$\mathfrak{t}$-dense subgroups of topological Abelian groups

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Abstract

For a subgroup H of a topological abelian group G denote by group $\mathcal{S}(H)$ the set of all sequences of integers (u_n) such that $u_n h \to 0$ for every $h \in H$; H is called $\mathfrak{t}$-*dense* if $\mathcal{S}(H) = \mathcal{S}(G)$. Motivated by a question of Raczkowski we explore the existence of small (with respect to size or measure) $\mathfrak{t}$-dense subgroups of topological abelian groups.

1 Introduction

For an Abelian topological group G and a sequence of integers $\underline{u} = (u_n) \in \mathbb{Z}^{\mathbb{N}}$ we denote by $t_{\underline{u}}(G)$ the subgroup consisting of the elements $x \in G$ (called *topologically* $\underline{u}$*-torsion*) such that $u_n x \to 0$ in G [12, 14]. For every subgroup H of G we define $\mathfrak{t}_G(H) = \bigcap\{t_{\underline{u}}(G) : \underline{u} \in \mathbb{Z}^{\mathbb{N}},\ H \leq t_{\underline{u}}(G)\}$ and we say that H is $\mathfrak{t}$-*closed* (resp. $\mathfrak{t}$-*dense*) if $H = \mathfrak{t}_G(H)$ (resp. $\mathfrak{t}_G(H) = G$) [12]. For every sequence $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$ the subgroups of the form $t_{\underline{u}}(G)$ are called *basic* $\mathfrak{t}$-*closed* subgroups of G.

The $\mathfrak{t}$-closure of subgroups of topological Abelian groups G was introduced in [13] in the framework of an appropriate Galois correspondence between subgroups of G and subgroups of $\mathcal{Z} := \mathbb{Z}^{\mathbb{N}}$ so that the Galois closed subgroups of G are precisely the $\mathfrak{t}$-closed ones (for another approach to this matter see [15]).

Basic $\mathfrak{t}$-closed subgroups of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ are studied in [2, 3, 6, 7, 12, 13, 14, 15, 18, 19, 23, 27, 29, 31] for their important applications to questions regarding number

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theory and totally bounded group topologies on the integers that make a given sequence $\underline{u}$ converge to 0. In particular, [6, Theorem 2] shows that *every countable subgroup H of $\mathbb{T}$ is basic $\mathfrak{t}$-closed*. On the other hand, it is proved in [14] that the weaker property of having all cyclic subgroups $\mathfrak{t}$-closed characterizes $\mathbb{T}$ among all *non-discrete locally compact groups*. A description of the locally compact Abelian groups G where every cyclic subgroup is $\mathfrak{t}$-dense, is given in [12, Theorem 4.12].

In this paper we consider the problem of the existence of small (with respect to size or measure) $\mathfrak{t}$-dense subgroups of arbitrary non-discrete Hausdorff Abelian groups G.

The motivation comes from a question raised by Raczkowski [31, Question 1] that, in terms of the $\mathfrak{t}$-closure, can be formulated as follows:

Question 1.1 Is it true that a Haar-null subgroup H of $\mathbb{T}$ is never $\mathfrak{t}$-dense?

A negative answer to Question 1.1 is given in [2] where, under the assumption of Martin's Axiom, $2^{\mathfrak{c}}$-many measure-zero $\mathfrak{t}$-dense subgroups of $\mathbb{T}$ are produced (cf. [2, Theorem 4.1]).

J. Hart and K. Kunen [23] recently obtained a negative answer to Question 1.1 in ZFC. For a topological group G and a filter $\mathcal{F}$ of infinite subsets of the naturals $\mathbb{N}$, they consider the subset of G defined as $\mathcal{D}_{\mathcal{F}}^G := \bigcup\{t_{\underline{u}}(G) : \underline{u} \in \mathcal{F}\}$. Clearly, $\mathcal{D}_{\mathcal{F}}^G$ is a subgroup of G when G is Abelian and the following theorem holds:

Theorem 1.2 [23, Theorem 1.6] *For every infinite compact group G there is an appropriate filter $\mathcal{F}$ such that $\mathcal{D}_{\mathcal{F}}^G$ is a Haar nullset of G. Moreover, if G is not totally disconnected, then $\mathcal{D}_{\mathcal{F}}^G$ is a $\mathfrak{t}$-dense subset of G.*

By Theorem 1.2 every Abelian non totally disconnected compact group G has a $\mathfrak{t}$-dense, Haar null subgroup of G (namely $\mathcal{D}_{\mathcal{F}}^G$). In fact, one can extend this property to all non-discrete locally compact Abelian groups (cf. Theorem 4.1 below).

The technical tools needed to face $\mathfrak{t}$-density are prepared in §2, pointing out the relation between topological properties of Hausdorff (locally compact) Abelian groups and topologically $\underline{u}$-torsion elements for some suitably chosen sequences $\underline{u} \in \mathcal{Z}$. In §3 we study the size of $\mathfrak{t}$-dense subgroups of locally compact Abelian groups. In Corollary 3.7 we prove that locally compact Abelian groups that are either totally disconnected or admit non-compact elements have countable (hence, measure zero) $\mathfrak{t}$-dense subgroups. Surprisingly, LCA groups G with large connected component $c(G)$ (i.e., $w(c(G)) \geq \mathfrak{c}$) have even a $\mathfrak{t}$-dense cyclic subgroup (Proposition 3.9). On the other hand, under [MA], a LCA group G without countable $\mathfrak{t}$-dense subgroups does not admit even $\mathfrak{t}$-dense subgroups of size less than $\mathfrak{c}$ (Trichotomy Law for $\mathfrak{t}$-dense subgroups, cf. Corollary 3.11).

In §4 we consider small (w.r.t. measure) $\mathfrak{t}$-dense subgroups of Abelian topological groups. In Theorem 4.1 we prove that *every non-discrete locally compact Abelian groups G contains a $\mathfrak{t}$-dense subgroup which is Haar-null on G* The counterexample answering Question 1.1 given in [2] was obtained by constructing (under MA) a $\mathfrak{t}$-dense subgroup H of $\mathbb{T}$ that trivially meets a first category subset F of $\mathbb{T}$ in which the Haar measure λ is concentrated. In Theorem 4.9 we extend this construction to an Abelian topological group G and give necessary and sufficient conditions (in terms of suitable subgroups of G) that guarantee the existence of a proper subgroup H of G avoiding a given first category subset and a given family T_ξ, $\xi < 2^{\mathfrak{c}}$, of proper subgroups of G. Since these conditions are fulfilled by $\mathbb{T}$ under MA, this gives as a corollary [2, Theorem 4.1].

In Theorem 4.12 we generalize the above mentioned property of the circle group $\mathbb{T}$ (in terms of its Haar measure λ) for certain measures on an arbitrary non-discrete Hausdorff Abelian group G. This is used to prove, via Theorem 4.9, that G contains a $\mathfrak{t}$-dense subgroup of measure 0 (see Corollaries 4.17 and 4.18). An important assumption here is that the subgroup nG is of second category in G whenever $nG \neq 0$. In Corollary 4.21 we show that for a compact group G the last condition is equivalent to "nG is open in G whenever $nG \neq 0$" (a complete description of the compact groups satisfying this condition is given in Theorem 4.23).

In §5 we study (following [15]) the counterpart ($\mathfrak{g}$-density) of the notion of $\mathfrak{t}$-density obtained by means of sequences of characters $u_n : G \to \mathbb{T}$ instead of sequences of integers (clearly both notions coincide for $G = \mathbb{T}$). One of the motivations to study $\mathfrak{t}$-dense subgroups of $\mathbb{T}$ is their relation to the totally bounded group topologies on $\mathbb{Z}$ without non-trivial convergent sequences (see [31], [2] and [9]; for the connection of $\mathfrak{g}$-density to totally bounded group topologies on arbitrary abelian groups without non-trivial convergent sequences see [15] and [9]). Further applications in this direction will be given in [4].

1.1 Notation and terminology

The symbols $\mathbb{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}_p$, $p \in \mathbb{P}$, are used for the set of primes, the set of positive integers, the group of integers, the group of rationals and the group of p-adic integers, respectively. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology.

Let G be an Abelian group. The subgroup of torsion (p-torsion) elements of G is denoted by $t(G)$ (resp., $t_p(G)$). For an integer n we denote by $G[n]$ the subgroup of all elements of G such that $nx = 0$ and we denote $\varphi_n(x) := nx$.

For a topological group (G, τ), we denote by $w(G)$ the *weight* of G (i.e., the minimal cardinality of a base for the topology on G). Moreover we will denote by $\mathcal{U}_\tau$ the filter of 0-neighborhoods in G and by $\chi(G)$ the *character* of G (that is, the minimal cardinality of a basis of $\mathcal{U}_\tau$). We denote by $c(G)$ the connected component of G.

For $p \in \mathbb{P}$ an element $x \in G$ is *quasi-p-torsion* if either the cyclic subgroup $\langle x \rangle$ of G generated by x is a finite p-group or $\langle x \rangle$, when endowed with the induced topology, is isomorphic to $\mathbb{Z}$ equipped with the p-adic topology τ_p ([33], see also [17, Chap. 4]). The subset of quasi-p-torsion elements of G is a subgroup and is denoted by $td_p(G)$. We denote by $td(G)$ the subgroup of G consisting of all elements x such that either x is torsion or the induced topology of $\langle x \rangle$ is non-discrete and linear (i.e., has a local base at 0 consisting of open subgroups of $\langle x \rangle$).

The symbol $\mathfrak{c}$ stands for the cardinality of the continuum and, as mentioned above, we will denote by $\mathcal{Z}$ the group $\mathbb{Z}^{\mathbb{N}}$ and by $\underline{u}$ sequences $(u_n) \in \mathbb{Z}^{\mathbb{N}}$ and finally, we will set $\mathcal{Z}_0 = \bigoplus_{\mathbb{N}} \mathbb{Z}$. For undefined terms see [17, 20, 25].

2 Topological torsion and $\mathfrak{t}$-density

To every topological Abelian group G we assign here a subgroup $\mathcal{S}(G)$ of $\mathcal{Z}$ carrying essential information about the $\mathfrak{t}$-dense subgroups of G.

Definition 2.1 Let G be a topological Abelian group and

$$\mathcal{S}(G) := \{\underline{u} \in \mathcal{Z} : t_{\underline{u}}(G) = G\} = \{\underline{u} \in \mathcal{Z} : u_n x \to 0 \text{ for every } x \in G\}.$$

Clearly, $\mathcal{S}(G)$ is a subgroup of $\mathcal{Z}$ containing $\mathcal{Z}_0$. The description of the group $\mathcal{S}(G)$ when G is a cyclic subgroup of $\mathbb{T}$ is an open problem set by D. Maharam and A. Stone [6]. In what follows we illustrate the connection of $\mathcal{S}(G)$ to $\mathfrak{t}$-density.

In the whole section let (G, τ) be a Hausdorff topological Abelian group.

Proposition 2.2 *Let H be a subgroup of G. Then*

(a) $\mathcal{S}(G) \subseteq \mathcal{S}(H)$;

(b) if $f : G \to G_1$ is a continuous homomorphism, then $\mathcal{S}(G) \subseteq \mathcal{S}(f(G))$;

(c) $\mathcal{S}(G) = \bigcap\{\mathcal{S}(C) : C \leq G$ is cyclic$\}$;

(d) H is $\mathfrak{t}$-dense if and only if $\mathcal{S}(G) = \mathcal{S}(H)$.

Proof. (a) and (b) are obvious; (c) easily follows from (a) and the definition of $\mathcal{S}(G)$.

(d) By definition, H is $\mathfrak{t}$-dense if and only if for any $\underline{u} \in \mathcal{Z}$ such that $H \leq t_{\underline{u}}(G)$ one has $\underline{u} \in \mathcal{S}(G)$. This occurs if and only if $\underline{u} \in \mathcal{S}(H)$ implies $\underline{u} \in \mathcal{S}(G)$. By item (a) this is equivalent to $\mathcal{S}(H) = \mathcal{S}(G)$. QED

Remark 2.3 (a) G is bounded torsion if and only if $\mathcal{S}(G)$ contains a non-zero constant sequence.

(b) If (G, τ) contains a discrete infinite cyclic subgroup, then $\mathcal{S}(G) = \mathcal{Z}_0$.

(c) If $\mathbb{Z}(p^\infty)$ is endowed with any Hausdorff group topology, then $\mathcal{S}(\mathbb{Z}(p^\infty))$ consists of the sequences $\underline{u} \in \mathcal{Z}$ converging to 0 with respect to the p-adic topology on $\mathbb{Z}$. Indeed, $\mathbb{Z}(p^\infty) = \bigcup_{n=1}^{\infty} \mathbb{Z}(p^n)$ where every $\mathbb{Z}(p^n)$ is finite, hence discrete. Now Proposition 2.2 (c) applies. (Clearly, $\mathbb{Z}(p^\infty)$ can be replaced by any unbounded p-group.)

(d) $\mathcal{S}(\mathbb{T}) = \mathcal{Z}_0$. Indeed, to prove the non-obvious inclusion $(\subseteq)$ let $\underline{u} \in \mathcal{S}(\mathbb{T})$. Then $u_n x \to 0$ for every $x \in \mathbb{T}$ and this implies that $u_n \to 0$ in $\mathbb{Z}$ endowed with the Bohr topology (i.e. the initial topology of all characters of $\mathbb{Z}$). Since the only sequences $\underline{u} \in \mathcal{Z}$ converging to 0 with respect to the Bohr topology are the trivial ones [21], one concludes that $\underline{u} \in \mathcal{Z}_0$, hence $\mathcal{S}(\mathbb{T}) = \mathcal{Z}_0$. For another proof of this fact see [2, Example 2.8] or [12, Example 2.11].

If $G=\mathbb{Z}$, then $\mathcal{S}(\mathbb{Z},\tau)=\{\underline{u}\in\mathcal{Z}: u_n \xrightarrow{\tau} 0\}$. The next remark shows that always $\mathcal{S}(G)$ consists exactly of the sequences $\underline{u}\in\mathcal{Z}$ converging to 0 with respect to a certain group topology on $\mathbb{Z}$.

Notation 2.4 *For $x\in G$, define $j_x:\mathbb{Z}\to G$ by $j_x(n):=nx$ and $j_x^{-1}(\tau)$ as the initial topology on $\mathbb{Z}$ induced by j_x. Put $\tau_{\mathbb{Z}}:=\sup_{x\in G} j_x^{-1}(\tau)$.*

Clearly $mx=0$ in G if and only if $\{m\mathbb{Z}\}$ is a 0-neighborhood basis of $j_x^{-1}(\tau)$.

Remark 2.5 $\tau_{\mathbb{Z}}$ is a group topology on $\mathbb{Z}$ and $\mathcal{S}(G)=\{\underline{u}\in\mathcal{Z}: u_n \xrightarrow{\tau_{\mathbb{Z}}} 0\}$. By Remark 2.3 (a), the topology $\tau_{\mathbb{Z}}$ is Hausdorff if and only if G is not bounded torsion.

Proposition 2.6 *Let (G,τ) be a Hausdorff topological Abelian group. Then*

(a) $\tau_{\mathbb{Z}}$ is linear and non-discrete if and only if $td(G)=G$.

(b) If $td(G)=G$, then $\mathcal{S}(G)$ contains the group $\mathcal{N}:=\bigcap_n(\mathcal{Z}_0+n\mathcal{Z})$, in particular $\mathcal{S}(G)\neq\mathcal{Z}_0$.

Proof. If $td(G)=G$, then $j_x^{-1}(\tau)$ is linear and non-discrete for any $x\in G$, hence coarser than the natural topology τ_0 on $\mathbb{Z}$. Therefore $\tau_{\mathbb{Z}}=\sup_{x\in G} j_x^{-1}(\tau)$ is linear and coarser than τ_0. It follows that

$$\mathcal{S}(G)=\{\underline{u}\in\mathcal{Z}: u_n \xrightarrow{\tau_{\mathbb{Z}}} 0\}\supseteq\{\underline{u}\in\mathcal{Z}: u_n \xrightarrow{\tau_0} 0\}=\mathcal{N}.$$

Suppose now that $\tau_{\mathbb{Z}}$ is linear and non-discrete. Let x be a non-torsion element of G. We have to show that τ induces on $\langle x\rangle$ a linear and non-discrete topology, or equivalently, that $j_x^{-1}(\tau)$ is linear and non-discrete. But this follows from the fact that $j_x^{-1}(\tau)$ is a group topology coarser than $\tau_{\mathbb{Z}}$ and from the following property:

Claim: *Any group topology σ on $\mathbb{Z}$ coarser than the natural topology on $\mathbb{Z}$ is linear.*

Proof We sketch the known proof: Let U be a closed 0-neighborhood in $(\mathbb{Z},\sigma)$. By assumption U contains a subgroup $H\neq 0$. Then its closure $\overline{H}$ is a subgroup contained in U, and $\overline{H}$ is open since $(\mathbb{Z}:H)<\infty$. QED

We now characterize the LCA groups G for which $\mathcal{S}(G)\neq\mathcal{Z}_0$. We recall that an element x of G is called *compact* if x is contained in a compact subgroup of G. The subset $B(G)$ of all compact elements of G is a subgroup of G. If G is a LCA group, then for every $x\in G\setminus B(G)$ the cyclic subgroup $\langle x\rangle$ is infinite and discrete (so topologically isomorphic to $\mathbb{Z}$), see [25] on p.84 and [17] on p.123.

Proposition 2.7 *Let (G,τ) be a LCA group. Then the following conditions are equivalent:*

(1) $\mathcal{S}(G)\neq\mathcal{Z}_0$;

(2) $\mathcal{S}(G)\supseteq\mathcal{N}$;

(3) $td(G) = G$*;*

(4) G *is totally disconnected and* $G = B(G)$*.*

Proof. (4) $\Rightarrow$ (3) If τ is totally disconnected, then τ is linear by [25, (7.7)]. Moreover, τ induces on $\langle x \rangle$, for every non-torsion element $x \in G$, a non-discrete topology if $B(G) = G$. Therefore $td(G) = G$.

(3) $\Rightarrow$ (2) follows from Proposition 2.6(b).

(2) $\Rightarrow$ (1) is obvious

(1) $\Rightarrow$ (4): If there exists an element $x \in G \setminus B(G)$, then the cyclic subgroup $\langle x \rangle$ is infinite and discrete, hence $\mathcal{S}(G) = \mathcal{Z}_0$ by Remark 2.3 (b).

Suppose now that $C = c(G) \neq 0$. Let $\chi : C \to \mathbb{T}$ be a non-trivial continuous character. Then χ is surjective (as $\mathbb{T}$ has no proper connected subgroups $\neq 0$). Therefore $\mathcal{S}(G) \subseteq \mathcal{S}(C) \subseteq \mathcal{S}(\mathbb{T}) = \mathcal{Z}_0$. Here we used Remark 2.3 (d) and item (b) of Proposition 2.2. QED

A description of $\mathcal{S}(G)$ can be obtained from Remark 2.5 and the following proposition.

Proposition 2.8 *Let* σ *be a non-discrete group topology on* $\mathbb{Z}$*. For* $p \in \mathbb{P}$*, let*

$$n_p(\sigma) := \sup\{r \in \mathbb{N} \cup \{0\} : p^r\mathbb{Z} \textit{ is open in } (\mathbb{Z}, \sigma)\};$$

for $\underline{u} \in \mathcal{Z}$ *put*

$$n_p(\underline{u}) := \sup\{r \in \mathbb{N} \cup \{0\} : p^r | u_n \textit{ eventually}\}.$$

If $\sigma \leq \tau$*, then* $n_p(\sigma) \leq n_p(\tau)$ *for every* p*. Moreover, when* σ *is linear* $\mathcal{S}(\mathbb{Z}, \sigma) = \{\underline{u} \in \mathcal{Z} : u_n \xrightarrow{\sigma} 0\}$ *coincides with* $\{\underline{u} \in \mathcal{Z} : n_p(\underline{u}) \geq n_p(\sigma)$ *for every* $p \in \mathbb{P}\}$*.*

It follows that if $\tau_{\mathbb{Z}}$ is linear, then $\mathcal{S}(G) = \{\underline{u} \in \mathcal{Z} : n_p(\underline{u}) \geq n_p(\tau_{\mathbb{Z}})$ for every $p \in \mathbb{P}\}$. We will compare $n_p(\tau_{\mathbb{Z}})$ with the invariants $n_p(G)$ defined as follows.

Definition 2.9 For every prime p let us define

$$n_p(G) := \begin{cases} n & \text{if } t_{\underline{p}}(G) \text{ is bounded torsion of exponent } p^n \\ \infty & \text{if } t_{\underline{p}}(G) \text{ is not bounded torsion} \end{cases}.$$

In other words, $n_p(G) = \sup\{n \in \mathbb{N} \cup \{0\} : (\exists x \in t_{\underline{p}}(G)),\ ord(x) \geq p^n\}$.

Lemma 2.10 *Let* G *be a Hausdorff topological Abelian group and let* $p \in \mathbb{P}$*.*

(a) Then $td(G) \cap t_{\underline{p}}(G) = td_p(G)$*.*

(b) If $td(G) = G$*, then* $n_p(G) \leq n_p(\tau_{\mathbb{Z}})$*.*

(c) If $td(G) = G$ *and* $B(G) = G$*, then* $n_p(G) = n_p(\tau_{\mathbb{Z}})$*.*

Proof. (a) follows from the fact that the only linear Hausdorff group topology on $\mathbb{Z}$ for which $(p^n)_{n\in\mathbb{N}}$ converges to 0 is the p-adic topology τ_p.

(b) If $td(G) = G$, then $\tau_{\mathbb{Z}}$ is non-discrete and linear by Proposition 2.6 (a). Moreover, by (a) we have $t_{\underline{p}}(G) = td_p(G)$. Therefore, if $x \in t_{\underline{p}}(G)$ has infinite order, τ induces on $\langle x\rangle$ the p-adic topology, hence $n_p(\tau_{\mathbb{Z}}) = +\infty$. If $x \in t_{\underline{p}}(G)$ has order p^n for some $n \in \mathbb{N}$, then $n = n_p(j_x^{-1}(\tau)) \leq n_p(\tau_{\mathbb{Z}})$.

(c) Let $k \in \mathbb{N}$ and $k \leq n_p(\tau_{\mathbb{Z}})$. Then $p^k\mathbb{Z}$ is open in $(\mathbb{Z}, \tau_{\mathbb{Z}})$. Therefore $p^k\mathbb{Z}$ is open in $(\mathbb{Z}, j_x^{-1}(\tau))$ for some $x \in G$. The closure $\overline{\langle x\rangle}$ is the (compact) completion of $\langle x\rangle$. By the Chinese Remainder theorem, $\overline{\langle x\rangle}$ is isomorphic to $\Pi_{q\in\mathbb{P}}G_q$ where $G_q = \mathbb{Z}_q$ if $n_q(j_x^{-1}(\tau)) = +\infty$ and $G_q = \mathbb{Z}/p^n\mathbb{Z}$ if $n_q(j_x^{-1}(\tau)) = n$. In particular, G contains a copy of G_p. Hence $n_p(G) \geq n_p(G_p) = n_p(j_x^{-1}(\tau)) \geq k$. QED

Corollary 2.11 *Let G be a Hausdorff topological Abelian group. If $B(G) = G$ and $td(G) = G$, then $\mathcal{S}(G) = \{\underline{u} \in \mathcal{Z} : n_p(\underline{u}) \geq n_p(G)$ for every $p \in \mathbb{P}\}$. If H is a subgroup of G and $n_p(H) = n_p(G)$ for every $p \in \mathbb{P}$, then H is $\mathfrak{t}$-dense in G.*

Proof. The first assertion follows from Proposition 2.8 and Lemma 2.10. For the second, let $\sigma := \tau|_H$. Then by hypothesis $n_p(G) = n_p(H)$. By Lemma 2.10(b) $n_p(H) \leq n_p(\sigma_{\mathbb{Z}})$. Since $\sigma_{\mathbb{Z}} \subseteq \tau_{\mathbb{Z}}$, we have $n_p(\sigma_{\mathbb{Z}}) \leq n_p(\tau_{\mathbb{Z}})$. By Lemma 2.10(c) $n_p(\tau_{\mathbb{Z}}) = n_p(G)$. Hence $n_p(H) = n_p(\sigma_{\mathbb{Z}}) = n_p(G)$. So $\mathcal{S}(H) = \{u \in \mathcal{Z} : n_p(\underline{u}) \geq n_p(\sigma_{\mathbb{Z}}))$ for every $p \in \mathbb{P}\} = \{u \in \mathcal{Z} : n_p(\underline{u}) \geq n_p(G)$ for every $p \in \mathbb{P}\} = \mathcal{S}(G)$. QED

In the rest of this section we compare density and $\mathfrak{t}$-density. Observe that even in compact groups $\mathfrak{t}$-density doesn't imply density and density doesn't imply $\mathfrak{t}$-density: For $p \in \mathbb{P}$, the diagonal $\Delta := \{(x, x) : x \in \mathbb{Z}_p\}$ is a closed proper subgroup of the compact group $\mathbb{Z}_p \times \mathbb{Z}_p$, but Δ is $\mathfrak{t}$-dense in $\mathbb{Z}_p \times \mathbb{Z}_p$ as $\mathcal{S}(\Delta) = \mathcal{S}(\mathbb{Z}_p \times \mathbb{Z}_p) = \{\underline{u} \in \mathcal{Z} : u_n \xrightarrow{\tau_p} 0\}$ (here τ_p denotes the p-adic topology on $\mathbb{Z}$). On the other hand, $\mathbb{Z}(p^\infty)$ is a dense subgroup of $\mathbb{T}$, but $\mathbb{Z}(p^\infty)$ is not $\mathfrak{t}$-dense as follows from Remark 2.3(c),(d); $\mathbb{Z}(p^\infty)$ is even $\mathfrak{t}$-closed in $\mathbb{T}$ since $\mathbb{Z}(p^\infty) = t_{\underline{p}}(\mathbb{T})$. Observe that the topology of $\mathbb{T}$ is not linear (cf. also [12, Theorem 4.12]).

Proposition 2.12 *In a group G endowed with a linear topology every dense subgroup is $\mathfrak{t}$-dense.*

Proof. Let H be a dense subgroup of G. We show that $\mathcal{S}(H) \subseteq \mathcal{S}(G)$. Let $\underline{u} \in \mathcal{S}(H)$ and $x \in G$. Let U be an open subgroup of G. Since H is dense in G, there is an element $y \in H$ with $x - y \in U$. Since $u_n y \to 0$, there exists $n_0 \in \mathbb{N}$ such that $u_n y \in U$ for $n \geq n_0$. Therefore $u_n x = u_n(x - y) + u_n y \in U$ for $n \geq n_0$. Hence $u_n x \to 0$. It follows $\underline{u} \in \mathcal{S}(G)$. QED

Corollary 2.13 *If G is endowed with a linear topology and $td(G) = B(G) = G$, then a subgroup H is $\mathfrak{t}$-dense in G if and only if $n_p(\overline{H}) = n_p(G)$ for every $p \in \mathbb{P}$.*

Proof. By Corollary 2.11 and Proposition 2.12 we have $\mathcal{S}(H) = \mathcal{S}(\overline{H}) = \{\underline{u} \in \mathcal{Z} : n_p(\underline{u}) \geq n_p(\overline{H})$ for every $p \in \mathbb{P}\}$ and $\mathcal{S}(G) = \{\underline{u} \in \mathcal{Z} : n_p(\underline{u}) \geq n_p(G)$ for every $p \in \mathbb{P}\}$. Therefore $n_p(\overline{H}) = n_p(G)$ for every $p \in \mathbb{P}$ if and only if $\mathcal{S}(H) = \mathcal{S}(G)$ if and only if H is $\mathfrak{t}$-dense in G, see 2.2. QED

Remark 2.14 (a) The assumption $B(G) = G$ is satisfied if G is complete and $td(G) = G$.

(b) Observe that in item (c) of Lemma 2.10 the assumption $G = B(G)$ cannot be removed. Indeed, let $G = \mathbb{Z}$ and let τ be the group topology on $\mathbb{Z}$ with 0-neighborhood basis given by $\{pq^n\mathbb{Z}\}_{n=1}^{\infty}$, where p and q are distinct primes. Then $G = td(G)$ and $B(G) \neq G$. Moreover, $n_p(G) = 0$ since $t_{\underline{p}}(G) = \{0\}$. On the other hand, $\tau_{\mathbb{Z}} = \tau$, hence $n_p(\tau_{\mathbb{Z}}) = 1$.

The same example shows that in Corollary 2.13 the assumption $n_p(\overline{H}) = n_p(G)$ cannot be weakened to $n_p(H) = n_p(G)$. Indeed, take $H = (\mathbb{Z}, \tau)$, where τ is the linear topology defined as above, and G the completion $\mathbb{Z}_q \times \mathbb{Z}(p)$ of H.

3 On the size of $\mathfrak{t}$-dense subgroups of the LCA groups

In analogy to the separable topological groups (that contain a dense countable subgroup), we introduce the following

Definition 3.1 A topological abelian group G is said to be $\mathfrak{t}$-*separable* if G admits a countable $\mathfrak{t}$-dense subgroup.

Any subgroup H of G with $\mathcal{S}(H) = \mathcal{Z}_0$ is $\mathfrak{t}$-dense in G. In particular, if G contains a discrete infinite cyclic subgroup $\langle x \rangle$, then $\langle x \rangle$ is $\mathfrak{t}$-dense in G and so G is $\mathfrak{t}$-separable. If (G, τ) is LCA and $x \in G \setminus B(G)$, then τ induces on $\langle x \rangle$ the discrete topology; so we have proved

Proposition 3.2 *If G is LCA and contains non-compact elements, then G is $\mathfrak{t}$-separable.*

Theorem 3.3 *Let (G, τ) be a Hausdorff topological Abelian group. If $td(G) = B(G) = G$, then G is $\mathfrak{t}$-separable.*

Proof. For every prime p choose a countable subgroup H_p of $t_{\underline{p}}(G)$ with the following property. If $t_{\underline{p}}(G)$ is non-torsion, just take any infinite cyclic subgroup H_p of $t_{\underline{p}}(G)$. If $t_{\underline{p}}(G)$ is bounded torsion of exponent p^n take H_p to be any cyclic subgroup of order p^n of $t_{\underline{p}}(G)$. Finally, if $t_{\underline{p}}(G)$ is non-bounded torsion take H_p to be any non-bounded countable subgroup of $t_{\underline{p}}(G)$. Define $H = \bigoplus_p H_p$. Then $n_p(H) = n_p(G)$ for every prime $p \in \mathbb{P}$, hence by Corollary 2.11 H is $\mathfrak{t}$-dense. QED

Remark 3.4 (a) If G is a totally disconnected LCA group, then the conditions $td(G) = G$ and $B(G) = G$ are equivalent as proved in Proposition 2.7.

(b) If G is a totally disconnected LCA group with $B(G) = G$, then G is $\mathfrak{t}$-separable by Theorem 3.3.

If G is separable and the topology of G is linear, then G is $\mathfrak{t}$-separable by Proposition 2.12. On the other hand, $\mathbb{T}$ is a compact separable group, which is not $\mathfrak{t}$-separable. This fact is contained in Corollary 3.7 which is based on Lemma 3.6 (see also [6, Theorem 2]).

In the sequel we denote by $\mathfrak{s}$ the least cardinal such that $\{0,1\}^{\mathfrak{s}}$ is not sequentially compact. Let μ be the least cardinal such that a product of μ sequentially compact spaces is not sequentially compact. It is known that $\aleph_0 < \mu \leq \mathfrak{s} \leq \mathfrak{c}$ (as a product of countably many sequentially compact spaces is sequentially compact while $\{0,1\}^{\mathfrak{c}}$ is not sequentially compact).

The following fact is probably well-known but for reader's convenience we give here a proof.

Lemma 3.5 *A compact group G is sequentially compact if and only if $w(G) < \mathfrak{s}$.*

Proof. It is a theorem of Kuz'minov that for an infinite compact group G there is a continuous surjection from $\{0,1\}^{w(G)}$ onto G (cf. [8, Theorem 1.43] or [25, 25.35]). Therefore, if $w(G) < \mathfrak{s}$, the sequential compactness of $\{0,1\}^{w(G)}$ implies that G is sequentially compact, too. On the other hand, every compact group G contains a homeomorphic copy of $\{0,1\}^{w(G)}$ (cf. [32]) hence the sequential compactness of G implies $w(G) < \mathfrak{s}$. QED

Lemma 3.6 *Let G be a Hausdorff topological Abelian group without infinite discrete cyclic subgroups. Let K be an open subgroup of G such that $K = D + C$, where C is a sequentially compact group and D is a compact totally disconnected group. Let A be a subset of G with $|A| < \mu$. Then there is a sequence $\underline{u} \in \mathcal{N}$ such that $A \subseteq t_{\underline{u}}(G)$.*

Proof. Let $a \in A$. Since by our assumptions $\langle a \rangle$ is finite or non-discrete and K is open, there is some $k_a \in \mathbb{N}$ such that $k_a a \in K$. Let $d(a) \in D$ and $c(a) \in C$ such that $k_a a = d(a) + c(a)$. Put

$$v_n(a) := \begin{cases} \frac{n!}{k_a} & \text{if } n \geq k_a \\ 0 & \text{if } n < k_a \end{cases}$$

and

$$d_n(a) := \begin{cases} v_n(a)d(a) & \text{if } n \geq k_a \\ 0 & \text{if } n < k_a \end{cases}, \quad c_n(a) := \begin{cases} v_n(a)c(a) & \text{if } n \geq k_a \\ n!a & \text{if } n < k_a \end{cases}.$$

Then

$$n!a = d_n(a) + c_n(a) \qquad \text{for every } n \in \mathbb{N} \text{ and every } a \in A$$

where $d_n(a) \in D$ and $c_n(a) \in C_a := C \cup \{m!a : m < k_a\}$ for every $a \in A$. By definition $(v_n(a)) \in \mathcal{N}$, hence Proposition 2.7(2) gives $d_n(a) = v_n(a)d(a) \to 0$ for every $a \in A$.

Moreover, since $|A| < \mu$ and the set C_a is sequentially compact for every $a \in A$, the product $\Pi_{a\in A}C_a$ is sequentially compact. Therefore, the sequence $((c_n(a)_{a\in A})_{n\in\mathbb{N}}$ has a convergent subsequence $(c_{n_m}(a)_{a\in A})_{m\in\mathbb{N}}$. Then $n_m!a = d_{n_m}(a) + c_{n_m}(a)$ converges in K for every $a \in A$. Let $u_m := n_{m+1}! - n_m!$. Then $u_m a = n_{m+1}!a - n_m!a \to 0$ for any $a \in A$, i.e., $A \subset t_{\underline{u}}(G)$. Finally, as $m!$ divides $n_{m+1}!$ and $n_m!$, $m!$ divides u_m, so $\underline{u} \in \mathcal{N}$. QED

Corollary 3.7 *Let G be a LCA group such that $w(c(G)) < \mathfrak{s}$. Then the following conditions are equivalent:*

(a) G is $\mathfrak{t}$-separable.

(b) G has a $\mathfrak{t}$-dense subgroup of size $< \mu$.

(c) G is totally disconnected or contains non-compact elements.

Proof. The implication (c) $\Rightarrow$ (a) was proved in Proposition 3.2, Theorem 3.3 and Remark 3.4 (b) and the implication (a) $\Rightarrow$ (b) is trivial.

To prove (b) $\Rightarrow$ (c), let H be a subgroup of G of size $< \mu$ and suppose that G is not totally disconnected and every element of G is compact. By [25, 24.30] G contains an open compact subgroup K. Clearly K contains the subgroup $c(G)$ and actually $c(G) = c(K)$. Since $w(c(G)) < \mathfrak{s}$ by hypothesis, the group $c(G)$ is sequentially compact. One can find a compact totally disconnected subgroup D of K such that $K = D + c(K)$ (cf. [17, §3.3]). By Lemma 3.6 there is a sequence $\underline{u} \in \mathcal{N}$ such that $H \subset t_{\underline{u}}(G)$ and therefore $\underline{u} \in \mathcal{S}(H)$. Since G is not totally disconnected, $\mathcal{S}(G) = \mathcal{Z}_0$ by Proposition 2.7. Thus H is not $\mathfrak{t}$-dense in G. QED

The next example shows that in the assumptions of Lemma 3.6 one cannot replace the sequential compactness by compactness and that in Corollary 3.7 the assumption $w(c(G)) < \mathfrak{s}$ cannot be removed.

Example 3.8 $\mathbb{T}^{\mathfrak{c}}$ contains a cyclic $\mathfrak{t}$-dense subgroup.

Proof. Put $\mathbb{T}_t := \mathbb{T}$ and $x_t := t$ for $t \in \mathbb{T}$. Then $x := (x_t)_{t\in\mathbb{T}} \in \Pi_{t\in\mathbb{T}}\mathbb{T}_t = \mathbb{T}^{\mathfrak{c}}$. By Remark 2.3(d), for any $\underline{u} \in \mathcal{Z} \setminus \mathcal{Z}_0$ there is an element $t \in \mathbb{T} \setminus t_{\underline{u}}(\mathbb{T})$. Thus $x \notin \Pi_{t\in\mathbb{T}}t_{\underline{u}}(\mathbb{T}_t) = t_{\underline{u}}(\Pi_{t\in\mathbb{T}}\mathbb{T}_t) = t_{\underline{u}}(\mathbb{T}^{\mathfrak{c}})$, i.e. $\mathcal{S}(\langle x\rangle) = \mathcal{Z}_0$. Therefore, $\langle x\rangle$ is $\mathfrak{t}$-dense in $\mathbb{T}^{\mathfrak{c}}$. QED

Now we generalize Example 3.8 as follows:

Proposition 3.9 *If G is a LCA group with $w(c(G)) \geq \mathfrak{c}$, then G has a $\mathfrak{t}$-dense cyclic subgroup.*

Proof. By hypothesis there exists a continuous surjective homomorphism $f : G \to \mathbb{T}^{\mathfrak{c}}$. By Example 3.8, there is an $x \in \mathbb{T}^{\mathfrak{c}}$ such that $\langle x\rangle$ is $\mathfrak{t}$-dense in $\mathbb{T}^{\mathfrak{c}}$. Choose an element $y \in G$

such that $f(y) = x$. Then $\mathcal{S}(\langle y \rangle) \subseteq \mathcal{S}(\langle x \rangle) = \mathcal{Z}_0$ by Proposition 2.2 (b) so that the cyclic subgroup $\langle y \rangle$ is $\mathfrak{t}$-dense in G. QED

The following corollary shows that there is no hope to improve (at least consistently) Proposition 3.9 by taking LCA groups with connected components of smaller weight.

Corollary 3.10 *[MA] Let G be a LCA group such that $w(c(G)) < \mathfrak{c}$. Then the following conditions are equivalent:*

(a) G is $\mathfrak{t}$-separable.

(b) G has a $\mathfrak{t}$-dense subgroup of size $< \mathfrak{c}$.

(c) G is totally disconnected or contains non-compact elements.

Proof. The implication (a) $\Rightarrow$ (b) is obvious and (c) $\Rightarrow$ (a) is true in ZFC by Corollary 3.7. To prove (b) $\Rightarrow$ (c) it suffices to note that $\mu = \mathfrak{c}$ under the assumption of MA (cf. [34],[35]), so that Corollary 3.7 applies again. QED

In other words, if a LCA group G does not admit $\mathfrak{t}$-dense cyclic subgroups, then (under MA), it is $\mathfrak{t}$-separable if and only if it is totally disconnected (if and only if G has a $\mathfrak{t}$-dense subgroup of size $< \mathfrak{c}$).

For reader's convenience we formulate separately the properties established so far, considering as a "parameter" the degree of connectedness of G, namely $w(c(G))$. Recall that when $G \neq B(G)$, then by Proposition 3.2 G is $\mathfrak{t}$-separable in a trivial way.

Corollary 3.11 (**Trichotomy Law for $\mathfrak{t}$-dense subgroups**)

For every LCA group G with $G = B(G)$ the following three possibilities occur:

(1) Under [MA]

- *(a_1) if G is totally disconnected, then G is $\mathfrak{t}$-separable;*
- *(b_1) if $\aleph_0 \leq w(c(G)) < \mathfrak{c}$, then G has no $\mathfrak{t}$-dense subgroups of size $< \mathfrak{c}$ (so in particular, G is not $\mathfrak{t}$-separable);*
- *(c_1) if $w(c(G)) \geq \mathfrak{c}$, then G has a $\mathfrak{t}$-dense cyclic subgroup.*

(2) Under [CH]

- *(a_2) if G is totally disconnected, then G is $\mathfrak{t}$-separable;*
- *(b_2) if $c(G) \neq 0$ is metrizable, then G is not $\mathfrak{t}$-separable;*
- *(c_2) if $c(G)$ is not metrizable, then G has a $\mathfrak{t}$-dense cyclic subgroup.*

Corollary 3.11 (1) answers (at least consistently) the next question raised in [9] in the general context of arbitrary compact abelian groups:

Question 3.12 [9, Question 6.5] What is the minimal size of a dense, $\mathfrak{t}$-dense subgroup of $\mathbb{T}$?

Since by [6, Theorem 2] (for a simpler direct proof see also [12] or [14]) every countable subgroup of $\mathbb{T}$ is $\mathfrak{g}$-closed, under the assumption of CH, the minimal size of a (necessarily dense) $\mathfrak{g}$-dense subgroup of $\mathbb{T}$ is precisely $\mathfrak{c} = 2^{|\mathbb{Z}|}$. On the other hand, the same conclusion is true under the weaker assumption of MA by Corollary 3.11 (b_1).

We can partially answer also the next question raised in [9] in the general context of arbitrary compact abelian groups:

Question 3.13 [9, Question 6.2] For which faithfully indexed sequence $\underline{u} \in Z^{\mathbb{N}}$ one has $|t_{\underline{u}}(\mathbb{T})| = |\mathbb{T}|$?

Indeed, $|t_{\underline{u}}(\mathbb{T})| = \mathfrak{c}$ for every sequence $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$ such that $\frac{u_{n+1}}{u_n} \to +\infty$ [2, Theorem 3.1]; whereas boundedness of the ratio $\frac{u_{n+1}}{u_n}$ yields $|t_{\underline{u}}(\mathbb{T})| \leq \omega$ ([2, Theorem 3.3]). More precisely, for every sequence $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$ satisfying a linear recurrence relation of second order $u_n = a_n u_{n-1} + b_n u_{n-2}$ the subgroup $t_{\underline{u}}(\mathbb{T})$ has size $\mathfrak{c}$ if and only if $\frac{u_{n+1}}{u_n}$ is unbounded (cf. [3, Theorem 4.12]).

4 Construction of $\mathfrak{t}$-dense subgroups of measure zero

Until now we have studied the question whether G contains a "small" $\mathfrak{t}$-dense subgroup, where "small" was intended with respect to size. Now we study this question for a different concept of smallness, namely a set is small if it is a Haar-null set.

The next theorem answers negatively Question 1.1 for all non-discrete LCA groups.

Theorem 4.1 *Let G be a non-discrete LCA group and let λ be a Haar measure of G. Then G contains a $\mathfrak{t}$-dense λ-null subgroup.*

Proof. (i) If G is totally disconnected, then Proposition 3.2 and Theorem 3.3 apply.

(ii) Suppose that the connected component $c(G)$ of G is either non-compact or non-open and non-trivial.

If $c(G)$ is non-compact, then G contains a homeomorphic copy of $\mathbb{R}$. In particular, G contains non-compact elements. By Proposition 3.2 there exists a $\mathfrak{t}$-dense subgroup which is countable, hence λ-null.

Assume that $c(G) \neq 0$ is compact and non-open. We show that $c(G)$ is a $\mathfrak{t}$-dense λ-null subgroup. By our assumption $c(G)$ is contained in an open compact subgroup K. As $c(G)$ is not open but closed, $(K : c(G))$ is infinite. Since $\lambda(K) < \infty$, this implies that $\lambda(c(G)) = 0$. By Proposition 2.7 $\mathcal{S}(c(G)) = \mathcal{Z}_0$. Therefore $c(G)$ is $\mathfrak{t}$-dense in G.

(iii) Consider the case when G has an open compact connected component $c(G)$. Let $f : c(G) \to \mathbb{T}$ be a continuous surjective homomorphism. By Theorem 1.2 $\mathbb{T}$ contains a λ-null subgroup A which is $\mathfrak{t}$-dense in $\mathbb{T}$. Put $H := f^{-1}(A)$. Since $\lambda(f^{-1}(\mathbb{T})) \neq 0$ (as

$f^{-1}(\mathbb{T}) = c(G)$ is open in G) and $\lambda(f^{-1}(\mathbb{T})) < +\infty$ (as $c(G)$ is compact), $B \mapsto \lambda(f^{-1}(B))$ defines a Haar measure on $\mathbb{T}$ and therefore H is a λ-null set. Since $\mathcal{S}(H) \subset \mathcal{S}(f(H)) = \mathcal{S}(A) = \mathcal{Z}_0$, H is $\mathfrak{t}$-dense. QED

We consider now the existence of $\mathfrak{t}$-dense subgroups of arbitrary topological Abelian groups and give another counterexample answering Question 1.1 under MA.

In the spirit of [11] we give the following definition.

Definition 4.2 Let $\mathfrak{a}$ be an uncountable cardinal number. A subspace Y of a topological space X is of *first* $\mathfrak{a}$*-category* in X if it can be written as the union of fewer than $\mathfrak{a}$ nowhere dense subsets of X. A subspace Y of X is of *second* $\mathfrak{a}$*-category in* X if it is not of first $\mathfrak{a}$-category in X.

Obviously, a space is of first (second) category in the usual sense if and only if it is of first (second) $\aleph_1$-category.

Moreover, if $\mathfrak{a}$, $\mathfrak{b}$ are uncountable cardinal numbers such that $\mathfrak{a} \leq \mathfrak{b}$, then every space of first $\mathfrak{a}$-category is obviously of first $\mathfrak{b}$-category, and every second $\mathfrak{b}$-category space is also of second $\mathfrak{a}$-category.

Definition 4.3 Let $\mathfrak{a}$ be an uncountable cardinal number. A topological group G is said to be $\mathfrak{a}$*-bounded* if for every non-empty open set U in G there is a subset $|A| < \mathfrak{a}$ of G with $G = A + U$.

Obviously, G is *totally bounded* if and only if G is $\aleph_0$-bounded.

Lemma 4.4 *Let $\mathfrak{a}$ be an uncountable cardinal number and let G be a topological Abelian group of second $\mathfrak{a}$-category. If F is a subset of first $\mathfrak{a}$-category in G and H is a proper subgroup of G, then $H \cup F \neq G$.*

Proof. Suppose that $H \cup F = G$. Let $x \in G \setminus H$. Then $x + H \subset G = H \cup F$. Since $(x + H) \cap H = \emptyset$, it follows that $x + H \subset F$. Hence $x + H$ is of first $\mathfrak{a}$-category in G. Thus H is of first $\mathfrak{a}$-category in G and consequently $H \cup F$ is of first $\mathfrak{a}$-category in G, a contradiction since $G = H \cup F$ is of second $\mathfrak{a}$-category. QED

Definition 4.5 Let X, Y be topological spaces. A map $f : X \to Y$ is said to be *almost open* if for every non-empty open subset A of X the closure $\overline{f(A)}$ has an inner point.

Remark 4.6 If $f : X \to Y$ is a homomorphism between topological groups, then f is almost open if and only if for any 0-neighborhood V in X the closure $\overline{f(V)}$ is a 0-neighborhood in Y.

Proposition 4.7 *Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map.*
(a) Then the following conditions are equivalent:

(1) f is almost open.

(2) If E is nowhere dense in Y, then $f^{-1}(E)$ is nowhere dense in X.

(3) If E is closed and nowhere dense in Y, then $f^{-1}(E)$ is closed and nowhere dense in X.

(4) If $\mathcal{O}$ is open dense in Y, then $f^{-1}(\mathcal{O})$ is open dense in X.

(b) Suppose f almost open. If E is of first $\mathfrak{a}$-category in Y, then $f^{-1}(E)$ is of first $\mathfrak{a}$-category in X.

Proof. (a) (1) $\Rightarrow$ (2): Let $E \subset Y$. Since $f^{-1}(\overline{E})$ is closed, we have $U := \overline{f^{-1}(E)} \subset f^{-1}(\overline{E})$ and $f(U) \subset \overline{E}$. If U had an inner point, then $\overline{f(U)}$, as well as $\overline{E}$, would have an inner point.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4): Let $\mathcal{O}$ be open dense in Y. Then $E := Y \setminus \mathcal{O}$ is closed and nowhere dense in Y so that, by hypothesis, $f^{-1}(E)$ is closed and nowhere dense in X. Hence $f^{-1}(\mathcal{O})$ is dense in X.

(4) $\Rightarrow$ (1): Let U be a non-empty open set in X. By way of contradiction, if $\overline{f(U)}$ was without inner points, then $\mathcal{O} := Y \setminus \overline{f(U)}$ should be open dense. Therefore $f^{-1}(\mathcal{O})$ is dense in X, by hypothesis. Then $X \setminus \overline{U}$ is dense since $f^{-1}(\mathcal{O}) = X \setminus f^{-1}(\overline{f(U)}) \subset X \setminus \overline{U}$, a contradiction.

(b) Follows from (a). QED

The following proposition can be considered as a straightforward generalization of [26, Proposition 32.11(a)].

Proposition 4.8 *Let $\mathfrak{a}$ be an uncountable cardinal number and let G be an $\mathfrak{a}$-bounded topological group. Let H be a topological group and let $f : G \to H$ be a homomorphism such that $f(G)$ is of second $\mathfrak{a}$-category in H. Then f is almost open.*

Proof. Let $\mathcal{O}$ be a non-empty open subset of G. Choose a subset A of G with $|A| < \mathfrak{a}$ and $G = A + \mathcal{O}$. Since $f(G)$ is of second $\mathfrak{a}$-category in H and $f(G) = \bigcup_{a\in A}(f(a) + f(\mathcal{O}))$, there is an $a \in A$ such that the closure of $f(a) + f(\mathcal{O})$ has an inner point. Then $\overline{f(\mathcal{O})}$ has an inner point. QED

Theorem 4.9 *Let $\mathfrak{a}$ be an uncountable cardinal number and let G be an $\mathfrak{a}$-bounded topological Abelian group. Then the following conditions are equivalent:*

(1) For any $n \in \mathbb{N}$, $nG = 0$ or nG is of second $\mathfrak{a}$-category in G.

(2) For every subset F of first $\mathfrak{a}$-category in G and every proper subgroup T of G there is an $x \in G \setminus T$ such that $\langle x \rangle \cap F \subseteq \{0\}$.

(3) *For every subset F of first $\mathfrak{a}$-category in G and every family $(T_\xi)_{\xi<\mathfrak{a}}$ of proper subgroups of G there is a subgroup H of G such that $H \not\subseteq T_\xi$ for every $\xi < \mathfrak{a}$ and $H \cap F \subseteq \{0\}$.*

Proof. If $G = \{0\}$, then there is nothing to prove. Hence we can assume that $G \neq \{0\}$. Then the set $I := \{n \in \mathbb{N} : nG \neq 0\} \neq \emptyset$.

(1) $\Rightarrow$ (3): We first define by transfinite induction a family $(F_\xi)_{\xi<\mathfrak{a}}$ of subsets of first $\mathfrak{a}$-category in G and a family $(x_\xi)_{\xi<\mathfrak{a}}$ with $x_\xi \in G \setminus (T_\xi \cup F_\xi)$ for $\xi < \mathfrak{a}$. By Proposition 4.8 the map φ_n is almost open for any $n \in I$. Hence, by Proposition 4.7 (b), $\varphi_n^{-1}(F)$ is of first $\mathfrak{a}$-category in G. Therefore $F_0 := \bigcup_{n\in I} \varphi_n^{-1}(F)$ is of first $\mathfrak{a}$-category in G. By Lemma 4.4 there exists an element $x_0 \in G \setminus (T_0 \cup F_0)$.

Suppose that $\xi < \mathfrak{a}$ and $(x_\eta)_{\eta<\xi}$ has been defined. Then the subgroup $H_\xi := \langle x_\eta : \eta < \xi \rangle$ has size $|H_\xi| \leq \aleph_0 \cdot \xi < \mathfrak{a}$ so that $F + H_\xi = \bigcup_{x\in H_\xi}(F + x)$ is of first $\mathfrak{a}$-category. As before we obtain by Propositions 4.8 and 4.7 (b) that $F_\xi := \bigcup_{n\in I} \varphi_n^{-1}(F + H_\xi)$ is of first $\mathfrak{a}$-category. Hence Lemma 4.4 entails that there exists an element $x_\xi \in G \setminus (T_\xi \cup F_\xi)$.

Let us show that the subgroup $H := \langle x_\xi : \xi < \mathfrak{a} \rangle$ has the desired properties. Obviously, for any $\xi < \mathfrak{a}$ we have $H \not\subseteq T_\xi$ since $x_\xi \in H \setminus T_\xi$. Suppose now that $x \in F \cap H$ and $x \neq 0$. Then x can be written as $x = \sum_{i=1}^n k_i x_{\xi_i}$ where $\xi_1 < \ldots < \xi_n$ and $k_i \in \mathbb{Z}$. Since $x \neq 0$, we may assume that $k_n x_{\xi_n} \neq 0$, hence $k_n \in I$. Then $k_n x_{\xi_n} = x - \sum_{i=1}^{n-1} k_i x_{\xi_i} \in F + H_{\xi_n}$ and therefore $x_{\xi_n} \in \varphi_{k_n}^{-1}(F + H_{\xi_n}) \subset F_{\xi_n}$, a contradiction. So we have proved that $F \cap H \subseteq \{0\}$.

(3) $\Rightarrow$ (2): (2) is a special case of (3) taking $T_\xi := T$ for any $\xi < \mathfrak{a}$.

(2) $\Rightarrow$ (1): Let $n \in \mathbb{N}$ with $nG \neq 0$. Then $T := G[n] \neq G$. Suppose that nG is of first $\mathfrak{a}$-category in G. Applying (2) with $F := nG$ we can find an element $x \in G \setminus T$ such that $\langle x \rangle \cap F \subseteq \{0\}$. Then $nx \in \langle x \rangle \cap nG \subseteq \{0\}$, hence $nx = 0$, i.e., $x \in G[n] = T$, a contradiction. QED

It is well-known – and contained in Theorem 4.12 – that the Haar measure on $\mathbb{T}$ is concentrated on a set F of first category, i.e. $\mathbb{T} \setminus F$ is a null set. Using this fact we get as an application of Theorem 4.9 the following corollary.

Corollary 4.10 *Let λ be a Haar measure on $\mathbb{T}$. Let $\mathfrak{a}$ be an uncountable cardinal number such that $\mathbb{T}$ is of second $\mathfrak{a}$-category. Then for every family $(T_\xi)_{\xi<\mathfrak{a}}$ of proper subgroups of $\mathbb{T}$ there is a λ-null subgroup H of $\mathbb{T}$ such that $H \not\subseteq T_\xi$ $(\xi < \mathfrak{a})$.*

Proof. Let F be a set of first category such that $\lambda(\mathbb{T} \setminus F) = 0$. As $\mathbb{T}$ is compact, it is $\mathfrak{a}$-bounded. Moreover, since $\mathbb{T}$ is divisible and, by hypothesis, of second $\mathfrak{a}$-category, every subgroup $n\mathbb{T}$, $n \in \mathbb{N}$, is obviously of second $\mathfrak{a}$-category. Then by Theorem 4.9 (1) $\Rightarrow$ (3) there is a proper subgroup H of $\mathbb{T}$ such that $F \cap H \subseteq \{0\}$ and $H \not\subseteq T_\xi$ for any $\xi < \mathfrak{a}$. To conclude it now suffices to note that $H \subset (F \cap H) \cup (\mathbb{T} \setminus F) \cap H$. QED

One can apply the last corollary to the family $(T_\xi)_{\xi<\mathfrak{c}} = \{t_{\underline{u}}(\mathbb{T}) : \underline{u} \in \mathscr{Z} \setminus \mathscr{Z}_0\}$ and under the assumption of MA obtain a negative answer to Question 1.1 (this was done in [2, Claim]).

The rest of this section is dedicated to an elaboration of the latter result. This is done in Corollaries 4.17 and 4.18 which are based on Theorem 4.12. For this, we need the following definition.

Definition 4.11 A $[0, +\infty]$-valued translation invariant Borel measure on a topological Abelian group G is said *locally finite* if G contains a 0-neighborhood of finite measure.

The following result can be found in [5] for diffused σ-finite Borel measures on metrizable and separable spaces. Here we provide a proof in a more general context that covers also the case of Haar measures. It should be mentioned here that the hypothesis of the next theorem, as well as that of Theorem 4.16, yields local total boundedness of the groups considered (cf. Proposition 4.13).

Theorem 4.12 *Let G be a non-discrete Hausdorff topological Abelian group and let μ be a $[0, +\infty]$-valued locally finite translation invariant σ-additive Borel measure on G. Then there exists an F_σ-set F of first category in G such that every $\aleph_1$-bounded subset of $G \setminus F$ is contained in a μ-null set.*

In Theorem 4.12 the set $G \setminus F$ is a μ-null set if μ is inner regular with respect to $\aleph_1$-bounded Borel sets, i.e. if for any Borel set A we have $\mu(A) = \sup\{\mu(B) : B$ is an $\aleph_1$-bounded Borel set, $B \subseteq A\}$. In particular, $\mu(G \setminus F) = 0$ if G is $\aleph_1$-bounded.

The following propositions will be used to reduce the proof of Theorem 4.12 to the case when G is first countable and separable.

The next fact – essentially contained in Theorem 443H of [22] – shows that a topological group carries a certain type of measures only if it is a subgroup of a locally compact group.

Proposition 4.13 *Let (G, τ) be a topological Abelian group and let μ be a $[0, +\infty]$-valued translation invariant σ-additive Borel measure on G. Suppose that μ is strictly positive on every non-empty open set. Then every open $V \in \mathcal{U}_\tau$ with $\mu(V + V) < \infty$ is totally bounded. In particular, G is locally totally bounded if μ is locally finite.*

Proof. Let V be an open 0-neighbourhood which not totally bounded. We show that $\mu(V + V) = \infty$. By our assumption there exists a sequence (x_n) in V and $W \in \mathcal{U}_\tau$ such that $x_i - x_j \notin W$ for every $i \neq j$. Let $U \in \mathcal{U}_\tau$ be open and symmetric with $U \subseteq V$ and $U + U \subset W$. Then $x_n + U$ are pairwise disjoint and contained in $V + U \subset V + V$. Hence $\mu(V + V) \geq \sum_{n=1}^{\infty} \mu(x_n + U) = \sum_{n=1}^{\infty} \mu(U) = \infty$. QED

Proposition 4.14 *Let σ, τ be group topologies on G with $\sigma \subset \tau$. Let $U_0 \in \mathcal{U}_\sigma$ be totally bounded in (G, τ). Then $\mathcal{V} := \{\overline{V}^\sigma : V \in \mathcal{U}_\tau\}$ is a basis of $\mathcal{U}_\sigma$. In particular, the identity map from (G, τ) onto (G, σ) is almost open.*

Proof. (i) First, we will show that every $U \in \mathcal{U}_\sigma$ contains a set of $\mathcal{V}$.

Since a topological group is regular, U contains a $V \in \mathcal{U}_\sigma$ which is closed in (G, σ). Hence $V \in \mathcal{U}_\tau$ with $\overline{V}^\sigma = V \subset U$.

(ii) Second, we will prove that $\mathcal{V} \subseteq \mathcal{U}_\sigma$.

Suppose that $\mathcal{V} \subsetneq \mathcal{U}_\sigma$, i.e. there exists $V \in \mathcal{U}_\tau$ such that $\overline{V}^\sigma \notin \mathcal{U}_\sigma$. Then pick a net $(x_\alpha)_{\alpha \in A} \in G \setminus \overline{V}^\sigma$ such that $x_\alpha \to 0$ in (G, σ). Since $U_0 \in \mathcal{U}_\sigma$, $x_\alpha \in U_0$ eventually. Moreover, since U_0 is totally bounded in (G, τ), there exists finitely many $z_1, \ldots, z_n \in G$ such that $U_0 \subseteq \bigcap_{i=1}^n z_i + V_0$ where $V_0 \in \mathcal{U}_\tau$ satisfies $V_0 - V_0 \subseteq V$. Then one of the sets, say $z_i + V_0$, contains a cofinal part $(y_\beta)_{\beta \in B}$ of $(x_\alpha)_{\alpha \in A}$. Hence, $y_\delta - y_\gamma \in V$ for $\delta, \gamma \in B$. Let $\eta \in B$. Then $y_\eta \notin \overline{V}^\sigma = \bigcap_{U \in \mathcal{U}_\sigma} (V + U)$. Hence there exists $U \in \mathcal{U}_\sigma$ such that $y_\eta \notin V + U$. On the other hand, since $y_\beta \to 0$ in (G, σ) there exists $\xi \in B$ such that $y_\xi \in U$. Then we have $y_\eta = (y_\eta - y_\xi) + y_\xi \in V + U$, a contradiction.

(iii) The last statement follows from (ii) and Remark 4.6. QED

Corollary 4.15 *Let (G, τ) be a locally totally bounded group such that $\overline{\{0\}}^\tau$ is not open.*

Then there exists a first countable group topology σ on G coarser than τ with the following properties:

(i) there exists $U \in \mathcal{U}_\sigma$ which is totally bounded in (G, τ);

(ii) the identity map from (G, τ) onto (G, σ) is almost open;

(iii) $\overline{\{0\}}^\sigma$ is not open in (G, τ);

Proof. Let $U_0 \in \mathcal{U}_\tau$ be totally bounded and let $U_n \in \mathcal{U}_\tau$ be symmetric with $U_n + U_n \subset U_{n-1}$ $(n \in \mathbb{N})$. Then $(U_n)_{n \in \mathbb{N}}$ is a basis for a first countable group topology σ on G coarser than τ. Moreover, we have (i) and, by Proposition 4.14, (ii).

We will prove that (iii) holds for a suitable choice of the sequence U_n.

If τ is not linear, then we can choose U_0 not containing any open subgroup of G. Then the subgroup $N := \overline{\{0\}}^\sigma = \bigcap_{n=0}^\infty U_n$ is not open.

If τ is linear, then we can choose U_n as open subgroups of G with $U_n \supsetneqq U_{n+1}$ $(n \in \mathbb{N} \cup \{0\})$. This is possible since $\overline{\{0\}}^\tau$ is not open and so it is different from U_n for every $n \in \mathbb{N} \cup \{0\}$. Now $(U_0 : N) \geq (U_0 : U_n) \to +\infty$. As U_0 is totally bounded, we get that N is not open. QED

Proof of Theorem 4.12

case 1. Suppose that G contains an open $U \in \mathcal{U}_\tau$ such that $\mu(U) = 0$. If M is an $\aleph_1$-bounded subset, then there is a sequence (a_n) in G such that $M \subset \bigcup_{n=1}^\infty (a_n + U)$. The set $\bigcup_{n=1}^\infty (a_n + U)$ is a μ-null set. So $F = \emptyset$ has the desired properties.

case 2. Suppose now that every open set in $\mathcal{U}_\tau$ has positive measure. Then (G, τ) is locally totally bounded by Proposition 4.13. Let σ be a group topology on G according to Corollary 4.15. By Corollary 4.15(i), there is an open symmetric $U \in \mathcal{U}_\sigma$ which is totally bounded in (G,τ). For $n \in \mathbb{N}$, let $U^{(n)}$ the sum of n copies of U. The sets $U^{(n)}$ are totally bounded subsets of (G,σ) and therefore separable since σ is first countable. It follows that $H := \bigcup_{n=1}^{\infty} U^{(n)}$ is a separable subgroup of (G,σ); moreover, H is open in (G,σ), hence closed.

We now show that $(H, \sigma_{|H})$ contains an F_σ-set F_0 of first category such that $\mu(H \setminus F_0) = 0$. Since $\overline{\{0\}}^{\sigma}$ is not open, there are an open set $V \in \mathcal{U}_{\sigma_{|H}}$ and an element $x \in G$ such that $V \cap (x+V) = \emptyset$ and $V \cup (x+V) \subset U$. Then $2\mu(V) = \mu(V) + \mu(x+V) = \mu(V \cup (x+V)) \leq \mu(U)$. From this it follows that there is a decreasing sequence $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_{\sigma_{|H}}$ of open sets with $\mu(U_n) \leq 2^{-n}$, $n \in \mathbb{N}$. Let $X = \{x_n : n \in \mathbb{N}\}$ be a dense subset in $(H, \sigma_{|H})$. Then, for $n \in \mathbb{N}$, the open set $E_n := \bigcup_{k\in\mathbb{N}}(x_k + U_{n+k})$ is dense in $(H, \sigma_{|H})$, hence $F_n := H \setminus E_n$ is closed and nowhere dense in $(H, \sigma_{|H})$; moreover, $\mu(E_n) \leq \sum_{k=1}^{\infty} \mu(x_k + U_{n+k}) \leq \sum_{k=1}^{\infty} 2^{-n-k} = 2^{-n}$. Let $E := \bigcap_{n=1}^{\infty} E_n$, then $\mu(E) = 0$. Therefore $F_0 := H \setminus E = \bigcup_{n=1}^{\infty}(H \setminus E_n) = \bigcup_{n=1}^{\infty} F_n$ is an F_σ-set of first category in (G,σ) and $\mu(H \setminus F_0) = \mu(E) = 0$.

Choose y_i, $i \in I$, such that $G = \bigcup_{i\in I}(y_i + H)$ is the disjoint union of the cosets $y_i + H$, $i \in I$. Then for every $n \in \mathbb{N}$ the set $B_n := \bigcup_{i\in I}(y_i + F_n)$ is closed and nowhere dense in (G,σ). Hence $F := \bigcup_{i\in I}(y_i + F_0) = \bigcup_{n\in\mathbb{N}} B_n$ is an F_σ-set of first category in (G,σ). It follows from 4.15 and 4.7 (b) that F is also an F_σ-set of first category in (G,τ).

Let now M be an $\aleph_1$-bounded subset of $G \setminus F$. Then there is a countable subset I_0 of I such that $M \subset \bigcup_{i\in I_0}(y_i + H)$. Therefore M is contained in the null set $\bigcup_{i\in I_0}[(y_i + H) \setminus F] = \bigcup_{i\in I_0}(y_i + E)$. QED

Theorem 4.16 *Let $\mathfrak{a}$ be an uncountable cardinal number, let G be an $\mathfrak{a}$-bounded non-discrete Hausdorff topological Abelian group and let μ be a $[0,+\infty]$-valued locally finite translation invariant σ-additive Borel measure on G which is inner regular with respect to $\aleph_1$-bounded Borel sets. Suppose that for any $n \in \mathbb{N}$, $nG = 0$ or nG is of second $\mathfrak{a}$-category in G. Then for every family $(T_\xi)_{\xi<\mathfrak{a}}$ of proper subgroups of G there is a μ-null subgroup H of G such that $H \not\subseteq T_\xi$ for every $\xi < \mathfrak{a}$.*

Proof. It is an immediate consequence of Theorem 4.9 (1) $\Rightarrow$ (3) and 4.12. QED

Corollary 4.17 *Let G be a $\mathfrak{c}$-bounded non-discrete Hausdorff topological Abelian group and let μ be a $[0,+\infty]$-valued locally finite translation invariant σ-additive Borel measure on G which is inner regular with respect to $\aleph_1$-bounded Borel sets. Suppose that for any $n \in \mathbb{N}$, $nG = 0$ or nG is of second $\mathfrak{c}$-category in G. Then G contains a $\mathfrak{t}$-dense μ-null subgroup.*

Proof. As in the proof of Corollary 4.10, we can enumerate by $\{T_\xi : \xi < \mathfrak{c}\}$ all proper subgroups of G of the form $t_{\underline{u}}(G)$. Then we can apply Theorem 4.16 with $\mathfrak{a} = \mathfrak{c}$. QED

This corollary generalizes Corollary 4.10 as $\mathbb{T}$ (as well as any compact group) is of second $\mathfrak{c}$-category under MA.

Corollary 4.18 *Let G be $\aleph_1$-bounded non-discrete Hausdorff topological Abelian group and let μ be a $[0,+\infty]$-valued locally finite translation invariant σ-additive Borel measure on G. Suppose that for any $n \in \mathbb{N}$, $nG = 0$ or nG is of second category in G. Under the continuum hypothesis G contains a $\mathfrak{t}$-dense μ-null subgroup.*

Note that the condition "$nG = 0$ or nG is of second category in G" of Corollary 4.18 is not necessary to produce $\mathfrak{t}$-dense μ-null subgroups. Indeed, in the compact group $G = \mathbb{Z}_p^{\mathbb{N}}$, the subgroup pG is nowhere dense but G is $\mathfrak{t}$-separable by Corollary 3.7.

Question 4.19 Does Corollary 4.18 hold in ZFC?

4.1 When the construction fails

All the examples of $\mathfrak{t}$-dense subgroups we gave in the previous section were based on the inductive construction contained in Theorem 4.9. An essential assumption there was that the subgroup nG is of second $\mathfrak{a}$-category in G whenever $nG \neq 0$. We are going to see that for a compact group G the last condition is equivalent to " nG is open in G whenever $nG \neq 0$". More generally, one can prove the following fact.

Proposition 4.20 *Let $\mathfrak{a}$ be an uncountable cardinal number and let G be a σ-compact group of second $\mathfrak{a}$-category. Then a closed subgroup H of G is open if and only if H is of second $\mathfrak{a}$-category in G.*

Proof. Let H be a closed subgroup of G. If H is not open, then H has no inner points and therefore is nowhere dense (since H is closed). Hence H is of first $\mathfrak{a}$-category.

If H is open, then the σ-compactness of G implies $(G : H) \leq \aleph_0$. Therefore there are countably many elements $\{x_1, \ldots, x_n, \ldots\}$ in G such that $G = \bigcup_{i=1}^{\infty}(x_i + H)$. Assume that H is of first $\mathfrak{a}$-category in G, i.e. $H = \bigcup_{\eta<\xi} F_\eta$ where F_η are closed nowhere dense subsets of G and $\xi < \mathfrak{a}$. It follows that $G = \bigcup_{i=1}^{\infty}(\bigcup_{\eta<\xi} x_i + F_\eta)$, hence G is of first $\mathfrak{a}$-category in G as well, a contradiction. QED

Corollary 4.21 *Let $\mathfrak{a}$ be an uncountable cardinal number, G be a compact group of second $\mathfrak{a}$-category and $n \in \mathbb{N}$. Then nG is open if and only if nG is of second $\mathfrak{a}$-category.*

This follows from Proposition 4.20 since nG is a compact, hence closed subgroup of G. This corollary together with condition (1) of Theorem 4.9 motivates the characterization of compact groups where the subgroups nG are open whenever $nG \neq 0$. To do this, we briefly recall some notions that will be helpful in the proof of the next theorem.

For an Abelian group G and a prime $p \in \mathbb{P}$, $s_p(G)$ will denote the *socle* of $t_p(G)$, i.e., the subgroup of elements x of G such that $px = 0$. Clearly, $s_p(G)$ is a linear space over the finite field $\mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z}$; its dimension is called the *p-rank* of G and is denoted by $r_p(G)$ (or simply by r_p when no confusion arises). One can prove that $r_p(G)$ coincides with the size of all maximal independent sets in G consisting of elements whose order is a power of p (cf. [25, Theorem A11]).

For an Abelian topological group (G,τ) we denote by $\widehat{G}$ the Pontryagin dual of G.

Proposition 4.22 [17, Proposition 3.3.15] *Let G be a LCA group. Then for every $p \in \mathbb{P}$, $s_p(G) \cong \widehat{\widehat{G}/p\widehat{G}}$. If G is compact, then $s_p(G) \cong \mathbb{Z}(p)^\alpha$ for some cardinal α.*

Theorem 4.23 *Let G be an infinite compact Abelian group. Then the following conditions are equivalent:*

(a) For any $n \in \mathbb{N}$, $nG = 0$ or nG is open in G.

(b) Either $G \cong \mathbb{Z}(p)^\alpha$ for some prime p and some infinite cardinal α, or $G \cong \prod_p \mathbb{Z}_p^{n_p} \times N$ where $n_p \in \mathbb{N} \cup \{0\}$ for every prime p and N is a subgroup of G containing $c(G)$ such that $N/c(G) \cong \prod_{p\in\mathbb{P}} F_p$ where F_p is a finite p-group for every prime p.

Proof. $(b) \Rightarrow (a)$ Fix an $n \in \mathbb{N}$. If $G \cong \mathbb{Z}(p)^\alpha$, then $nG = 0$ if $p|n$; otherwise $nG = G$.

Assume that $G \cong A \times N$ where $A := \prod_p \mathbb{Z}_p^{n_p}$ and n_p, N are as in (b). Then $nG \cong nA \times nN \neq 0$ and we have to prove that nG is open in G. Since nG is closed by the compactness of G, this is equivalent to prove that $(G : nG) < \infty$. Since $G/nG \cong A/nA \times N/nN$, it suffices to prove that $|A/nA| < \infty$ and $|N/nN| < \infty$.

Let $n = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ be the factorization of n in distinct primes. Since $\mathbb{Z}_q$ is p-divisible for every prime $p \neq q$, $A/nA \cong \prod_{i=1}^t (\mathbb{Z}_{p_i}/{p_i}^{\alpha_i}\mathbb{Z}_{p_i})^{n_{p_i}} \cong \prod_{i=1}^t \mathbb{Z}({p_i}^{\alpha_i})^{n_{p_i}}$, hence it is finite.

Since $c(G)$ is divisible (as every compact connected Abelian group), $c(G) \subseteq nN$, hence it follows that $(N : nN) = (N/c(G) : n(N/c(G)))$. Then the isomorphism $N/c(G) \cong \prod_{q\in\mathbb{P}} F_q$ yields immediately that

$$(N : nN) = \left| \prod_{q\in\mathbb{P}} F_q : n \prod_{q\in\mathbb{P}} F_q \right| < \infty.$$

$(a) \Rightarrow (b)$: Assume first that there exists $m \in \mathbb{N}$ such that $mG = 0$. We can assume that m is the exponent of G. Let $p \in \mathbb{P}$ such that $p|m$. Then $m = pk$ with $k \in \mathbb{N}$. We will prove that $k = 1$. Since $k < m$, we have that $kG \neq 0$ and consequently kG is open in G. Therefore $|G/kG| < +\infty$. As $kG \subseteq G[p]$, $|G/G[p]| < +\infty$. The isomorphism $G/G[p] \cong pG$ entails $|pG| < +\infty$, hence pG is not open. So $pG = 0$, i.e. $p = m$. Then $G = s_p(G) \cong \mathbb{Z}(p)^\alpha$ by Proposition 4.22.

From now on we are left with the case when $nG \neq 0$ for every $n \in \mathbb{N}$.

Let $K = G/c(G)$. As a compact totally disconnected group, K admits a decomposition of the form $K = \prod_{p\in\mathbb{P}} K_p$ where $K_p = td_p(K)$ are compact totally disconnected groups (cf. [17, Example 4.1.3 (a)]). Let $p \in \mathbb{P}$. By hypothesis pG is open in G and this entails that also pK is open in K. Note that

$$K/pK = \prod_{q\in\mathbb{P}} K_q/pK_q = K_p/pK_p$$

as K_q is p-divisible for every prime $p \neq q$. Since K/pK is compact and discrete, it follows that $|K/pK| = |K_p/pK_p| < \infty$. On the other hand, by [25, Theorem A.27]

$$K_p/pK_p \cong \mathbb{Z}(p)^{r_p} \text{ for some } r_p \in \mathbb{N} \cup \{0\}$$

so that by Proposition 4.22 $\widehat{K_p}$ has finite socle. Since $\widehat{K_p}$ is a p-group (cf. [17, Lemma 3.5.8]), by [17, Exercise 3.8.10] it decomposes as $\widehat{K_p} \cong \mathbb{Z}(p^\infty)^{n_p} \times F_p$ where F_p is a finite p-group (note that, being a divisible group, $\mathbb{Z}(p^\infty)^{n_p}$ is a direct summand of $\widehat{K_p}$). Finally, by applying Pontryagin duality, one gets $K_p \cong \mathbb{Z}_p^{n_p} \times F_p$. Hence $G/c(G) \cong (\prod_p \mathbb{Z}_p^{n_p}) \times (\prod_p F_p)$.

Let us consider now the short exact sequence

$$0 \longrightarrow c(G) \longrightarrow G \longrightarrow G/c(G) \longrightarrow 0$$

and let $\pi : G \to G/c(G)$ be the canonical homomorphism. If we set $N := \pi^{-1}(\prod_{p \in \mathbb{P}} F_p)$, then $N \geq \ker(\pi) = c(G)$ and gives rise to the short exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow G/N = \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{n_p} \longrightarrow 0. \tag{1}$$

By applying Pontryagin duality to (1) one gets the exact sequence

$$0 \longrightarrow \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)^{n_p} \longrightarrow G^* \longrightarrow N^* \longrightarrow 0 \tag{2}$$

that splits since $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)^{n_p}$ is divisible. Hence (1) splits too and G decomposes as $G = \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{n_p} \times N$ with $c(G) \leq N$ and $N/c(G) \cong \prod_{p \in \mathbb{P}} F_p$. QED

Remark 4.24 By applying Pontryagin duality it is easy to see that for every unbounded compact Abelian group G satisfying the equivalent conditions of Theorem 4.23 one has $r_p(\widehat{G}) < \infty$ for every $p \in \mathbb{P}$.

Theorem 4.23 shows that there are many metrizable compact Abelian groups where the subgroups $nG \neq 0$ are not of second $\mathfrak{a}$-category (as $\mathbb{Z}_p^{\mathbb{N}}$), hence we cannot apply Corollaries 4.18 and 4.17 to produce $\mathfrak{t}$-dense subgroups.

Remark 4.25 [16, Example 2] shows that in Theorem 4.23 (b) $c(G)$ is not always a topological direct summand of N (hence of G).

5 $\mathfrak{g}$-dense subgroups of LCA groups

Let us recall the following

Definition 5.1 [15, Definition 2.1] For an Abelian topological group G and a sequence $(u_n) = \underline{u}$ in $\widehat{G}$, set

$$s_{\underline{u}}(G) := \{x \in G : u_n(x) \longrightarrow 0 \text{ in } \mathbb{T}\}. \tag{3}$$

Since $\mathbb{Z} = \widehat{\mathbb{T}}$, one has that $s_{\underline{u}}(\mathbb{T}) = t_{\underline{u}}(\mathbb{T})$ for every sequence $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$.

Analogously to the case of the $\mathfrak{t}$-closure, for a subgroup H of a topological group G the $\mathfrak{g}$-*closure* of H is defined as $\mathfrak{g}_G(H) := \bigcap\{s_{\underline{u}}(G) : \underline{u} \in \widehat{G}^{\mathbb{N}},\ H \leq s_{\underline{u}}(G)\}$ (see [15, Definition 2.4]) and H is said, respectively, $\mathfrak{g}$-*closed* if $H = \mathfrak{g}_G(H)$, $\mathfrak{g}$-*dense* if $\mathfrak{g}_G(H) = G$.

The properties of the $\mathfrak{g}$-closure in the category of topological Abelian groups are studied in [15] where a characterization of the class of maximally almost periodic Abelian groups is obtained in terms of the $\mathfrak{g}$-closure (*a topological Abelian group G is maximally almost periodic if and only if every cyclic subgroup of G is $\mathfrak{g}$-closed*, [15, Theorem 1.2]).

For a compact Abelian group K the measure properties of the subgroups $s_{\underline{u}}(K)$ are studied in [10] where the following theorem is proved:

Theorem 5.2 [10, Lemma 3.10] *Let K be a compact Abelian group and let λ be the Haar measure of K. Let $\underline{u} = (u_n)_{n<\omega}$ be a faithfully indexed sequence in $\widehat{K}$. Then $s_{\underline{u}}(K)$ is a λ-measurable subgroup of G and $\lambda(s_{\underline{u}}(K)) = 0$.*

The above theorem was extended to all locally compact Abelian groups in [31, Theorem 6] as follows

Theorem 5.3 *Let G be a LCA group and let λ denote a Haar measure on G. Let $(u_n)_{n<\omega}$ be a faithfully indexed sequence in $\widehat{G}$ with no accumulation points. Then $s_{\underline{u}}(G)$ is a locally λ-null*[1] *subgroup of G.*

Since for a LCA group G, the group $\widehat{G}$ is σ-compact if and only if G is metric (see [8, 3.14]), taking into account that locally λ-null sets of a σ-compact group are λ-null, as a consequence of Theorem 5.3 one obtains :

Corollary 5.4 [31, Corollary 8] *If G is LCA and metric and $s_{\underline{u}}(\widehat{G})$ is defined as above, then $\lambda(s_{\underline{u}}(\widehat{G})) = 0$.*

In Theorem 4.1 and Corollary 4.17 we answered a general version of Question 1.1 by proving that Hausdorff (locally compact) Abelian groups admit $\mathfrak{t}$-dense subgroups of measure-zero. In the next corollary we apply Theorem 4.16 to answer another general version of Question 1.1 regarding $\mathfrak{g}$-density.

Corollary 5.5 *Let G be a $\mathfrak{c}$-bounded non-discrete Hausdorff topological Abelian group and let λ be a $[0, +\infty]$-valued locally finite translation invariant σ-additive Borel measure on G which is inner regular with respect to $\aleph_1$-bounded Borel sets. Suppose that for any $n \in \mathbb{N}$, $nG = 0$ or nG is of second $\mathfrak{c}$-category in G. If $|\widehat{G}| \leq \mathfrak{c}$, then G contains a $\mathfrak{g}$-dense λ-null subgroup.*

Proof. Since $|\widehat{G}| \leq \mathfrak{c}$, we can enumerate by $\{T_\xi : \xi < \mathfrak{c}\}$ all proper subgroups of G of the form $s_{\underline{u}}(G)$. Then we can apply Theorem 4.16 with $\mathfrak{a} = \mathfrak{c}$. QED

[1] i.e., $A \cap F$ is λ-null for every compact set F in $\widehat{G}$.

Corollary 5.6 *Let K be an infinite compact Abelian group of second $\mathfrak{c}$-category satisfying the equivalent conditions of Theorem 4.23 and let λ be the Haar measure of K. If $w(K) \leq \mathfrak{c}$, then K admits a dense, $\mathfrak{g}$-dense measurable subgroup H such that $\lambda(H) = 0$.*

Proof. We first observe that the group K is separable since by [8, Theorem 3.1] the minimal cardinality $d(K)$ of a dense subset of K (i.e., the *density character* of K) coincides with the minimal cardinal κ such that $2^\kappa \geq w(K)$ while $w(K) \leq \mathfrak{c} = 2^\omega$ by hypothesis. By Corollary 4.21 either $nK = 0$ or nK is of second $\mathfrak{c}$-category for any $n \in \mathbb{N}$. Since $w(K) = |\widehat{K}|$ (cf. [25, Theorem 24.15]), the assumption $w(K) \leq \mathfrak{c}$ allows us to apply Corollary 5.5 to get a $\mathfrak{g}$-dense λ-null subgroup H of K. If H is dense in K, then we are done. If H is not dense in K, then take any countable dense subgroup D of K. Clearly, $D + H$ is a dense, $\mathfrak{g}$-dense subgroup of K with $\lambda(H + D) = 0$ (as countable union of measure-zero subsets of K). QED

In particular, under the assumption of MA, every infinite compact Abelian group K with $w(K) \leq \mathfrak{c}$ satisfying the equivalent conditions of Theorem 4.23, admits a dense, $\mathfrak{g}$-dense measurable subgroup H such that $\lambda(H) = 0$. Since the compact Abelian group K in question may be also *totally disconnected*, this is not covered by the following ZFC-result announced by Hart and Kunen [24]: if a compact Abelian group K is not totally disconnected, then K admits (in ZFC) a dense, $\mathfrak{g}$-dense measurable subgroup H such that $\lambda(H) = 0$.

Corollary 5.6 and Theorem 4.23 allow us to partially answer a question raised in [9] that can be formulated as follows in terms of $\mathfrak{g}$-density (see [15] for the equivalence of the present form of the question with the original one):

Question 5.7 [9, Question 6.1] Let K be an infinite compact Abelian group and let λ be the Haar measure of K. Does K admit a dense, $\mathfrak{g}$-dense measurable subgroup H such that $\lambda(H) = 0$?

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