HAHN DECOMPOSITION OF MODULAR MEASURES AND APPLICATIONS

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Abstract. We prove a Hahn decomposition for σ-additive modular measures on σ-complete D-lattices. As a consequence, we establish an isomorhism between the space of all bounded real-valued modular measures on a D-lattice and the space of all completely additive measures on a suitable Boolean algebra. Another consequence is a Uhl type theorem concerning relative compactness and convexity of the range of nonatomic modular measures with values in Banach spaces.

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Introduction. Effect algebras have been introduced by D.J. Foulis and M.K. Bennett in 1994 (see [B-F]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arises in quantum physics (see [B-C]) and in Mathematical Economics (see [E-Z], [M], [B-K]), in particular of orthomodular lattices in non-commutative measure theory and MV-algebras in fuzzy measure theory. After 1994, there have been a great number of papers concerning effect algebras (see [D-P] for a bibliography).

In this paper we study modular measures on lattice ordered effect algebras.

The starting point of our paper is the paper $[W_4]$, in which a characterization of modular functions on complemented lattices which have a Hahn decomposition is given. An important consequence of this result is that, if two order-continuous modular functions on a complemented complete lattice coincide on the centre of the lattice, then they coincide on every element of the lattice. Similar result also holds for measures on MV-algebras (see [B-L-W]).

In this paper we prove that similar results also hold in the more general context of modular measures on lattice ordered effect algebras. Precisely, in Section 2 we give a characterization of modular measures which have a Hahn decomposition and we derive that also in complete effect algebras the values of order-continuous modular measures only depends on the values on the centre (see (2.5) and (2.7)). As a consequence, we obtain, as in $[W_4]$, an isomorphism between the space of all bounded real-valued modular measures on an effect algebra and the space of all completely additive real-valued measures on a suitable Boolean algebra (see (4.3)).

These results are essential tools to study properties of the range of modular measures with values in Banach spaces. We prove that the closure of the range of every nonatomic modular measure of bounded variation with values in a Banach space with the Radon-Nikodym property or in a B-convex Banach space is compact and convex (see (5.2) and (6.3)).

This extends results of Uhl (see [U]) and Kadets (see [K]) for measures on σ-algebras, and of $[A_1]$ for modular functions on complemented lattices.

1. Preliminaries.

An *effect algebra* $(L, \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for every $a, b, c \in L$:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(3) For every $a \in L$, there exists a unique $a^\perp \in L$ such that $a \oplus a^\perp$ is defined and $a \oplus a^\perp = 1$.
(4) If $a \oplus 1$ is defined, then $a = 0$.

In every effect algebra a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a.

Moreover we can define a binary relation on L by $a \leq b$ if and only if there exists $c \in L$ such that $c \oplus a = b$ and $\leq$ is a partial ordering in L, with 0 as smallest element. We say that two elements $a, b \in L$ are *orthogonal,* and we write $a \perp b$, if $a \oplus b$ exists. Then $a \perp b$ if and only if $a \leq b^\perp$. Moreover, for every $a \in L$, we have $a^\perp = 1 \ominus a$.

If $(L, \leq)$ is a lattice, we say that the effect algebra is a *lattice ordered effect algebra* or a *D-lattice.*

Effect algebras are a common generalization of orthomodular posets and MV-algebras. For a study, we refer to [D-P].

In this paper, we denote by L a D-lattice.

We write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) whenever $\{a_n\}$ is an increasing sequence in L and $a = \sup_n a_n$ (respectively, $\{a_n\}$ is decreasing and $a = \inf_n a_n$).

L is said to be *complete* (*σ-complete*) if every (countable) set in L has a supremum and an infimum. We set $\Delta = \{(a, b) \in L \times L : a = b\}$. If $a, b \in L$ and $a \leq b$, we set $[a, b] = \{c \in L : a \leq c \leq b\}$.

If $a_1, ..., a_n \in L$, we inductively define $a_1 \oplus ... \oplus a_n = (a_1 \oplus ... \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, ..., a_n\}$ of L is *orthogonal* if $a_1 \oplus ... \oplus a_n$ exists.

For a sequence $\{a_n\}$, we say that it is *orthogonal* if, for every n, $\bigoplus_{i \leq n} a_i$ exists. If, moreover, $\sup_n \bigoplus_{i \leq n} a_i$ exists, we set $\bigoplus_{n \in \mathcal{N}} a_n = \sup_n \bigoplus_{i \leq n} a_i$.

It is helpful to recall from [D-P, 1.1.2, 1.1.6, 1.8.2, 1.8.5 and 1.8.9] the following basic results.

Proposition (1.1). *Assume a, b, c elements of L.*

1. *If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.*
2. *If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $a \oplus (c \ominus b) = c \ominus (b \ominus a)$.*
3. *If $a \leq b^\perp \leq c^\perp$, then $a \oplus (b \ominus c) = (a \oplus b) \ominus c$.*
4. *If $a \leq c$ and $b \leq c$, then $c \ominus a = c \ominus b$ iff $a = b$.*
5. *If $a \perp b$ and $a \oplus b \leq c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.*
6. *If $a \leq b$, then $b = a \oplus (b \ominus a)$.*
7. *If $c \geq a$ and $c \geq b$, then $c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$.*
8. *If $a \perp b$ and $a \wedge b = 0$, then $a \oplus b = a \vee b$.*
9. *$[(a \ominus (a \wedge b)] \wedge [b \ominus (a \wedge b)] = 0$.*

An element $a \in L$ is said to be *principal* if $b \perp c$, $b \leq a$ and $c \leq a$ imply $b \oplus c \leq a$.

An element $a \in L$ is said to be *central* if, for every $b \in L$, $b = (b \wedge a) \vee (b \wedge a^\perp)$. The set $C(L)$ of all central element of L is said to be *the centre* of L. By $[R_1]$ (5.5), $a \in L$ is central if and only if a is principal and, for every $b \in L$, $b = (b \wedge a) \oplus (b \wedge a^\perp)$.

It is helpful to recall the following properties of the centre (see 4.13 and 5.1 of [M-M], 1.9.14 of [D-P] and III.2.5 of [G]).

Proposition (1.2).

1. *$C(L)$ is a Boolean algebra.*
2. *If $a \in C(L)$, $\{b_\alpha\} \subseteq L$ and $b = \sup_\alpha b_\alpha$, then $\sup_\alpha (a \wedge b_\alpha) = a \wedge b$.*
3. *If $a_1, \cdots, a_n \in C(L)$ and $a \in L$, then $a \wedge \bigvee_{i=1}^n a_i = \bigvee_{i=1}^n (a \wedge a_i)$.*

If G is an Abelian group, a function $\mu : L \to G$ is said to be a *measure* if $a \perp b$ implies $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure if and only if $a \leq b$ implies $\mu(b \ominus a) = \mu(b) - \mu(a)$.

If G is a topological group, μ is said to be *σ-additive* if, for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus_n a_n$ exists, $\mu(\bigoplus_n a_n) = \sum_{n=1}^\infty \mu(a_n)$, *exhaustive* if, for every monotone sequence $\{a_n\}$ in L, $\{\mu(a_n)\}$ is a Cauchy sequence in G, *σ-order continuous*

(σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ in L implies $\lim_n \mu(a_n) = \mu(a)$, and *order-continuous* (o.c.) if the same condition holds for nets. By 2.4 of [A-B], a measure μ is σ-additive if and only if μ is σ-o.c.

μ is said to be *nonatomic* if, for every $a \in L$ and every neighbourhood W of 0 in G, there exists an orthogonal family $\{a_1, \cdots, a_n\}$ in L such that $\bigoplus_{i=1}^n a_i = a$ and $\mu(b) \in W$ whenever $b \leq a_i$ for some $i \leq n$.

μ is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$. By [F-T], every modular function on any lattice generates a *lattice uniformity* $\mathcal{U}(\mu)$, i.e. a uniformity which makes uniformly continuous the lattice operations $\vee$ and $\wedge$, and $\mathcal{U}(\mu)$ is the weakest lattice uniformity which makes μ uniformly continuous. Moreover, by 4.2 of [A-B], if μ is a modular measure on L, then $\mathcal{U}(\mu)$ is a *D-uniformity*, i.e. $\mathcal{U}(\mu)$ makes $\ominus$ and $\oplus$ uniformly continuous, too, and a base of $\mathcal{U}(\mu)$ is the family consisting of the sets $\{(a, b) \in L \times L : \mu(c) \in W$ for every $c \leq (a \vee b) \ominus (a \wedge b)\}$, where W is a neighbourhood of 0 in G.

A D-uniformity is said to be *exhaustive* if every monotone sequence in L is a Cauchy sequence in $\mathcal{U}$, *σ-order-continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ in L implies that $\{a_n\}$ converges to a in $\mathcal{U}$, and *order-continuous* (o.c.) if the same condition holds for nets.

If G is a seminormed group, the total variation of μ is the function $|\mu| : L \to [0, +\infty]$ defined by $|\mu|(a) = \sup\{\sum_{i=0}^{n-1} \|\mu(a_{i+1}) - \mu(a_i)\| : n \in N, a_0 \leq a_1 \leq \cdots \leq a_n = a\}$ for $a \in L$. By (1.3.10) of $[W_2]$, if μ is a modular function, $|\mu|$ is a modular function, too. Moreover, if μ is a modular measure, it is possible to see that $|\mu|$ is a modular measure in a similar way as in 3.11 of [B].

2. Hahn decomposition.

In this section $\mu : L \to R$ is a modular measure.

For $a \in L$, we denote by μ_a the function defined by $\mu_a(b) = \mu(b \wedge a)$ for $b \in L$. Moreover we set

$$N(\mu) = \{(a, b) \in L \times L : \mu(c) = 0 \text{ for every } c \leq (a \vee b) \ominus (a \wedge b)\}.$$

We recall that $N(\mu) = \bigcap\{U : U \in \mathcal{U}(\mu)\}$ (see Section 1) and therefore, by (2.5) and (3.1) of $[W_3]$, the quotient $\hat{L} = L/N(\mu)$ is a modular lattice. Moreover, by 4.3 of [A-B], $\hat{L}$ is a D-lattice, too.

Lemma (2.1). *If $a \in L$ is central and $b, c \in L$ are orthogonal, then $(b \oplus c) \wedge a = (b \wedge a) \oplus (c \wedge a)$.*

Proof. Since a is principal and $b \wedge a, c \wedge a$ are orthogonal elements less then or equal to a, then $(b \wedge a) \oplus (c \wedge a) \leq a$. Since $(b \wedge a) \oplus (c \wedge a) \leq b \oplus c$, then $(b \wedge a) \oplus (c \wedge a) \leq (b \oplus c) \wedge a$. Moreover, since a is central, we have $b \oplus c = ((b \oplus c) \wedge a) \vee ((b \oplus c) \wedge a^{\perp})$, from which $(b \oplus c) \wedge a = (b \oplus c) \ominus ((b \oplus c) \wedge a^{\perp})$. By the first part of the proof, we have $(b \oplus c) \wedge a^{\perp} \geq (b \wedge a^{\perp}) \oplus (c \wedge a^{\perp})$. Then $(b \oplus c) \wedge a) \leq (b \oplus c) \ominus [(b \wedge a^{\perp}) \oplus (c \wedge a^{\perp})]$. By (1.1)-5 and 3, we get $(b \oplus c) \wedge a \leq [(b \oplus c) \ominus (b \wedge a^{\perp})] \ominus (c \wedge a^{\perp}) = [c \oplus (b \ominus (b \wedge a^{\perp}))] \ominus (c \wedge a^{\perp}) = [b \ominus (b \wedge a^{\perp})] \oplus [(c \ominus (c \wedge a^{\perp})]$. Since $b = (b \wedge a) \oplus (b \wedge a^{\perp})$ and $c = (c \wedge a) \oplus (c \wedge a^{\perp})$, we have $b \ominus (b \wedge a^{\perp}) = b \wedge a$ and $c \ominus (c \wedge a^{\perp}) = c \wedge a$. Then $a \wedge (b \oplus c) \leq (b \wedge a) \oplus (c \wedge a)$.

Corollary (2.2). *If a is central, then μ_a is a modular measure and $\mu = \mu_a + \mu_{a^\perp}$. Moreover, if μ is o.c., μ_a and $\mu_{a^\perp}$ are o.c., too.*

Proof. It is trivial by (2.1) and (1.2).

Proposition (2.3). *Let $p \in L$ be such that $\mu(p) = \sup \mu(L)$. Then:*

(1) *If $a \leq p$, then $\mu(a) \geq 0$. If $a \leq p^\perp$, then $\mu(a) \leq 0$.*
(2) *If $N(\mu) = \Delta$ and $\mu(r) = \sup \mu(L)$, then $r = p$.*
(3) *If $N(\mu) = \Delta$, then $p^\perp$ is the unique complement of p.*

Proof. (1) If $a \leq p$, then by (1.1)-6 we have $p = a \oplus (p \ominus a)$, from which $\mu(a) + \mu(p \ominus a) = \mu(p) \geq \mu(p \ominus a)$ and therefore $\mu(a) \geq 0$. If $a \leq p^\perp$, then $\mu(p) \geq \mu(a \oplus p) = \mu(a) + \mu(p)$, from which $\mu(a) \leq 0$.

(2) First suppose $r \leq p$. Set $a = p \ominus r$ and let $b \leq a$. Then $a = b \oplus (a \ominus b)$, from which

$$0 = \mu(a) = \mu(b) + \mu(a \ominus b). \tag{*}$$

Since $b \leq a \leq p$ and $a \ominus b \leq p$, by (1) we have $\mu(b) \geq 0$ and $\mu(a \ominus b) \geq 0$. Then, by (*), we get $\mu(b) = \mu(a \ominus b) = 0$. Since $b \leq a$ is arbitrary and $N(\mu) = \Delta$, we get $p = r$.

Now we remove the assumption $r \leq p$. Set $m = \sup \mu(L)$. Since $\mu(p \wedge r) \leq m$, $\mu(p \vee r) \leq m$ and $\mu(p \wedge r) + \mu(p \vee r) = \mu(p) + \mu(r) = 2m$, we have $\mu(p \vee r) = \mu(p \wedge r) = m$. By the first part of the proof, we get $p \vee r = p \wedge r = p$, from which $p = r$.

(3) By (1) we get $\mu(c) = 0$ for every $c \leq p \wedge p^\perp$. Since $N(\mu) = \Delta$, we have $p \wedge p^\perp = 0$. Then, since $p, p^\perp$ are disjoint and orthogonal, by (1.1)-8 we obtain $p \vee p^\perp = p \oplus p^\perp = 1$.

Now let q be a complement of p. Then we get $\mu(q^\perp) = \sup \mu(L)$ since, if $a \in L$, then $\mu(a) = \mu(1) - \mu(a^\perp) \geq \mu(1) - \mu(p) = \mu(q^\perp)$ since $p \vee q = 1$ and $p \wedge q = 0$. By (2) we get $q^\perp = p$, from which $q = p^\perp$.

Theorem (2.4). *If $N(\mu) = \Delta$ and $\mu(p) = \sup \mu(L)$, then p is central.*

Proof. (i) First we prove that, for every $a \in L$, $a \wedge p = 0$ implies $a \perp p$.

Suppose that a is not orthogonal to p. Then $w = a \ominus (a \wedge p^\perp) \neq 0$. We prove that $\mu(w) > 0$. By (2.3)-(2), we have $\mu(a^\perp \wedge p) < \mu(p)$. Then $\mu(a^\perp \vee p) = \mu(a^\perp) + \mu(p) - \mu(a^\perp \wedge p) > \mu(a^\perp)$. Therefore we get $0 < \mu(a^\perp \vee p) - \mu(a^\perp) = \mu(1) - \mu(a \wedge p^\perp) - \mu(1) + \mu(a) = \mu(a) - \mu(a \wedge p^\perp) = \mu(w)$. Moreover, since $w \leq a$, we have $w \wedge p \leq a \wedge p = 0$. Then $\mu(w) = \mu(w \vee p) + \mu(w \wedge p) - \mu(p) = \mu(w \vee p) - \mu(p) \leq 0$, a contradiction. Therefore $a \leq p^\perp$.

(ii) Let $b \in L$. We prove that $b = (b \wedge p) \vee (b \wedge p^\perp)$.

Set $a = b \ominus (b \wedge p)$, $c = a \vee p$, $d = b \vee p$, $e = (a \vee p) \wedge p^\perp$ and $f = (b \vee p) \wedge p^\perp$. We prove that

$$\mu(a \wedge p) = \mu(f) - \mu(e).$$

We have $\mu(a \wedge p) = \mu(a) + \mu(p) - \mu(a \vee p) = \mu(b) - \mu(b \wedge p) + \mu(p) - \mu(a \vee p) = \mu(b \vee p) - \mu(a \vee p)$. Since $a \leq b$, we have $c \leq d$. Therefore, since L is modular, we have $c \vee (p^\perp \wedge d) = (c \vee p^\perp) \wedge d = (a \vee p \vee p^\perp) \wedge d = d$. Then $\mu(d) = \mu(c) + \mu(p^\perp \wedge d) - \mu(c \wedge p^\perp)$, since $c \leq d$. Hence we obtain

$$\mu(a \wedge p) = \mu(d) - \mu(c) = \mu(p^\perp \wedge d) - \mu(p^\perp \wedge c) = \mu(f) - \mu(e).$$

Since $e \leq f$, we have $f = e \oplus (f \ominus e)$, from which $\mu(f) = \mu(e) + \mu(f \ominus e)$. Since $f \ominus e \leq f \leq p^\perp$, by (2.3)-(1) we have $\mu(f \ominus e) \leq 0$. Then $\mu(a \wedge p) \leq 0$. On the other hand, since $a \wedge p \leq p$, by (2.3)-(1) we have $\mu(a \wedge p) \geq 0$. Hence $\mu(a \wedge p) = 0$. Therefore $\mu(p \ominus (a \wedge p)) = \mu(p) - \mu(a \wedge p) = \mu(p)$. By (2.3)-(3), we get $p \ominus (a \wedge p) = p$, from which $a \wedge p = 0$. By (i), we get $a \leq p^{\perp}$. Since $a \leq b$, we have $a \leq b \wedge p^{\perp}$. Since $p \wedge p^{\perp} = 0$ by (2.3)-(2), we obtain $b = a \oplus (b \wedge p) \leq (b \wedge p^{\perp}) \oplus (b \wedge p) = (b \wedge p^{\perp}) \vee (b \wedge p) \leq b$.

Corollary (2.5) (Hahn decomposition). *Let $p \in L$ and $q = p^{\perp}$. Then the following conditions are equivalent:*

(1) $\mu(p) = \sup \mu(L)$.
(2) $\mu(q) = \inf \mu(L)$.
(3) $\mu = \mu_p + \mu_q$, μ_p *and* μ_q *are modular measures,* $\mu_p \geq 0$ *and* $\mu_q \leq 0$.

Moreover, if (1) holds, then the equivalence classes $\hat{p}$ and $\hat{q}$ of p and q in the quotient $\hat{L} = L/N(\mu)$ are central, $\hat{q}$ is the unique complement of $\hat{p}$ and $\mu(w) = \sup \mu(L)$ implies $\hat{w} = \hat{p}$.
If L is σ-complete and μ is σ-additive, then there exists $p \in L$ such that (1) is satisfied.

Proof. It is clear that we can suppose $N(\mu) = \Delta$.

The equivalence of (1) and (2) is trivial.

(1) $\Rightarrow$ (3) By (2.4), p is central. By (2.2), μ_p and μ_q are modular measures, and $\mu = \mu_p + \mu_q$. By (2.3)-(1), we have $\mu_p \geq 0$ and $\mu_q \leq 0$.

(3) $\Rightarrow$ (1) If $a \in L$, we have $\mu(a) = \mu_p(a) + \mu_q(a) \leq \mu_p(a) \leq \mu_p(1) = \mu(p)$. Hence $\mu(p) = \sup \mu(L)$.

The other conclusions follow by (2.3), (2.4) and 1.2.2 of $[W_3]$ (in which it is proved that every σ-o.c. modular function on any σ-complete lattice attains its supremum).

Lemma (2.6). *The following conditions are equivalent:*

(1) μ *is bounded.*
(2) μ *is exhaustive.*
(3) μ *has bounded variation.*

Proof. By 2.7 of $[W_3]$, the equivalence between (1), (2) and (3) holds for any modular function which satisfies the following condition: for every finite chain $a_0 \leq ... \leq a_n$ in L and for every $I \subseteq \{1, ..., n\}$,

$$(*) \qquad \sum_{i \in I} [\mu(a_i) - \mu(a_{i-1})] \in \mu(L) - \mu(L).$$

Therefore we have only to observe that (*) holds for any measure on L. For every $i \leq n$, set $b_i = a_i \ominus a_{i-1}$. By 2.1 of [A-B], for every $i \leq n$, $\{b_1, ..., b_i\}$ is orthogonal and $b_1 \oplus ... \oplus b_i = a_i \ominus a_0$. Let $I \subseteq \{1, ..., n\}$. Then $\sum_{i \in I} [\mu(a_i) - \mu(a_{i-1})] = \sum_{i \in I} [\mu(a_i \ominus a_0) - \mu(a_{i-1} \ominus a_0)] = \sum_{i \in I} [\mu(\oplus_{j \leq i} b_i) - \mu(\oplus_{j \leq i-1} b_j)] = \sum_{i \in I} [\sum_{j \leq i} \mu(b_j) - \sum_{j \leq i-1} \mu(b_j)] = \sum_{i \in I} \mu(b_i) = \mu(\oplus_{i \in I} b_i) \in \mu(L)$.

In the next result we use that the closure of $\{0\}$ in $\mathcal{U}(\mu)$ is the set $I(\mu) = \{a \in L : \mu(b) = 0 \text{ for every } b \leq a\}$ and $(a, b) \in N(\mu)$ if and only if $(a \vee b) \ominus (a \wedge b) \in I(\mu)$ (see Section 1).

Corollary (2.7). *If L is complete and μ is o.c., then there exists a central element $p \in L$ such that $\mu(p) = \sup \mu(L)$.*

Proof. We can suppose $\mu \neq 0$. Observe that μ is exhaustive. In fact, by 3.5 and 3.6 of $[W_3]$ μ is o.c. (exhaustive) if and only if $\mathcal{U}(\mu)$ is o.c. (exhaustive) and, by 8.1.2 of

$[W_1]$, every σ-o.c. lattice uniformity on a σ-complete lattice is exhaustive. Then, by (2.6), the total variation ν of μ is bounded, and therefore, by 1.3.11 of $[W_2]$, μ and ν generate the same topology. Then we have $I(\mu) = \{a \in L : \nu(a) = 0\}$. By 5.1 of $[R_2]$, there exists a central element $c \in L$ such that $I(\mu) = [0, c]$. Then, for every $a \in L$, $a = (a \wedge c) \oplus (a \wedge c^\perp)$, from which $\mu(a) = \mu(a \wedge c) + \mu(a \wedge c^\perp) = \mu(a \wedge c^\perp)$. Hence $\sup \mu(L) = \sup \mu([0, c^\perp])$. Denote by μ' the restriction of μ to $[0, c^\perp]$. Observe that $N(\mu') = \Delta$ since $I(\mu') = \{a \in [0, c^\perp] : \mu'(b) = 0 \ \forall\, b \le a\} \subseteq I(\mu) \cap [0, c^\perp] = \{0\}$, since $c \wedge c^\perp = 0$. Moreover it is clear that μ' is o.c. and $[0, c^\perp]$ is a complete D-lattice. Then, by (2.5), there exists a central element $p \in [0, c^\perp]$ such that $\mu'(p) = \sup \mu'(L) = \sup \mu(L)$. Since $c^\perp$ is central in L, it is easy to see that p is central in L, too.

3. An extension theorem.

In this section we prove an extension theorem which we need in Section 4 and 5.

We denote by $\mathcal{A}$ a Boolean subalgebra of $C(L)$, by $\nu : L \to [0, +\infty[$ a measure and by X a Banach space. Moreover we set $S_\nu(\mathcal{A}) = \{\sum_{i=1}^n x_i \nu_{a_i} : n \in N, x_i \in X, a_i \in \mathcal{A}\}$.

The following result can be proved in a similar way as 3.3 of $[A_1]$.

Lemma (3.1).

(1) *Every $\gamma \in S_\nu(\mathcal{A})$ is an X-valued measure on L. If ν is modular, then every element of $S_\nu(\mathcal{A})$ is modular. Moreover, if ν is o.c. or nonatomic, then every $\gamma \in S_\nu(\mathcal{A})$ is o.c. or nonatomic, respectively.*

(2) *For every $\gamma \in S_\nu(\mathcal{A})$, there exist $s \in N$, $y_1, \cdots, y_s \in X$ and a disjoint family $\{b_1, \cdots, b_s\}$ in $\mathcal{A}$ with $\bigvee_{i=1}^s b_i = 1$ such that $\gamma = \sum_{i=1}^s y_i \nu_{b_i}$.*

(3) *If $\gamma = \sum_{i=1}^s x_i \nu_{a_i} \in S_\nu(\mathcal{A})$ and $\{a_1, \cdots, a_s\}$ is a disjoint family in $\mathcal{A}$ with $\bigvee_{i=1}^s a_i = 1$, then the total variation of $\gamma_{|\mathcal{A}}$ in 1 is $\sum_{i=1}^s \|x_i\| \nu(a_i)$.*

Proof. (1) follows by (2.1) and (1.2).

(2) Let $\gamma = \sum_{i=1}^n x_i \nu_{a_i} \in S_\nu(\mathcal{A})$. Since $\{a_1, \cdots, a_n\}$ is contained in a Boolean algebra, we can find a disjoint family $\{b_1, \cdots, b_s\} \subseteq \mathcal{A}$ with the following property: for every $j \le n$, there exist $r_1, \cdots, r_{h_j} \in \{1, \cdots, s\}$ such that $a_j = \bigvee_{i=1}^{h_j} b_{r_i}$. By (1.2) we have $\nu_{a_j}(a) = \nu(\bigvee_{i=1}^{h_j} (b_{r_i} \wedge a))$ for every $a \in L$. Since $a \wedge b_{r_i} \wedge \bigvee_{j<i} (a \wedge b_{r_j}) = a \wedge \bigvee_{j \le i} (b_{r_i} \wedge a \wedge b_{r_j}) = 0$ for every $i \le s$ and ν is modular, we have that $\nu_{a_j}(a) = \sum_{i=1}^{h_j} \nu_{b_{r_i}}(a)$. Then we obtain the assertion.

(3) Denote by $\tilde{\mathcal{A}}$ the Stone algebra of $\mathcal{A}$, by Ω the Stone space of $\mathcal{A}$ and by $h : \mathcal{A} \to \tilde{\mathcal{A}}$ the Stone map. Set, for $A \in \tilde{\mathcal{A}}$, $\tilde{\gamma}(A) = \gamma(h^{-1}(A))$ and $\tilde{\nu}(A) = \nu(h^{-1}(A))$. Set $f = \sum_{i=1}^n x_i \chi_{h(a_i)}$. If $A \in \tilde{\mathcal{A}}$, then

$$\tilde{\gamma}(A) = \sum_{i=1}^n x_i \nu(a_i \wedge h^{-1}(A)) = \sum_{i=1}^n x_i \tilde{\nu}_{h(a_i)}(A) = \int_A f \, d\tilde{\nu}.$$

Hence, if we set $\gamma' = \gamma_{|\mathcal{A}}$, by [D-S](III.2.20), we obtain

$$|\gamma'|(1) = |\tilde{\gamma}|(\Omega) = \int_\Omega \|f\| \, d\tilde{\nu} = \sum_{i=1}^n \|x_i\| \tilde{\nu}(h(a_i)) = \sum_{i=1}^n \|x_i\| \nu(a_i).$$

We also need the following result of $[A_1]$ (3.4), which is an extension of the Radon-Nikodym theorem to Boolean algebras (see also 3.1 of $[W_4]$).

Recall that X is said to have *the Radon-Nikodym property* if, for every σ-algebra of sets Σ, for every σ-additive real-valued measure ν on Σ and for every X-valued ν-continuous measure μ of bounded variation, there exists a ν-integrable function f such that $\mu(A) = \int_A f\, d\nu$ for all $A \in \Sigma$.

Lemma 3.2. *Suppose that X has the Radon-Nikodym property and let $\lambda : \mathcal{A} \to X$ be a $\nu-$continuous measure of bounded variation. Then there exists a sequence $\{\lambda_n\} \subseteq S_\nu(\mathcal{A})$ such that $\lambda(a) = \sum_{n=1}^\infty \lambda_n(a)$ uniformly with respect to $a \in \mathcal{A}$ and $\sum_{n=1}^\infty |\lambda_n|(1) < +\infty$.*

Theorem (3.3). *Suppose that X has the Radon-Nikodym property and let $\lambda : \mathcal{A} \to X$ be a measure of bounded variation and absolutely continuous with respect to the restriction of ν to $\mathcal{A}$. Then there exists a sequence $\{\nu_n\} \subseteq S_\nu(\mathcal{A})$ such that the function μ defined by $\mu(a) = \sum_{n=1}^\infty \nu_n(a)$ for $a \in L$ is an X-valued measure on L which extends λ.*

Proof. By (3.2), we can find a sequence $\{\nu_n\} \subseteq S_\nu(\mathcal{A})$ such that $\lambda(a) = \sum_{n=1}^\infty \nu_n(a)$ uniformly for $a \in L$, and $\sum_{n=1}^\infty |\nu'_n|(1) < +\infty$, where ν'_n is the restriction of ν_n to $\mathcal{A}$ and $|\nu'_n|$ is the total variation of ν_n. Moreover, by (3.1), we have $\|\nu_n(a)\| \leq |\nu'_n|(1)$ for every $a \in L$. Then the series $\sum_{n=1}^\infty \|\nu_n(a)\|$ converges in X uniformly with respect to $a \in L$. Hence we can define $\mu(a) = \sum_{n=1}^\infty \nu_n(a)$ for $a \in L$. Trivially $\mu : L \to X$ is a measure which extends λ.

4. An isomorphism theorem.

In this section we establish an isomorhism between the space $B(L)$ of all bounded real-valued modular measures on L and the space of all completely additive real-valued measures on a suitable Boolean algebra. We follow the idea of $[W_4]$.

Denote by $A(L)$ the space of all o.c. real-valued modular measures on L. It is easy to see that $B(L)$ and $A(L)$ are ordered linear spaces with respect to the pointwise order and $B(L)$ is a Banach space with respect to the norm defined by $\|\mu\|_\infty = \sup\{|\mu(a)| : a \in L\}$. Moreover, if L is σ-complete, by 8.1.2 of $[W_1]$ $A(L)$ is a closed subspace of $B(L)$.

For a modular measure $\mu : L \to R$, we set $N(\mu) = \{(a, b) \in L \times L : \mu(c) = 0$ for every $c \leq (a \vee b) \ominus (a \wedge b).\}$ Recall that $N(\mu) = \bigcap\{U : U \in \mathcal{U}(\mu)\}$, where $\mathcal{U}(\mu)$ is the uniformity generated by μ.

Theorem (4.1). *Suppose that L is complete and $\bigcap\{N(\mu) : \mu \in A(L), \mu \geq 0\} = \Delta$. Then the map $\phi : \mu \to \mu_{|C(L)}$ is a linear isometry preserving the order between $A(L)$ and $A(C(L))$.*

Proof. Observe that $\bigcap\{N(\mu) : \mu \in A(L), \mu \geq 0\}$ is the congruence generated by $\sup\{\mathcal{U}(\mu) : \mu \in A(L), \mu \geq 0\}$. Then, by 3.4 of $[W_5]$ $C(L)$ is a complete sublattice of L and $a_\alpha \uparrow a$ ($a_\alpha \downarrow a$) in $C(L)$ imply $a_\alpha \uparrow a$ ($a_\alpha \downarrow a$) in L. Then, if $\mu \in A(L)$, we have $\phi(\mu) \in A(C(L))$. Trivially ϕ is linear. Moreover ϕ is an isometry since, by (2.7), every $\mu \in A(L)$ attains its supremum on $C(L)$.

We have to prove that every $\mu \in A(C(L))$ can be extended to a $\mu \in A(L)$.

Since for the elements of $A(C(L))$ the Hahn decomposition holds, we can restrict our attention to the positive elements of $A(C(L))$. Set $V = \{\mu \in A(L) : \mu \geq 0\}$. First observe that, by 5.1 of $[R_2]$, for every $\nu \in V$ we can find $a_\nu \in C(L)$ such that $\{a \in L :$

$\nu(a) = 0\} = [0, a_\nu]$. Set $s = \inf\{a_\nu : \nu \in V\}$. We have $s = 0$ since $0 \le \nu(s) \le \nu(a_\nu) = 0$ for every $\nu \in V$, from which $(0, s) \in \bigcap\{N(\nu) : \nu \in V\} = \Delta$.

Now let $\mu_0 \in A(C(L))$, with $\mu_0 \ge 0$, and let $b \in C(L)$ be such that $\{a \in C(L) : \mu_0(a) = 0\} = [0, b] \cap C(L)$. Since μ_0 is strictly positive on $[0, b^\perp] \cap C(L)$, then the Boolean algebra $[0, b^\perp] \cap C(L)$ has the countable chain condition. Therefore we can find a sequence $\{\nu_n\}$ in V such that $\inf_n a_{\nu_n} \le b$ and then we can find a disjoint sequence $\{c_n\} \subseteq C(L)$ such that $\inf_n c_n^\perp = b$ and $c_n \le a_{\nu_n}^\perp$. For each integer n, choose $\varepsilon_n > 0$ such that $\varepsilon_n \nu_n(1) \le 2^{-n}$, and set $\lambda_n(a) = \nu_n(a \wedge c_n)$ for $a \in L$. By (2.2) $\lambda_n \in V$. Then, if we set $\nu = \sum_{n=1}^\infty \varepsilon_n \lambda_n$, we have that $\nu \in V$. We see that μ_0 is absolutely continuous with respect to the restriction of ν to $C(L)$. Let $a \in C(L)$ be such that $\nu(a) = 0$. Then we have $\nu_n(a \wedge c_n) = 0$ for each n and therefore $a \wedge c_n \le a_{\nu_n}$. Since $c_n \le a_{\nu_n}^\perp$, we get $a \wedge c_n = 0$ for each n. Since $a, c_n \in C(L)$, we get $a \le c_n^\perp$ for each n, from which $a \le \inf_n c_n^\perp = b$. Then $\mu_0(a) = 0$. Moreover, since μ_0 is bounded, μ_0 has bounded variation. Then, by (3.3), we can find $\mu \in V$ which extends μ_0.

For the next result, we recall by $[W_4]$ (4.2) and [A-B] (4.3) the following result.

Theorem (4.2). *Let $\mathcal{U}$ be a lattice uniformity on L. Then:*

1. $N(\mathcal{U}) = \bigcap\{U : U \in \mathcal{U}\}$ *is a congruence, the quotient* $\hat{L} = L/N(\mathcal{U})$ *is a lattice and the quotient uniformity* $\hat{\mathcal{U}}$ *is a Hausdorff lattice uniformity.*
2. *If we denote by* $(\tilde{L}, \tilde{\mathcal{U}})$ *the uniform completion of* $(\hat{L}, \hat{\mathcal{U}})$ *in (1), then* $\tilde{L}$ *is a lattice with respect to the continuous extension of the lattice operations of* $\hat{L}$ *to* $\tilde{L}$ *and* $\tilde{\mathcal{U}}$ *is a lattice uniformity. Moreover, if* $\mathcal{U}$ *is exhaustive, then* $\tilde{\mathcal{U}}$ *is exhaustive and o.c. and* $(\tilde{L}, \le)$ *is complete.*
3. *If* $\mathcal{U}$ *is a D-uniformity, then* $\hat{L}$ *and* $\tilde{L}$ *are D-lattices, and* $\hat{\mathcal{U}}, \tilde{\mathcal{U}}$ *are D-uniformity.*
4. *If* μ *is a modular function on* L *which is uniformly continuous with respect to* $\mathcal{U}$, *then the function* $\hat{\mu}$ *defined by* $\hat{\mu}(\hat{a}) = \mu(a)$ *is a well defined modular function on* $\hat{L}$ *and, if* $\mathcal{U}$ *is exhaustive, there exists a unique uniformly continuous extension of* $\hat{\mu}$ *to a modular function* $\tilde{\mu}$ *on* $\hat{L}$, *such that* $\mu(L)$ *is dense in* $\tilde{\mu}(\tilde{L})$.
5. *If* $\mathcal{U}$ *is a D-uniformity and* μ *is a modular measure, then* $\hat{\mu}, \tilde{\mu}$ *are modular measures.*

Now we can prove the isomorphism theorem.

Corollary (4.3). *Let* $\mathcal{U} = \sup\{\mathcal{U}(\mu) : \mu \in B(L)\}$. *Let* $(\hat{L}, \hat{\mathcal{U}})$, $(\tilde{L}, \tilde{\mathcal{U}})$ *and, for* $\mu \in B(L)$, $\hat{\mu}$ *and* $\tilde{\mu}$ *be as in (4.2). Then the spaces* $B(L), B(\hat{L}), A(\tilde{L})$ *and* $A(C(\tilde{L}))$ *are isometric Banach spaces and isomorphic as ordered linear spaces by the following maps:*

1. $\mu \in B(L) \to \hat{\mu} \in B(\hat{L})$,
2. $\tilde{\mu} \in A(\tilde{L}) \to \tilde{\mu}_{|\hat{L}} \in B(\hat{L})$,
3. $\tilde{\mu} \in A(\tilde{L}) \to \tilde{\mu}_{|C(\tilde{L})} \in A(C(\tilde{L}))$.

Proof. The first map is trivially a linear isometry which preserves the order. For the second map the assertion follows since every $\tilde{\mu} \in A(\tilde{L})$ is bounded and the continuous extension of every $\hat{\mu} \in B(\hat{L})$ is in $A(\tilde{L})$ since $\hat{\mathcal{U}}$ is o.c. For the last map the assertion follows by (4.1), since $\tilde{\mathcal{U}} = \sup\{\mathcal{U}(\tilde{\mu}) : \tilde{\mu} \in A(\tilde{L})\} = \sup\{\mathcal{U}(\tilde{\mu}) : \tilde{\mu} \in A(\tilde{L}), \tilde{\mu} \ge 0\}$ (see 1.3.11 of $[W_2]$).

5. Uhl's theorem.

In this section X is a Banach space and $\mu : L \to X$ is a modular measure.

As consequence of the results of section 2 and 3, we prove a Uhl type theorem concerning compactness and convexity of the closure of the range of μ. We follow the idea of $[A_1]$.

Lemma (5.1). *Suppose that, for every $\varepsilon > 0$ and every $a \in L$, there exists $b \leq a$ such that $\|\mu(b) - \frac{1}{2}\mu(a)\| < \varepsilon$. Then $\overline{\mu(L)}$ is convex.*

Proof. It is sufficient to prove that, for every $a, b \in L$ and $\varepsilon > 0$, there exists $s \in L$ such that $\|\mu(s) - \frac{\mu(a)+\mu(b)}{2}\| < \varepsilon$.

Let $a, b \in L$ and $\varepsilon > 0$. Set $e = a \wedge b$, $c = a \ominus e$ and $d = b \ominus e$. Then we have $a = c \oplus e$ and $b = d \oplus e$, from which $\mu(a) = \mu(c) + \mu(e)$ and $\mu(b) = \mu(d) + \mu(e)$. By assumption, we can choose $c_\varepsilon \leq c$ and $d_\varepsilon \leq d$ such that $\|\mu(c_\varepsilon) - \frac{1}{2}\mu(c)\| < \varepsilon$ and $\|\mu(d_\varepsilon) - \frac{1}{2}\mu(d)\| < \varepsilon$. We see that $e \perp c_\varepsilon \vee d_\varepsilon$. In fact, we have $a \wedge b \leq c^\perp \leq c_\varepsilon^\perp$ and $a \wedge b \leq d^\perp \leq d_\varepsilon^\perp$, from which $a \wedge b \leq c_\varepsilon^\perp \wedge d_\varepsilon^\perp = (c_\varepsilon \vee d_\varepsilon)^\perp$ by (1.1)-7. Set $s = (c_\varepsilon \vee d_\varepsilon) \oplus e$. Then we have $\mu(s) = \mu(c_\varepsilon \vee d_\varepsilon) + \mu(e) = \mu(c_\varepsilon) + \mu(d_\varepsilon) + \mu(e)$, since, by (1.1)-9, we have $c_\varepsilon \wedge d_\varepsilon \leq c \wedge d = 0$. Therefore we obtain $\|\mu(s) - \frac{\mu(a)+\mu(b)}{2}\| = \|\mu(c_\varepsilon) + \mu(d_\varepsilon) + \mu(e) - \frac{1}{2}\mu(c) - \frac{1}{2}\mu(e) - \frac{1}{2}\mu(d) - \frac{1}{2}\mu(e)\| = \|\mu(c_\varepsilon) - \frac{1}{2}\mu(c)\| + \|\mu(d_\varepsilon) - \frac{1}{2}\mu(d)\| < 2\varepsilon$.

Theorem (5.2). *Suppose that X has the Radon-Nikodym property and μ has bounded variation. Then $\overline{\mu(L)}$ is compact. Moreover, if μ is also nonatomic, then $\overline{\mu(L)}$ is also convex.*

Proof. Denote by ν the total variation of μ.

(1) First suppose that L is complete and μ is o.c. Recall that, since ν is bounded, by (1.3.11) of $[W_2]$ μ and ν generate the same topology. Hence ν is o.c., too, and if μ is nonatomic, ν is nonatomic, too, since, by (2.6) of $[A_2]$, a modular measure is nonatomic if and only if the topology generated by it is nonatomic.

(i) First we prove that there exists a sequence $\{\nu_n\} \subseteq S_\nu(C(L))$ such that $\mu(a) = \sum_{n=1}^\infty \nu_n(a)$ uniformly with respect to $a \in L$ (see notation of Section 3).

(b) Set $\lambda = \mu_{|C(L)}$. Since $\|\mu(a)\| \leq \nu(a)$ for every $a \in L$, then λ is a $\nu_{|C(L)}$−continuous measure on the Boolean algebra $C(L)$ and its total variation $|\lambda|$ is bounded, since $|\lambda| \leq \nu$. By (3.3), there exists a sequence $\{\nu_n\} \subseteq S_\nu(C(L))$ such that the function $\mu' : L \to X$ defined by $\mu'(a) = \sum_{n=1}^\infty \nu_n(a)$ is a modular measure which extends λ. We prove that $\mu = \mu'$. Since ν is o.c., by (3.1) every ν_n is o.c. and therefore μ' is o.c, too. Hence, if f is a linear continuous functional on X, $f \circ \mu$ and $f \circ \mu'$ are o.c. modular measures on L such that $f \circ \mu = f \circ \mu'$ on $C(L)$. Therefore, by (2.7), $f \circ \mu = f \circ \mu'$ on L. Hence, by Hahn-Banach theorem, we obtain $\mu = \mu'$ on L.

(ii) We prove that $\overline{\mu(L)}$ is compact. Let $\varepsilon > 0$ and, by (i), $\gamma \in S_\nu(C(L))$ such that $\|\mu(a) - \gamma(a)\| < \varepsilon/2$ for every $a \in L$. Since $\gamma(L)$ is contained in a finite dimensional linear subspace of X, we can find $x_1, ..., x_s$ in X such that $\gamma(L) \subseteq \bigcup_{i=1}^s B(x_i, \varepsilon/2)$, where, for $x_0 \in X$ and $\varepsilon > 0$, $B(x_0, \varepsilon) = \{x \in X : \|x - x_0\| < \varepsilon\}$. Hence, if $a \in L$, we can choose $j \leq s$ such that $\|\gamma(a) - x_j\| < \varepsilon/2$, from which $\|\mu(a) - x_j\| < \varepsilon$. Therefore $\mu(L) \subseteq \bigcup_{i=1}^s B(x_i, \varepsilon)$.

(iii) Now suppose that μ is nonatomic. We prove that $\overline{\mu(L)}$ is convex. Let $\varepsilon > 0$ and $a \in L$. By (5.1), it is sufficient to prove that there exists $b \leq a$ such that $\|\mu(b) - \frac{1}{2}\mu(a)\| <$

ε. By (i), we can choose $\gamma \in S_\nu(C(L))$ such that $\|\mu(c) - \gamma(c)\| < \frac{\varepsilon}{2}$ for every $c \in L$. Since ν is nonatomic, γ is nonatomic by (3.1). Moreover $\gamma(L)$ is contained in a finite dimensional linear subspace of X. Then, since L is complete, $\gamma([0,a])$ is convex by 3.12 of [B]. Hence we can choose $b \le a$ such that $\gamma(b) = \frac{1}{2}\gamma(a)$. Therefore

$$\|\mu(b) - \frac{1}{2}\mu(a)\| \le \|\mu(b) - \gamma(b)\| + \frac{1}{2}\|\gamma(a) - \mu(a)\| < \varepsilon.$$

(2) Now we remove the assumptions that L is complete and μ is o.c.

We use (4.2). Denote by $\mathcal{U}$ the uniformity generated by ν, by $(\hat{L}, \hat{\mathcal{U}})$ the quotient of L with respect to $N(\mathcal{U})$ and by $(\tilde{L}, \tilde{\mathcal{U}})$ its uniform completion. Since $\mathcal{U}$ is exhaustive, then $\tilde{\mathcal{U}}$ is Hausdorff, exhaustive and o.c. and $(\tilde{L}, \le)$ is a complete D-lattice.

Since $\|\mu(a) - \mu(b)\| \le \nu(a) - \nu(b)$ if $a \ge b$, then $\mathcal{U}(\mu) \le \mathcal{U}$. Therefore μ and ν are uniformly continuous with respect to $\mathcal{U}$. Then, if we set $\hat{\mu}(\hat{a}) = \mu(a)$ and $\hat{\nu}(\hat{a}) = \nu(a)$ for $a \in \hat{a} \in \hat{L}$, we have that $\hat{\mu}$ and $\hat{\nu}$ are well defined uniformly continuous modular functions on $(\hat{L}, \hat{\mathcal{U}})$ and there exist unique uniformly continuous o.c. modular measures $\tilde{\mu}$ and $\tilde{\nu}$ on $(\tilde{L}, \tilde{\mathcal{U}})$, which extend, respectively, $\hat{\mu}$ and $\hat{\nu}$, and $\overline{\mu(L)} = \overline{\hat{\mu}(\hat{L})} = \overline{\tilde{\mu}(\tilde{L})}$ (see (4.2)). Moreover, as remarked in the proof of 1.3.11 of $[W_2]$, $\tilde{\nu}$ is the total variation of $\tilde{\mu}$ and then $\tilde{\mu}$ has bounded variation.

Now we prove that, if μ is nonatomic, then $\tilde{\mu}$ is nonatomic. First remark that, since $\tilde{\mathcal{U}}$ is Hausdorff, the greatest element 1 of $\hat{L}$ is also the greatest element of $\tilde{L}$. Let $\varepsilon > 0$. Since $\hat{\mu}$ is nonatomic, we can choose a partition $\{a_1, ..., a_n\}$ of 1 in $\hat{L}$ such that $\|\hat{\mu}(b)\| < \frac{\varepsilon}{2}$ if $b \le a_i$ and $b \in \hat{L}$. Now let $b \in \tilde{L}$, with $b \le a_i$ and choose $\{b_\alpha\}$ in $\hat{L}$ such that $b_\alpha \to b$ in $(\tilde{L}, \tilde{\mathcal{U}})$. We can suppose $b_\alpha \le a_i$, since $b_\alpha \wedge a_i \in \hat{L}$ and $b_\alpha \wedge a_i \to b \wedge a_i = b$ in $(\tilde{L}, \tilde{U})$. Hence $\|\hat{\mu}(b_\alpha)\| < \frac{\varepsilon}{2}$. Moreover, by the continuity of $\tilde{\mu}$ in $(\tilde{L}, \tilde{\mathcal{U}})$, we can choose α such that $\|\hat{\mu}(b_\alpha) - \tilde{\mu}(b)\| < \frac{\varepsilon}{2}$. Then

$$\|\tilde{\mu}(b)\| \le \|\tilde{\mu}(b) - \hat{\mu}(b_\alpha)\| + \|\hat{\mu}(b_\alpha)\| < \varepsilon.$$

By (2.5) of $[A_2]$, $\tilde{\mu}$ is nonatomic. Hence the assertion follows by (1).

Corollary (5.3). *If $X = R^n$ and μ is nonatomic, then $\overline{\mu(L)}$ is convex and compact.*

Proof. We can apply (5.2) since, for $X = R^n$, nonatomic implies bounded variation.

Remark. If L is σ-complete and μ is σ-additive, we obtain that in 5.2 the assumption that μ is nonatomic, as in the classical Uhl theorem in [U], can be replaced with the assumption that μ is atomless, since by (2.8) of $[A_2]$ in this case atomless and nonatomic are equivalent.

6. Kadets theorem.

In this section, we prove (see (6.3)) that the result of the Theorem 5.2 concerning the convexity of $\overline{\mu(L)}$ also holds if we replace the assumption that X has the Radon-Nikodym property by the assumption that X is B-convex.

Denote by X a Banach space and by $\mu : L \to X$ a modular measure.

X is said to be *B-convex* if there exist an integer $n \ge 2$ and a real number $0 < k < 1$ such that, for every $x_1, ..., x_n$ in X, $\min_{\alpha_i = \pm 1} \|\sum_{i=1}^n \alpha_i x_i\| \le kn \sup_{i \le n} \|x_i\|$.

The notion of B-convexity derives from the characterization in $[B_1]$ of those Banach spaces in which the strong law of large numbers holds. For a study of these spaces, see [D-J-T].

The result of (6.3) is independent on the result of (5.2) since the sequences space l_1 is an example of space with the Radon-Nikodym property which is not B-convex and, conversely, not all B-convex spaces have the Radon-Nikodym property, otherwise, by [D-J-T] (13.6) and by [S] (p. 88), we would have that all B-convex spaces are reflexive, which is false by the example in [J].

We need the following result, which is a special case of 13.5 in [D-J-T].

Lemma (6.1). *Suppose that X is B-convex and let n and k be as in the definition of B-convex space. If $s \in N$, $m = n^s$ and $x_1, ..., x_m \in X$, then* $\min_{\alpha_i=\pm 1} \| \sum_{i=1}^m \alpha_i x_i\| \leq mk^s \sup_{i\leq m} \|x_i\|$.

Lemma (6.2). *Let $\nu : L \to [0, +\infty[$ be a measure. Suppose that, for every $a \in L$ and α such that $0 < \alpha < \nu(a)$, there exists $b \leq a$ such that $\nu(b) = \alpha$. Then, for every $a \in L$ and $m \in N$, there exists an orthogonal family $\{a_1, \cdots, a_m\}$ in L such that $\bigoplus_{i=1}^m a_i = a$ and $\nu(a_i) = \frac{1}{m}\nu(a)$ for each $i \leq m$.*

Proof. The assertion is trivial for $m = 2$. Let $a \in L$ and $m \geq 2$.

(i) As a first step, we prove that, for each positive integer $s \leq m - 1$, there exists an orthogonal family $\{a_1, \cdots, a_s\}$ in L with the following properties:

(1) $\bigoplus_{j=1}^i a_j \leq a$ for each $i \in \{2, \cdots, s\}$.

(2) $a_i \leq a \ominus \bigoplus_{j=1}^{i-1} a_j$ for each $i \in \{2, \cdots, s\}$.

(3) $\nu(a_i) = \frac{1}{m}\nu(a)$ for each $i \leq s$.

This is trivial if $s = 2$. Now let $s > 2$ and suppose that (1), (2), (3) are satisfied for $s-1$. Since $\nu(a\ominus\bigoplus_{j=1}^{s-1} a_j) = \nu(a) - \sum_{j=1}^{s-1} \nu(a_j) = \nu(a) - \frac{s-1}{m}\nu(a) \geq \frac{1}{m}\nu(a)$, we can find $a_s \leq a \ominus \bigoplus_{j=1}^{i-1} a_j$ such that $\nu(a_s) = \frac{1}{m}\nu(a)$. Then $\{a_1, \cdots, a_s\}$ is an orthogonal family in L since $a_s \leq a\ominus\bigoplus_{j=1}^{s-1} a_j \leq 1\ominus\bigoplus_{j=1}^{s-1} a_j = (\bigoplus_{j=1}^{s-1} a_j)^\perp$. Moreover, by (1.1) we obtain $\bigoplus_{i=1}^s a_i = \bigoplus_{i=1}^{s-1} a_i\oplus a_s \leq \bigoplus_{i=1}^{s-1} a_i\oplus(a\ominus\bigoplus_{i=1}^{s-1}) = a\ominus(\bigoplus_{i=1}^{s-1} a_i\ominus\bigoplus_{i=1}^{s-1} a_i) = a\ominus 0 = a$, i.e. (1), (2), (3) for s.

(ii) Choose $a_1, \cdots a_{m-1}$ as in (i). Since $\bigoplus_{i=1}^{m-1} a_i \leq a$, then $a_m = a \ominus \bigoplus_{i=1}^{m-1}$ exists. Then $\{a_1, \cdots, a_m\}$ is an orthogonal family in L since $a_m \leq (\bigoplus_{i=1}^{m-1} a_i)^\perp$. Moreover, by (1.1), we obtain

$$\bigoplus_{i=1}^{m} a_i = \bigoplus_{i=1}^{m-1} a_i \bigoplus a_m = \bigoplus_{i=1}^{m-1} a_i \oplus (a \ominus \bigoplus_{i=1}^{m-1} a_i) = a \ominus (\bigoplus_{i=1}^{m-1} a_i \ominus \bigoplus_{i=1}^{m-1} a_i) = a$$

and

$$\nu(a_m) = \nu(a) - \sum_{i=1}^{m-1} \nu(a_i) = \nu(a) - \frac{m-1}{m}\nu(a) = \frac{1}{m}\nu(a).$$

The following result has been proved in [K] for σ-additive measures on σ-algebras.

Theorem (6.3). *Suppose that X is B-convex and μ is nonatomic with bounded variation. Then $\overline{\mu(L)}$ is convex.*

Proof. As in (5.2), we can suppose that L is complete.

Let $\varepsilon > 0$ and $a \in L$. By (5.1), it is sufficient to prove that there exists $b \leq a$ such that $\|\mu(b) - \frac{1}{2}\mu(a)\| < \varepsilon$. Then we can suppose $\mu(a) \neq 0$. Choose n and k as in the definition of B-convex space and $s \in N$ such that $k^s|\mu|(a) < \varepsilon$. Set $r = n^s$. Since $|\mu|$ is bounded, by 1.3.11 of $[W_2]$ μ and $|\mu|$ generate the same topology. Hence $|\mu|$ is nonatomic, too, by (2.6) of $[A_2]$. Then, since L is σ-complete, by 3.12 of [B] $|\mu|([0,a])$ is convex. Therefore, by (6.1), we can find an orthogonal family $\{a_1, \cdots, a_r\}$ in L such that $\bigoplus_{i=1}^r a_i = a$ and $|\mu|(a_i)| = \frac{1}{r}|\mu|(a)$ for each $i \leq r$. For $I \subseteq \{1, \cdots, r\}$, set $a_I = \bigoplus_{i \in I} a_i$. We have $\mu(a_I) = \sum_{i \in I} \mu(a_i)$ and $\mu(a) = \sum_{i=1}^r \mu(a_i)$. Then we obtain $\|\mu(a_I) - \frac{1}{2}\mu(a)\| = \frac{1}{2}\| \sum_{i \in I} \mu(a_i) - \sum_{i \notin I} \mu(a_i)\|$, from which we have $\min_I \|\mu(a_I) - \frac{1}{2}\mu(a)\| = \min_{\alpha_i = \pm 1} \| \sum_{i=1}^r \alpha_i \mu(a_i)\|$. Choose $b \leq a$ such that $\|\mu(b) - \frac{1}{2}\mu(a)\| = \min_I \|\mu(a_I) - \frac{1}{2}\mu(a)\|$. Then we get $\|\mu(b) - \frac{1}{2}\mu(a)\| \leq k^s r \sup_{i \leq r} \|\mu(a_i)\| < k^s r \sup_{i \leq r} |\mu|(a_i) = k^s |\mu|(a) < \varepsilon$.

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