\,\mathrm{yr}^{-1}$, $p = 0.25$, and $\delta = 0.05\,\mathrm{yr}^{-1}$.

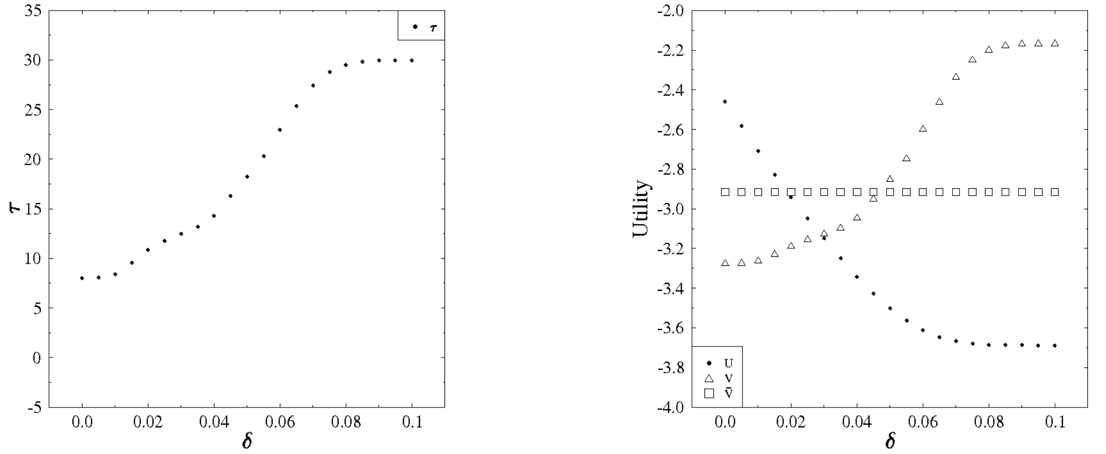

Figure 6: Variation of δ, where $T = 30\,\mathrm{yr}$, $N = 15$, $K = 15$, $I_0 = 1$, $\mu = 0.04775\,\mathrm{yr}^{-1}$, $\sigma = 0.06435\mathrm{yr}^{-1}$, $R_0 = 0.05\,\mathrm{yr}^{-1}$, $\overline{R} = 0.025\,\mathrm{yr}^{-1}$, $\alpha = 2.6$, $\beta = 0.06\,\mathrm{yr}^{-1}$, $\overline{S} = 0.02\,\mathrm{yr}^{-1}$, $p = 0.25$, and $\gamma = 0.10\,\mathrm{yr}^{-1}$.

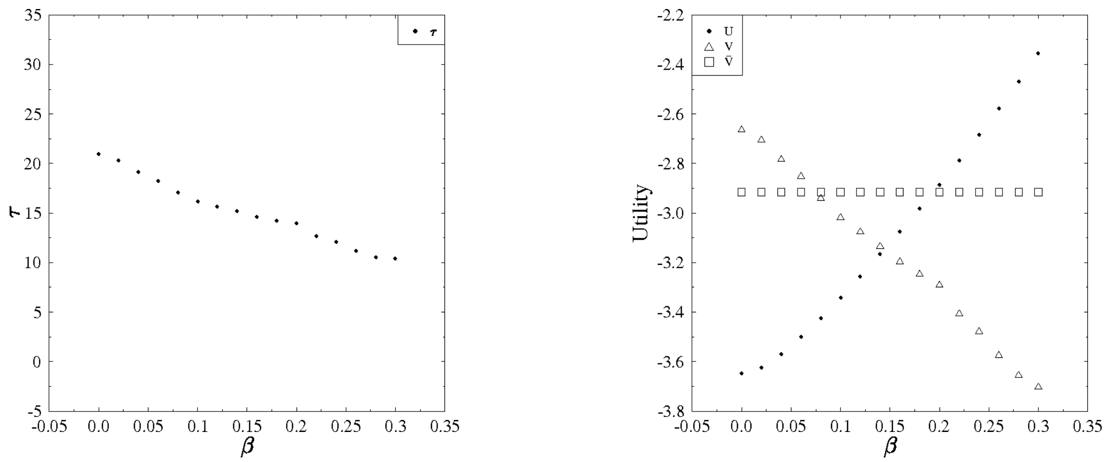

Figure 7: Variation of β, where $T = 30\,\mathrm{yr}$, $N = 15$, $K = 15$, $I_0 = 1$, $\mu = 0.04775\,\mathrm{yr}^{-1}$, $\sigma = 0.06435\mathrm{yr}^{-1}$, $R_0 = 0.05\,\mathrm{yr}^{-1}$, $\overline{R} = 0.025\,\mathrm{yr}^{-1}$, $\alpha = 2.6$, $\overline{S} = 0.02\,\mathrm{yr}^{-1}$, $p = 0.25$, $\delta = 0.05\,\mathrm{yr}^{-1}$, $\gamma = 0.10\,\mathrm{yr}^{-1}$.

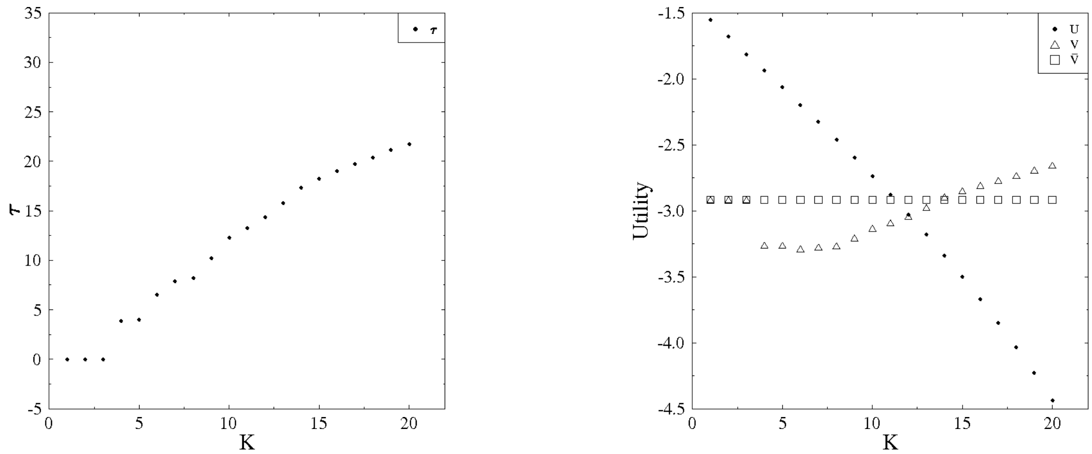

Figure 8: Variation of K, where $T = 30\,\mathrm{yr}$, $N = 15$, $I_0 = 1$, $\mu = 0.04775\,\mathrm{yr}^{-1}$, $\sigma = 0.06435\mathrm{yr}^{-1}$, $R_0 = 0.05\,\mathrm{yr}^{-1}$, $\overline{R} = 0.025\,\mathrm{yr}^{-1}$, $\alpha = 2.6$, $\beta = 0.06\,\mathrm{yr}^{-1}$, $\overline{S} = 0.02\,\mathrm{yr}^{-1}$, $p = 0.25$, $\delta = 0.05\,\mathrm{yr}^{-1}$, $\gamma = 0.10\,\mathrm{yr}^{-1}$.