A Lyapunov theorem for T_s-measures

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1 Introduction

The celebrated classical theorem of A.A. Lyapunov states that the range of a finite dimensional vector measure is compact, and if in addition the measure is atomless, its range is moreover convex.

Many applications spring up from this classical theorem. In optimal control theory it is equivalent to the "bang-bang" principle, see [8, chapter 8 and 12]. The Lyapunov theorem has been used to advantage in Banach space theory, see e.g. [7] (for further information see Notes and remarks, Chapter IX of [6] on p.273). But it's actually in economics that its applications are widely utilized. It has been applied to problems in bargaining and even in politics of dividing land (see the references contained in [1]). It plays a fundamental role for allowing the use of the separation theorems in proving the existence of prices in general equilibrium models, in particular for large economy (in the setting of perfect competition). The importance of the Lyapunov theorem for correspondences is pointed out in [3].

Generalizing the classical Lyapunov theorem, Butnariu and Klement proved in chapter III.14 of their book [5] "Triangular-norm based measures and games with fuzzy coalitions" a Lyapunov type theorem for T_∞-measures and posed in [5, 30.5] the problem whether such a theorem also holds true for T_s-measures, $0 < s < \infty$. As main result of [4], we furnish this Lyapunov type theorem for T_s-measures, $0 < s < \infty$, and for this we make use of the classical Lyapunov theorem valid in the Boolean case and of a representation theorem for T_s-measures.

The concept of a T-measure is based on triangular norms on the unit interval $[0, 1]$. In Menger's [9],[10] and Schweizer–Sklar's [11] opinion, triangular norms (and their dual conorms) are useful in continuum physics and in the theory of probabilistic metric spaces: In such spaces the distance between two points is measured by probability distribution functions rather than by real numbers, and the triangular norms (defined for functions pointwise) allow to generalize the triangle inequality. Our source of information is the book [5] by Butnariu and Klement.

Moreover, in their book [5] the authors emphasize that triangular norm-based measures are powerful tools in exploring the coalitional behaviour in games and they stress that a crucial point in game theoretical considerations is the validity of a Lyapunov type theorem for a large class of triangular norm-based measures. In fact, in this case, as an application, they could extend the existence of Aumann-Shapley values to games based on these measures, provided the range of these measures is compact and convex. These values generalize the well-known Shapley values. As contained in [2], the Shapley value has a very broad spectrum of applications. It applies to political (e.g. voting games) and economic contexts: e.g. it has been used to analyze taxation problems. Following these results, Butnariu and Klement in [5] show that Aumann-Shapley values are also useful in solving real-life problems such as rate problems for services in bulk.

2 The result

We follow the terminology of Butnariu and Klement [5]. A triangular norm (t-norm for short) is a function $T : [0,1] \times [0,1] \to [0,1]$ which is monotone, associative, commutative and satisfies $T(x,1) = x$. If T is a triangular norm, then the corresponding triangular conorm (t-conorm for short) is the function $S : [0,1] \times [0,1] \to [0,1]$ defined by $S(x,y) := 1 - T(1-x, 1-y)$. A special role is played in [5] by the Frank t-norms T_s, $0 \le s \le \infty$, (called in [5] fundamental t-norms) which are defined as follows:

$$T_0(x,y) := x \wedge y$$
$$T_s(x,y) := \log_s[1 + \tfrac{(s^x-1)(s^y-1)}{s-1}] \text{ for } 0 < s < \infty \; s \neq 1$$
$$T_1(x,y) := xy$$
$$T_\infty(x,y) := (x+y-1) \vee 0$$

The t-norms T_s, $0 \le s \le \infty$, are solutions of the functional equation $T(x,y)+S(x,y) = x+y$, and the t-norms T_s, $0 < s < \infty$, are the only continuous solutions of this functional equation satisfying $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$. The t-conorm corresponding to T_s is denoted by S_s.

A T_s-tribe on a set Ω is a σ-sublattice $\mathcal{T}$ of the space $[0,1]^\Omega$ of "Fuzzy subsets of Ω" which contains the zero function, $1-f$ and fT_sg for all f, $g \in \mathcal{T}$. Here fT_sg is pointwise defined by $(fT_sg)(w) := T_s(f(w), g(w))$. A function $\mu : \mathcal{T} \to \mathbb{R}^l$ is called a T_s-measure if $\mathcal{T}$ is a T_s-tribe, $\mu(0) = 0$, $\mu(fT_sg) + \mu(fS_sg) = \mu(f) + \mu(g)$ for all f, $g \in \mathcal{T}$ and $\lim \mu(f_n) = \mu(f)$ whenever $(f_n) \in \mathcal{T}^{\mathbb{N}}$ and $f_n \uparrow f$.

The concepts of T_s-tribes and T_s-measures are natural generalizations of σ-algebras and measures: Identifying sets A with their characteristic functions χ_A, a system $\mathcal{A}$ of $\{0,1\}$-valued functions on Ω is a T_s-tribe iff it is a σ-algebra, and a function $\mu : \mathcal{A} \to \mathbb{R}^l$ on a σ-algebra is a T_s-measure iff it is a σ-additive measure (in the classical sense).

For a T_s-measure μ on $\mathcal{T}$, we define a pseudometric $d_\mu(a,b) := \sup\{\|\mu(f) - \mu(g)\| : f, g \in \mathcal{T},\ a \wedge b \le f \le g \le a \vee b\}$. We say that $\mathcal{T}$ is μ-chained if for every $\varepsilon > 0$ and every f, $g \in \mathcal{T}$ with $f \le g$ there is a finite increasing sequence $f = f_0 \le \cdots \le f_n = g$ in $\mathcal{T}$ such that $d_\mu(f_{i-1}, f_i) \le \varepsilon$ for $i = 1, \ldots n$, or equivalently if for every $\varepsilon > 0$ there are $f_1, \ldots, f_n \in \mathcal{T}$ such that $\sum_1^n f_i = 1$ and $d_\mu(f_i, 0) \le \varepsilon$ for $i = 1, \ldots, n$. This condition is stronger than the condition that μ is atomless, which means that for every $f \in \mathcal{T}$ with $\mu(f) \neq 0$ there is a $g \in \mathcal{T}$ such that $\mu(g) \notin \{0, \mu(f)\}$.

Now we are in a position to formulate and to comment the main result of [4].

THEOREM 1 *Let $s \in (0,\infty)$, $l \in \mathbb{N}$ and $\mu : \mathcal{T} \to \mathbb{R}^l$ a T_s-measure on a T_s-tribe. If $\mathcal{T}$ is μ-chained , then $\mu(\mathcal{T})$ is convex.*

We point out that the assumption that $\mathcal{T}$ is μ-chained cannot be replaced by the weaker condition that μ is atomless, as shown by the following example.

EXAMPLE 1 *Let I be the closed real unit interval . Define $\mu : I \to \mathbb{R}$ by $\mu(\alpha) = \alpha + 1$ if $0 < \alpha \le 1$ and $\mu(0) = 0$. Then μ is an atomless T_s-measure for any $s \in (0,\infty)$, but its range is not convex.*

Moreover, the fact that the condition "$\mathcal{T}$ μ-chained" is equivalent to the convexity of $\mu([f,g])$ for any f, $g \in \mathcal{T}$ with $f \le g$ shows that this assumption of theorem 1 is (in a certain sense) the best possible.

So far we haven't considered compactness of the range. In example 1, the range of μ is not compact. Also if $\mu : \mathcal{T} \to \mathbb{R}^l$ is a T_s-measure such that $\mathcal{T}$ is μ-chained, its range need not be compact as the following example shows.

EXAMPLE 2 *Let $\mathcal{T}$ be the space of $[0,1]$-valued Lebesgue measurable functions on $[0,1]$, λ the Lebesgue measure on $[0,1]$ and μ defined by $\mu(f) := -\int_0^1 f(x)dx + \lambda(suppf)$. Then, for any $s \in (0,\infty)$, $\mathcal{T}$ is a T_s-tribe, μ is a T_s-measure and $\mathcal{T}$ is μ-chained. But $\mu(\mathcal{T})$ is not compact. Indeed, $\sup_{f\in\mathcal{T}} \mu(f) = 1$ and for all $f \in \mathcal{T}$ we have $\mu(f) < 1$.*

THEOREM 2 *Let $s \in (0,\infty)$, $l \in I\!N$ and $\mu : \mathcal{T} \to I\!R^l$ a T_s-measure. If $(\mathcal{T}, d_\mu)$ is complete, then $\mu(\mathcal{T})$ is compact.*

The proofs of theorems 1 and 2 are based on the representation of a T_s-measure μ as a sum of a "support measure" and a "linear integral measure": $\mu(f) = \rho(\text{supp } f) + \int f d\lambda$ for $f \in \mathcal{T}$, where ρ and λ are σ-additive measures on the σ-algebra $\{A \subset \Omega : \chi_A \in \mathcal{T}\}$ of "crisp sets"of $\mathcal{T}$.

The proof of theorems 1 and 2 and of this representation theorem, which answers the question [5, 30.3] of Butnariu and Klement, are contained in [4].

To compare theorems 1 and 2 with the classical Lyapunov theorem we observe that for any σ-additive measure $\mu : \mathcal{A} \to \mathbb{R}^l$ on a σ-algebra the pseudometric space $(\mathcal{A}, d_\mu)$ is complete, and that $\mathcal{A}$ is μ-chained iff μ is atomless.

References

[1] C.D. Aliprantis and K.C. Border: Infinite dimensional analysis, Studies in Economic Theory, Springer, (1994)

[2] R. Aumann: The Shapley value, In Game Theory and Applications, T.Ichiishi, A.Neyman and Y.Tauman (Eds.), Academic Press, pp 158–165 (1990)

[3] A. Avallone and A. Basile: On the Liapunov-Richter theorem in the finitely additive setting, Journal of Mathematical Economics 22 (1993) 557–561

[4] G. Barbieri and H. Weber: A representation theorem and a Lyapunov theorem for T_s-measures: the solution of two problems of Butnariu and Klement, preprint (1998)

[5] D. Butnariu and E.P. Klement: Triangular Norm Based Measures and Games with Fuzzy Coalitions, Kluwer, Acad. Publ., Dordrecht, Holland (1993)

[6] J. Diestel and J.J. Uhl: Vector measures, American Mathematical Society, Providence, Rhode Island (1977)

[7] J.Hagler and C.Stegall: Banach spaces whose duals contain complemented subspaces isomorphic to $C[0,1]$, J. Functional Analysis 13 (1973), 233–251

[8] H. Hermes and J. H. LaSalle: Functional Analysis and Time optimal control, Academic Press (1969)

[9] K.Menger: Probabilistic theories of relations, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 171–180

[10] K.Menger: Probabilistic geometry, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 226–229

[11] B.Schweizer and A.Sklar: Probabilistic metric spaces, North-Holland series in probability and applied mathematics, New York, (1983)

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