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On Dieudonné's Boundedness Theorem

GIUSEPPINA BARBIERI

Abstract. – *We generalize the classical Dieudonné boundedness theorem for modular measures on lattice ordered effect algebras.*

1. – Introduction.

A famous theorem of Dieudonné [3] states that for compact metric spaces the pointwise boundedness of a family of Borel regular measures on open sets implies its uniform boundedness on all Borel sets.

In this note we furnish an abstract formulation of the boundedness Dieudonné theorem for group-valued modular measures on lattice ordered effect algebras. We use an abstract concept of regularity (see Definition 2.6) where $\mathcal{F}$ and $\mathcal{G}$ play the role of compact sets and open sets, respectively. We generalize Guariglia's work [5].

Effect algebras have been introduced by Foulis and Bennett in 1994, they are a generalization of orthomodular lattices and MV-algebras, in particular of Boolean algebras.

2. – Notation and Preliminaries.

In this section we shall give some basic definitions and fix some notations.

DEFINITION 2.1. – Let $(L, \leq)$ be a poset with a smallest element 0 and a greatest element 1 and let $\ominus$ be a partial operation on L such that $b \ominus a$ is defined if and only if $a \leq b$ and for all a, b, $c \in L$:

If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.

If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Then $(L, \leq, \ominus)$ is called a *difference poset (D-poset* for short), or a *difference lattice (D-lattice* for short) if L is a lattice.

If not otherwise specified, let L be a D-lattice and $(G, |\ |)$ *be a seminormed Abelian group.* We recall that a real-valued function $|\ |$ on the group G is a *seminorm* if and only if $|0| = 0$, $|-x| = |x|$ and $|x + y| \leq |x| + |y|$ for any x, $y \in G$.

One defines in L a partial operation $\ominus$ as follows:

$$a \oplus b \text{ is defined and } a \oplus b = c \text{ if and only if } c \ominus b \text{ is defined and } c \ominus b = a.$$

The operation $\ominus$ is well-defined by the cancellation law [4, page 13] ($a \le b, c$ and $b \ominus a = c \ominus a$ implies $b = c$), and $(L, \oplus, 0, 1)$ is an effect algebra (see [4, Theorem 1.3.4]).

Write $a^{\perp} = 1 \ominus a$ for $a \in L$. We say that a and b are *orthogonal* if $a \le b^{\perp}$ and we write $a \perp b$. Therefore $a \oplus b$ is defined if and only if $a \perp b$. If $a_1, \dots, a_n \in L$ we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ if the right-hand side exists. The sum is independent on any permutation of the elements. We say that a finite family $(a_i)_{i=1}^n$ of (not necessarily different) elements of L is *orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists.

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ of L is orthogonal if the set $\{a_1, \dots a_n\}$ is orthogonal, for every $n \in \mathbb{N}$. If $(a_n)_{n\in\mathbb{N}}$ is an orthogonal sequence, we set $\oplus_{n\in\mathbb{N}} a_n := \sup\{\oplus_{a\in F} a_a : F \text{ finite subset of } \mathbb{N}\}$ provided the right hand side exists.

A function ϕ on L with values in G is called a *measure* if for every $a, b \in L$, with $a \perp b$,

$$\phi(a \oplus b) = \phi(a) + \phi(b).$$

A *modular measure* is a measure which also satisfies the modular law, that is for all $a, b \in L$

$$\phi(a \vee b) + \phi(a \wedge b) = \phi(a) + \phi(b).$$

For the rest of the paper let $\mathcal{F}, \mathcal{G}$ be two sublattices of L. Moreover, we suppose that $f^{\perp} \in \mathcal{G}$ for every $f \in \mathcal{F}$, $\mathcal{G}$ is closed under finite sums and that $g \ominus (f \wedge g) \in \mathcal{G}$ for every $g \in \mathcal{G}, f \in \mathcal{F}$.

According to Avallone and Vitolo, we give the following definition:

Definition 2.2. – We say that L has the SIP (Subsequential Interpolation Property) if for every orthogonal sequence $(g_n)_{n\in\mathbb{N}}$ in L and every infinite $M \subseteq \mathbb{N}$, there exist an infinite $A \subseteq M$ and an element $b \in L$ such that $b \ge \oplus_{n\in F} g_n$ for every finite $F \subseteq A$ and $b \perp \oplus_{n\in G} g_n$ for every finite $G \subseteq \mathbb{N} \setminus A$.

As observed in [2], the previous definition corresponds to the one introduced in the Boolean case.

Imitating Guariglia's work we say that

Definition 2.3. – $\mathcal{G}$ is a (D)-SIP lattice if, for every orthogonal sequence g_n in $\mathcal{G}$, there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ and a sub-effect algebra of $\mathcal{G}$ with SIP containing the g_{n_k}'s

Definition 2.4. – A modular measure ϕ on L is called $\mathcal{G}$-exhaustive if for every orthogonal sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ we have $\lim_n \phi(g_n) = 0$.

Notation 2.5. – *Let ϕ be a modular measure on L, $\mathcal{C} \subseteq L$ and $a \in L$. We put*

$$\tilde{\phi}(a) := \sup\{|\phi(h)| : h \in L,\ h \le a\}$$

and

$$\mathcal{C}_a := \{h \in \mathcal{C},\ h \le a\}.$$

Definition 2.6. – We say that a modular measure ϕ is *regular* if for every $\varepsilon > 0$ and

- for every $a \in L$, there exist $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that
$$f \le a \le g \quad \text{and} \quad \tilde{\phi}(g \ominus f) < \varepsilon$$
- for every $f \in \mathcal{F}$, there exist $e \in \mathcal{G}$, $h \in \mathcal{F}$, $g \in \mathcal{G}$ such that
$$f \le e \le h \le g \quad \text{and} \quad \tilde{\phi}(g \ominus f) < \varepsilon$$

3. – The theorems.

Lemma 3.1. – *Let Φ be a set of regular modular measures from L to G such that*

- *(α) $\sup_{\phi\in\Phi} |\phi(g)| < +\infty$ for every $g \in \mathcal{G}$;*
- *(β) for every sequence $(\phi_n)_{n\in\mathbb{N}}$ in Φ and every othogonal sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ there exists an infinite subset M of $\mathbb{N}$ such that $\sup\{|\phi_n(g_n)| : n \in M\} < +\infty$.*

Then $\sup\{|\phi(a)| : \phi \in \Phi, a \in L\} < +\infty$.

Proof. – Assume that $\sup\{|\phi(a)| : \phi \in \Phi, a \in L\} = +\infty$.

We will show that Φ satisfies the following property:

(γ) For every $a \in \mathcal{G}$ with $\sup\{\tilde{\phi}(a) : \phi \in \Phi\} = +\infty$ and for every $n \in \mathbb{N}$ there exist $\phi \in \Phi$ and $g, a^* \in \mathcal{G}_a$ such that $|\phi(g)| \ge n$, $g \le a^{*\perp}$ and $\sup\{\tilde{\phi}(a^*) : \phi \in \Phi\} = +\infty$.

Fix $a \in \mathcal{G}$ with $\sup\{\tilde{\phi}(a) : \phi \in \Phi\} = +\infty$ and $n \in \mathbb{N}$.

Let $h \in \mathbb{N}$ such that $\sup\{|\phi(a)| : \phi \in \Phi\} \le h$. There are two possibilities.

Case 1. – There exist $\phi \in \Phi$ and $t \in \mathcal{F} \wedge a$ such that $|\phi(t)| \ge 2(n+h)$ and

In this case take $g^*, g^{**} \in \mathcal{G}, f^* \in \mathcal{F}$ such that $t \leq g^* \wedge a \leq f^* \wedge a \leq a \wedge g^{**}$ and $\tilde{\phi}((a \wedge g^{**}) \ominus t) \leq n + h$, so $|\phi(f^* \wedge a)| \geq n + h$.

Then if we put

$$g := a \ominus (f^* \wedge a) \quad \text{and} \quad a^* := g^* \wedge a,$$

one can check that $(\phi, g, a^*) \in \Phi \times \mathcal{G}_a \times \mathcal{G}_a$ are as desired.

CASE 2. – For every $\phi \in \Phi$ and $t \in \mathcal{F} \wedge a$, $|\phi(t)| \geq 2(n+h)$ implies $\sup\{\tilde{\phi}(t) : \phi \in \Phi\} < +\infty$.

In this case, let $f \in \mathcal{F}_a$ and $\phi^* \in \Phi$ such that $|\phi^*(f)| \geq 4(n+h)$; then we can find $g^*, g^{**} \in \mathcal{G}, f^* \in \mathcal{F}$ such that $f \leq g^* \leq f^* \leq g^{**}$ and $\tilde{\phi}^*(g^{**} \ominus f) \leq n + h$. So $|\phi^*(f^* \wedge a)| \geq 2(n+h)$ and $|\phi^*(g^* \wedge a)| \geq 2(n+h)$. Therefore by assumptions $\sup\{\tilde{\phi}(f^* \wedge a) : \phi \in \Phi\} < +\infty$.

Put

$$g := g^* \wedge a \quad \text{and} \quad a^* := a \ominus (f^* \wedge a),$$

then they are as desired.

We can choose $a_1 \in \mathcal{G}$ such that $\sup\{\tilde{\phi}(a_1) : \phi \in \Phi\} = +\infty$ (Simply, take $a_1 = 1$). By (γ), we can find $(\phi_1, g_1, a_2) \in \Phi \times \mathcal{G}_{a_1} \times \mathcal{G}_{a_1}$ such that $|\phi_1(g_1)| \geq 1$, $g_1 \leq a_2^{\perp}$, $\sup\{\tilde{\phi}(a_2) : \phi \in \Phi\} = +\infty$.

Continuing we can find, for every $n \in \mathbb{N}$, $|\phi_n(g_n)| \geq n$, $g_n \leq a_{n+1}^{\perp}$ and $\sup\{\tilde{\phi}(a_{n+1}) : \phi \in \Phi\} = +\infty$. Then we obtain a sequence $(\phi_n)_{n \in \mathbb{N}} \in \Phi$ and an orthogonal sequence $(g_n)_{n \in \mathbb{N}} \in \mathcal{G}$ such that $|\phi_n(g_n)| \geq n$ for every $n \in \mathbb{N}$, a contradiction with (β). □

THEOREM 3.2. – *Suppose that $\mathcal{G}$ is a (D)-SIP lattice and Φ is a set of $\mathcal{G}$-exhaustive regular modular measures from L to G such that $\sup_{\phi \in \Phi} |\phi(g)| < +\infty$ for every $g \in \mathcal{G}$. Then $\sup\{|\phi(a)| : \phi \in \Phi, a \in L\} < +\infty$.*

PROOF. – It suffices to prove (β) of Lemma 3.1. For this, let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in Φ and $(g_n)_{n \in \mathbb{N}}$ be an orthogonal sequence in $\mathcal{G}$; then by assumptions there exists a subsequence $(g_{n_h})_{h \in \mathbb{N}}$ of $(g_n)_{n \in \mathbb{N}}$ contained in a sub-effect algebra E of L with SIP. By [2, 4.6] we have

$$\sup_{h \in \mathbb{N}} |\phi_{n_h}(g_{n_h})| \leq \sup\{|\phi_n(g)| : n \in \mathbb{N}, g \in E\} < +\infty,$$

and this completes the proof. □

We now offer a version of the theorem for measures with values in a topological Abelian group. First we give the definition of boundedness in this framework.

DEFINITION 3.3. – Let G be topological Abelian group. A subset M of G is called bounded in G if, for every 0-neighbourhood U in G, there is a finite subset F of G and an integer $n \in \mathbb{N}$ with $M \subset F + U^n$.

PROPOSITION 3.4. – ([6, 6.8]) *Let G be a topological Abelian group. A subset M of G is bounded if and only if, for every continuous seminorm $|\ |$ on G, $\sup_{y \in M} |y| < +\infty$*

DEFINITION 3.5. – Let G be a topological Abelian group and $\mathcal{U}(0)$ be the set of the neighborhoods of the neutral element in G. A modular measure on L with values in G is regular if for every $U \in \mathcal{U}(0)$ and

- for every $a \in L$, there exist $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that

$$f \leq a \leq g \quad \text{and} \quad \phi(r) \in U \quad \text{for} \quad r \in L_{g \ominus f}$$

- for every $f \in \mathcal{F}$, there exist $e \in \mathcal{G}$, $h \in \mathcal{F}$, $g \in \mathcal{G}$ such that

$$f \leq e \leq h \leq g \quad \text{and} \quad \phi(r) \in U \quad \text{for} \quad r \in L_{g \ominus f}$$

THEOREM 3.6. – *Suppose that G is a topological Abelian group, $\mathcal{G}$ is a (D)-SIP lattice and Φ is a set of $\mathcal{G}$-exhaustive pointwise bounded regular modular measures from L to G. Then Φ is uniformly bounded.*

PROOF. – Observe that the topology of G is generated by a family of seminorms and a modular measure is regular if and only if it is regular with respect to this family of seminorms. Apply Proposition 3.4 and Theorem 3.2 to complete the proof. □

Theorem 3.6 generalizes the main result contained in Guariglia's paper [5].

We continue offering a version valid for $\mathcal{G}$ satisfying the Subsequential Completeness Property:

DEFINITION 3.7. – We say that $\mathcal{G}$ has the SCP (Subsequential Completeness Property) if for any orthogonal sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{G}$, there exists an infinite subset A of $\mathbb{N}$ such that $\oplus_{n \in A} g_n$ exists in $\mathcal{G}$.

We recall that a Boolean algebra R has the SCP if for any disjoint sequence $(a_n)_{n \in \mathbb{N}}$ in R, there is an infinite subset A of $\mathbb{N}$ such that $\vee_{n \in A} a_n$ exists in R.

THEOREM 3.8. – *Suppose that G is a topological Abelian group, $\mathcal{G}$ has the SCP and Φ is a set of $\mathcal{G}$-exhaustive regular modular measures from L to G such that $\Phi(g) := \{\phi(g) : \phi \in \Phi\}$ is bounded for every $g \in \mathcal{G}$. Then $\Phi(L) := \{\phi(a) : \phi \in \Phi, a \in L\}$ is bounded.*

PROOF. – As in Theorem 3.6 we may reduce to the seminormed case. It suffices to prove (β) of Lemma 3.1. For this, let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in Φ and subsequence $(g_{n_h})_{h\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ such that $\oplus_{h\in\mathbb{N}} g_{n_h}$ exists and belongs to $\mathcal{G}$. Define $\nu_n(A) := \phi_n(\oplus_{h\in A}\, g_{n_h})$ whenever $\oplus_{h\in A} g_{n_h} \in \mathcal{G}$. With the aid of [1, 2.5], one can check that they form a sequence of finitely additive measures. Moreover, the set $\mathcal{A} := \{A \subseteq \mathbb{N} : \oplus_{h\in A} g_{n_h} \in \mathcal{G}\}$ is a Boolean algebra with SCP. By [6, 7.1.2] we have

$$\sup_{h\in\mathbb{N}} |\phi_{n_h}(g_{n_h})| \leq \sup\{|\nu_n(A)| : n \in \mathbb{N}, A \in \mathcal{A}\} < +\infty,$$

and this completes the proof. □

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Dipartimento di Matematica e Informatica, Università di Udine
Via delle Scienze 206, 33100 Udine, Italy
E-mail: giuseppina.barbieri@dimi.uniud.it

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