Convergent sequences in precompact group topologies

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Abstract

We study the sequences of integers (u_n) that converge to 0 in some precompact group topology on $\mathbb{Z}$ and the properties of the finest topology with this property when (u_n) satisfies a linear recurrence relation with bounded coefficients. Some of the results are extended to the case of sequences in arbitrary Abelian groups.

1 Introduction

Given a sequence (u_n) of an Abelian group G, one may ask when there exists a Hausdorff group topology τ on G such that $u_n \to 0$ in (G, τ). For the group of the integers $\mathbb{Z}$ this question has been studied in [8, 21, 30, 31]. The general case of an Abelian group G is faced in [33] where the authors call a sequence (u_n) of G a *T-sequence* if (u_n) converges to zero for some Hausdorff group topology on G. The counterpart of the above question for *precompact* group topologies on $\mathbb{Z}$ is studied by Raczkowski [34]. A more general approach to the question is considered in [1] where the authors propose a counterpart of the notion of T-sequence for precompact group topologies by calling a sequence (u_n) of an Abelian group G a *TB-sequence* if there exists a precompact group topology τ on G such that $u_n \longrightarrow 0$ in (G, τ). For every TB-sequence of integers $\underline{u} := (u_n) \in \mathcal{Z} := \mathbb{Z}^{\mathbb{N}}$ there exists a *finest* precompact group topology $\sigma_{\underline{u}}$ on $\mathbb{Z}$ that makes $\underline{u}$ converging to 0. A crucial role for the study of TB-sequences of $\mathbb{Z}$ is played by suitable subgroups of the circle group $\mathbb{T}$. Indeed, for a sequence $\underline{u} \in \mathcal{Z}$, one can consider the subgroup $t_{\underline{u}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \to 0 \text{ in } \mathbb{T}\}$ of *topologically $\underline{u}$-torsion* elements of $\mathbb{T}$ (cf. [13, 15]). In these terms $\underline{u}$ is a TB-sequence if and only if $t_{\underline{u}}(\mathbb{T})$ is infinite, moreover the weight of $\sigma_{\underline{u}}$ coincides with the cardinality of $t_{\underline{u}}(\mathbb{T})$ (see [1, Proposition 2.4]). This connection, along with motivation from other fields, justified the study of the subgroups $t_{\underline{u}}(\mathbb{T})$ of $\mathbb{T}$ in [1, 2, 6, 13, 18, 27, 28, 34].

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In this paper we face the problem of describing precompact group topologies on the integers that make a fixed sequence $\underline{u} \in \mathcal{Z}$ converge to 0, paying particular attention to the properties of the topology $\sigma_{\underline{u}}$. In §5 we consider the problem also in arbitrary Abelian groups.

In §1.2 we recall some basic facts on precompact group topologies on Abelian groups and the relationship between the topology $\sigma_{\underline{u}}$ of $\mathbb{Z}$ and the subgroups $t_{\underline{u}}(\mathbb{T})$ of $\mathbb{T}$.

§2 is dedicated to some general properties of $\sigma_{\underline{u}}$ and TB-sequences. One of them is a dychotomy rule: $\sigma_{\underline{u}}$ is always either metrizable or it has weight $\mathfrak{c}$ (see Theorem 2.2). The other properties are mainly in relation to the ratio $\frac{u_n}{u_{n-1}}$:

- if $\frac{u_n}{u_{n-1}} \to \infty$, then the weight of $\sigma_{\underline{u}}$ is $\mathfrak{c}$,
- if $\frac{u_n}{u_{n-1}}$ is bounded, then $\sigma_{\underline{u}}$ is metrizable.

We extend to TB-sequences a result proved in [33] for T-sequences by showing that for every real number $\theta \geq 1$ there exists a TB-sequence $\underline{u}$ such that $\frac{u_n}{u_{n-1}} \to \theta$ (cf. Theorem 2.3).

In §3 we recall known properties of $t_{\underline{u}}(\mathbb{T})$ when the sequence $\underline{u}$ satisfies a linear recurrence relation with bounded coefficients.

The properties of the topology $\sigma_{\underline{u}}$ and the completion $K_{\underline{u}}$ of $(\mathbb{Z}, \sigma_{\underline{u}})$, with particular emphasis on metrizability, will be discussed in §4. Every metrizable precompact group topology on the integers is of the form $\sigma_{\underline{u}}$ for some sequence $\underline{u} \in \mathbb{Z}$ (cf. Theorem 4.1). On the other hand, for every TB-sequence $\underline{u}$ satisfying a linear recurrence relation with bounded coefficients, $\sigma_{\underline{u}}$ is metrizable and the structure of $K_{\underline{u}}$ can be completely described (cf. Theorem 4.4). We provide sufficient conditions for sequences satisfying a second order recurrence relation $u_n = a_n u_{n-1} + b_n u_{n-2}$ ($a_n \in \mathbb{N}$, $b_n \in \mathbb{N} \cup \{0\}$) to be TB-sequences (namely, $a_n \geq b_n$). Moreover, the asymptotic behaviour of $\frac{u_n}{u_{n-1}}$ determines metrizability of $\sigma_{\underline{u}}$ whenever $a_n \geq b_n$ (cf. Theorem 4.6). In §4.2 we study TB-sequences satisfying a second order recurrence relation $u_n = a_n u_{n-1} + b_n u_{n-2}$ where a_n, b_n are *fixed* sequences with $a_n \geq b_n > 0$ and the initial values (u_1, u_2) vary in $\mathbb{N}^2$. We show that under suitable assumptions on the sequences $(a_n), (b_n)$, there exists a single precompact group topology on $\mathbb{Z}$ that makes all these sequences converge to 0 (cf. Theorem 4.17).

The Fibonacci's sequence $\underline{u}$, defined by $u_n = u_{n-1} + u_{n-2}$ for $n > 2$ and $u_1 = 1$, $u_2 = 1$, is a T-sequence [33, Theorem 14]. The authors of [33] set the problem of characterizing all T-sequences satisfying a second order recursion with constant coefficients recursion $u_n = a u_{n-1} + b u_{n-2}$. The counterpart if this question for TB-sequences is solved in Theorem 4.11.

TB-sequences as well as precompact group topologies without nontrivial convergent sequences on *arbitrary* Abelian groups are considered in §5. In this connection let us recall also the following quantitative result proved in [10]:

Theorem 1.1 [10, Theorems 4.1 and 5.5]

(a) Every infinite Abelian group G admits a family $\mathcal{A}$ of $2^{2^{|G|}}$-many pairwise nonhomeomorphic precompact group topologies such that no nontrivial sequence in G converges in any of the topologies $\tau \in \mathcal{A}$.

(b) Every infinite Abelian group G admits a family $\mathcal{B}$ of $2^{2^{|G|}}$-many pairwise nonhomeomorphic precompact group topologies, with $w(G, \tau) = 2^{|G|}$ for all $\tau \in \mathcal{B}$, such that some fixed faithfully indexed sequence in G converge to 0 in each $\tau \in \mathcal{B}$.

1.1 Notation

We denote by $\mathbb{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$ and v_p, $p \in \mathbb{P}$, the set of primes, the set of positive integers, the group of integers, the group of rationals, the group of p-adic integers and the p-adic valuation, respectively. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology. We will denote by $\varphi : \mathbb{R} \to \mathbb{T}$ the canonical map. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum so $\mathfrak{c} = 2^{\aleph_0}$ and we denote by $\mathcal{Q}_0$ the subgroup $\bigoplus_{\mathbb{N}} \mathbb{Q}$ of $\mathbb{Q}^{\mathbb{N}}$.

For $\underline{u} \in \mathcal{Z} := \mathbb{Z}^{\mathbb{N}}$, the subgroup $t_{\underline{u}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \to 0 \text{ in } \mathbb{T}\}$ is a proper subgroup of $\mathbb{T}$ if and only if $\underline{u} \in \mathcal{Z}^* := \mathcal{Z} \setminus \mathcal{Q}_0$ (cf. [1, Example 2.8]).

To study the group $t_{\underline{u}}(\mathbb{T})$, it is sometimes convenient to deal with real numbers instead of elements of $\mathbb{T}$. That is why, for every sequence $\underline{u} \in \mathcal{Z}$, we consider also the subgroup $\tau_{\underline{u}}(\mathbb{R}) := \varphi^{-1}(t_{\underline{u}}(\mathbb{T}))$ of $\mathbb{R}$. Then $\mathbb{Z} \subseteq \tau_{\underline{u}}(\mathbb{R})$ and $\varphi(\tau_{\underline{u}}(\mathbb{R})) = t_{\underline{u}}(\mathbb{T})$.

Let G be an Abelian group. The subgroup of torsion (p-torsion) elements of G is denoted by $t(G)$ (resp., by $t_p(G)$). We denote by $r(G)$ the free rank of G. For a positive integer n we denote by $\mathbb{Z}(n)$ the cyclic group of order n. For a topological group (G, τ), we denote by $w(G)$ the *weight* of G, i.e. the minimal cardinality of a base for the topology on G.

For undefined terms see [17, 19, 24, 25].

1.2 Background on precompact group topologies

Let G be an Abelian group. A group topology τ on G is said to be *totally bounded* if for every non-empty open set U of (G, τ) there exists a finite subset F of G such that $G = U + F$. If τ is Hausdorff and totally bounded, then τ is called *precompact* because the completion of (G, τ) is a compact group [36].

We denote by $Hom(G, \mathbb{T})$ the group of all characters of G, i.e. the group of all homomorphisms from G to the circle group $\mathbb{T}$. When (G, τ) is an Abelian topological group, the set of τ-continuous homomorphisms $\chi : G \to \mathbb{T}$ is a subgroup of $Hom(G, \mathbb{T})$ and is denoted by $\widehat{G}$. When no topology is specified on G we assume that G is discrete, so that $\widehat{G} = Hom(G, \mathbb{T})$.

For an Abelian group G and a subgroup $H \leq Hom(G, \mathbb{T})$, let T_H be the weakest topology on G that makes all characters of G from H continuous with respect to T_H. One easily shows that T_H is a totally bounded group topology on G, called the *topology generated* by H, and $\widehat{(G, T_H)} = H$. Moreover, T_H is Hausdorff if and only if H separates the points of G, i.e. for every $g \in G \setminus \{0\}$ there exists $h \in H$ such that $h(g) \neq 0$.

It was proved by Comfort and Ross that conversely every precompact group topology τ on an Abelian group G is generated by some suitable point-separating subgroup of characters $H \leq Hom(G, \mathbb{T})$ of G; moreover the correspondence $H \longleftrightarrow T_H$ is an order isomorphism between the family of all subgroups of $\widehat{G}$ and the family of all totally bounded group topologies on G (cf. [11, Theorem 1.2]).

A group topology τ on G is called *linear* if it has a local base at 0 consisting of open subgroups of G. It is easy to see that a linear group topology on G is totally bounded if and only if every open subgroup of G has finite index. Therefore, every non-discrete linear group topology on $\mathbb{Z}$ is totally bounded.

It follows from [1, Corollary 2.3] that for $\underline{u} \in \mathcal{Z}$, $\sigma_{\underline{u}} := T_{t_{\underline{u}}(\mathbb{T})}$ is the finest totally bounded group topology on $\mathbb{Z}$ for which $\underline{u}$ converges to 0. The next proposition gives a relationship between $\sigma_{\underline{u}}$ and $t_{\underline{u}}(\mathbb{T})$ (compare with Theorem 5.3).

Proposition 1.2 [1, Proposition 2.4] *Let* $\underline{u} \in \mathcal{Z}$. *Then:*

(a) $w(\mathbb{Z}, \sigma_{\underline{u}}) = |t_{\underline{u}}(\mathbb{T})|$;

(b) $\sigma_{\underline{u}}$ *is Hausdorff (i.e.,* $\underline{u}$ *is a TB-sequence) if and only if* $t_{\underline{u}}(\mathbb{T})$ *is infinite;*

(c) $\sigma_{\underline{u}}$ *is metrizable if and only if* $|t_{\underline{u}}(\mathbb{T})| = \aleph_0$;

(d) $\sigma_{\underline{u}}$ *is linear if and only if the subgroup* $t_{\underline{u}}(\mathbb{T})$ *is torsion.*

2 The asymptotic behaviour of $\frac{u_n}{u_{n-1}}$

By Proposition 1.2 (a), for every sequence $\underline{u} \in \mathcal{Z}$, the weight of the topology $\sigma_{\underline{u}}$ coincides with the size of the corresponding subgroup $t_{\underline{u}}(\mathbb{T})$. This shows the importance of determing the cardinality of $t_{\underline{u}}(\mathbb{T})$.

The next theorem shows that for $\underline{u} \in \mathbb{N}^{\mathbb{N}}$, the behaviour at infinity of the ratios $\frac{u_n}{u_{n-1}}$ determines specific restraints on $|t_{\underline{u}}(\mathbb{T})|$.

Theorem 2.1 [1, Theorems 3.1 and 3.3] *Let* $\underline{u}$ *be a sequence in* $\mathbb{Z} \setminus \{0\}$.

(a) *If* $\frac{u_n}{u_{n-1}} \to \infty$, *then* $|t_{\underline{u}}(\mathbb{T})| = \mathfrak{c}$ *and therefore* $w(\mathbb{Z}, \sigma_{\underline{u}}) = \mathfrak{c}$.

(b) *If* $\frac{u_n}{u_{n-1}}$ *is bounded, then* $|t_{\underline{u}}(\mathbb{T})| \leq \aleph_0$ *and therefore* $w(\mathbb{Z}, \sigma_{\underline{u}}) \leq \aleph_0$.

It follows from [1, Remark 3.5] that neither in (a) nor in (b) of Theorem 2.1 the implications can be replaced by an equivalence. On the other hand, for sequences satisfying a suitable recursive relation, the asymptotic behaviour of $\frac{u_n}{u_{n-1}}$ completely determines the size of $t_{\underline{u}}(\mathbb{T})$ (cf. Theorem 3.7 below).

The weight of a precompact group topology on $\mathbb{Z}$ can vary arbitrarily between $\aleph_0$ and $\mathfrak{c}$. However, for those of the form $\sigma_{\underline{u}}$ the following dycotomy holds:

Theorem 2.2 *For every sequence* $\underline{u} \in \mathcal{Z}$ *the following conditions are equivalent:*

(a) $\sigma_{\underline{u}}$ *is Hausdorff and not metrizable;*

(b) *the weight of* $\sigma_{\underline{u}}$ *is* $\mathfrak{c}$;

(c) *there are* $2^{\mathfrak{c}}$*-many pairwise non homeomorphic precompact group topologies of weight* $\mathfrak{c}$ *that make* $\underline{u}$ *converge to 0.*

Proof. (c) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b) Since for every sequence $\underline{u} \in \mathcal{Z}$, $t_{\underline{u}}(\mathbb{T})$ is a Borel subset of $\mathbb{T}$, the subgroup $t_{\underline{u}}(\mathbb{T})$ is either countable or it has size $\mathfrak{c}$ (see [26, Theorem 13.6]). Therefore it follows from Proposition 1.2 that the weight of $\sigma_{\underline{u}}$ is $\mathfrak{c}$ if $\sigma_{\underline{u}}$ is Hausdorff and not metrizable.

(b) $\Rightarrow$ (c) By Proposition 1.2 (a) the subgroup $H = t_{\underline{u}}(\mathbb{T})$ has size $\mathfrak{c}$. Since $t(H) \leq \mathbb{Q}/\mathbb{Z}$ is countable, it follows that $r(H) = \mathfrak{c}$, so H contains a subgroup $H_1 = \oplus_{i<\mathfrak{c}} C_i$, where $C_i \cong \mathbb{Z}$ for every $i < \mathfrak{c}$. The family $[\mathfrak{c}]^{\mathfrak{c}}$ of subsets B of $\mathfrak{c}$ of size $\mathfrak{c}$ has size $2^{\mathfrak{c}}$. Moreover, for each $B \in [\mathfrak{c}]^{\mathfrak{c}}$ the subgroup $H_B = \oplus_{i \in B} C_i$ of H has size $\mathfrak{c}$. Clearly the correspondence $B \mapsto H_B$ is one-to-one so we get a family of size $2^{\mathfrak{c}}$ of infinite subgroups of H of size $\mathfrak{c}$. Thus the family $\mathcal{T} = \{T_{H_B} : B \in [\mathfrak{c}]^{\mathfrak{c}}\}$ has size $2^{\mathfrak{c}}$ as well. Moreover, $\underline{u}$ converges to 0 in T_{H_B} as $H_B \leq t_{\underline{u}}(\mathbb{T})$ for every $B \in [\mathfrak{c}]^{\mathfrak{c}}$ (cf. [1, Proposition 2.2]). Finally, T_{H_B} has weight $|H_B| = \mathfrak{c}$ for each $B \in [\mathfrak{c}]^{\mathfrak{c}}$. To finish the proof argue as in [10] to see that at most $\mathfrak{c}$ many members of $\mathcal{T}$ can be pairwise homeomorphic (due to the fact that there at most $\mathfrak{c}$ many permutations of $\mathbb{Z}$), while $\mathcal{T}$ has size $2^{\mathfrak{c}}$. QED

Now combining this proposition with Theorem 2.1 we see practically the source of $2^{\mathfrak{c}}$ many precompact group topologies on $\mathbb{Z}$ of weight $\mathfrak{c}$ having a nontrivial convergent sequence (compare with Theorem 1.1).

For a sequence $\underline{u} \in \mathcal{Z}$ and a prime $p \in \mathbb{P}$, we set

$$n_p(\underline{u}) := \sup\{m \in \mathbb{N} : p^m | u_n \text{ eventually}\},$$

i.e. $n_p(\underline{u}) = \liminf v_p(u_n) \in \mathbb{N} \cup \{0, \infty\}$. The parameter $n_p(\underline{u})$ describes the p-torsion subgroup of $t_{\underline{u}}(\mathbb{T})$; in fact we have $t(t_{\underline{u}}(\mathbb{T})) \cong \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^{n_p(\underline{u})})$ for any $\underline{u} \in \mathcal{Z}$ (cf. [2, Proposition 2.3]).

In [33, Theorem 13] it is proved that for every real number $\theta \geq 1$ there exists a T-sequence $\underline{u} \in \mathcal{Z}$ such that $\frac{u_n}{u_{n-1}} \to \theta$. In the next theorem we show that $\underline{u}$ can be chosen to be a TB-sequence.

Theorem 2.3 *Let $\theta \in \mathbb{R}$ and $\theta \geq 1$. Then there exists a sequence $\underline{u} \in \mathbb{N}^{\mathbb{N}}$ such that $\frac{u_n}{u_{n-1}} \to \theta$ and $t_{\underline{u}}(\mathbb{T}) \supseteq \mathbb{Q}/\mathbb{Z}$. In particular, $\underline{u}$ is a TB-sequence.*

Proof. (i) Let $\theta = 1$. Consider the sequence $l_m = (p_1 \dots p_m)^m$ where $p_1, p_2, p_3, \dots$ are all prime numbers and $l_0 = 1$. For every n let s_n be the greatest index such that $l_{s_n} \leq n$, so that $n < l_{s_n+1}$. Then $\lim_n s_n = \infty$. Let r_n be the rest of n^2 modulo l_{s_n}, i.e. $l_{s_n} | n^2 - r_n$ and $0 \leq r_n < l_{s_n}$. In particular, $r_n < l_{s_n} \leq n$. Now define $v_n := n^2 - r_n$. Hence $l_{s_n} | v_n$ and $n^2 \geq v_n > n^2 - n$. Therefore, for every $n > 1$,

$$\frac{v_n}{v_{n-1}} \geq \frac{n^2 - n}{(n-1)^2} \to 1$$

and

$$\frac{v_n}{v_{n-1}} \leq \frac{n^2}{(n-1)^2 - (n-1)} \to 1.$$

This proves that $\frac{v_n}{v_{n-1}} \to 1$. Finally, since $l_m | v_n$ for $n, m \in \mathbb{N}$ with $m \geq s_n$, it follows that $n_p(\underline{v}) = \infty$ for every $p \in \mathbb{P}$. Hence, $t_{\underline{v}}(\mathbb{T}) \supseteq \mathbb{Q}/\mathbb{Z}$.

(ii) If $\theta > 1$, choose $(w_n) \in \mathbb{N}^{\mathbb{N}}$ such that $\frac{w_n}{w_{n-1}} \to \theta$. Set $u_n := v_n \cdot w_n$, where v_n is defined as in (i). Then obviously $\frac{u_n}{u_{n-1}} \to \theta$ and $n_p(\underline{u}) \geq n_p(\underline{v})$ for every $p \in \mathbb{P}$, therefore $t_{\underline{u}}(\mathbb{T}) \supseteq t(t_{\underline{u}}(\mathbb{T})) \supseteq t(t_{\underline{v}}(\mathbb{T})) = \mathbb{Q}/\mathbb{Z}$. QED

It was proved in [33, Theorem 12] that for every algebraic number $\theta \geq 1$ there exists a sequence u_n satisfying $\frac{u_n}{u_{n-1}} \to \theta$ without being a T-sequence. On the other hand, if θ is a transcendental number and $\frac{u_n}{u_{n-1}} \to \theta$, then $\underline{u}$ is a T-sequence by [33, Theorem 11].

Question 2.4 If the sequence $\underline{u} \in \mathcal{Z}$ satisfies $\frac{u_n}{u_{n-1}} \to \theta > 1$ with θ transcendental, can we assert that $\underline{u}$ is always a TB-sequence?

In other words, we ask whether also in the case of TB-sequences one can distinguish in this respect between transcendental and algebraic numbers.

3 The structure of the groups $t_{\underline{u}}(\mathbb{T})$ for $\underline{u}$ satisfying a linear recurrence

3.1 Linear recurrences of order k

Let $k \in \mathbb{N}$ and $a_n^{(i)} \in \mathbb{Z}$ for $n, i \in \mathbb{N}$, $i \leq k$.

In this section we recall the structure theorems for $t_{\underline{u}}(\mathbb{T})$ proved in [2] when the sequence $\underline{u}$ satisfies the following linear recurrence relation:

$$u_n = a_n^{(1)} u_{n-1} + a_n^{(2)} u_{n-2} + \ldots + a_n^{(k)} u_{n-k} \ \ (n > k).$$

We denote by $\mathcal{R}_k$ the group of all sequences $\underline{r} \in \mathcal{Z} + \mathcal{Q}_0$ satisfying $r_n = a_n^{(1)} r_{n-1} + a_n^{(2)} r_{n-2} + \ldots + a_n^{(k)} r_{n-k}$. Clearly, $\mathcal{R}_k$ is a group of rank k.

Theorem 3.1 [2, Theorem 3.10] *Let $\{a_n^{(i)} : n \in \mathbb{N}, i = 1, ..., k\}$ be bounded and $a_n^{(k)} = 1$ $(n \in \mathbb{N})$. If $\underline{u} \in \mathcal{R}_k \cap \mathcal{Z}^*$, then $t_{\underline{u}}(\mathbb{T})$ is a finitely generated group. More precisely, if d is the greatest common divisor of $u_1, u_2, \ldots, u_k$, then $t_{\underline{u}}(\mathbb{T}) \cong \mathbb{Z}^r \times \mathbb{Z}(d)$ with $r < k$. In particular, $t_{\underline{u}}(\mathbb{T})$ is free if and only if $d = 1$.*

Theorem 3.2 [2, Theorems 3.11 and 3.12] *Let $\{a_n^{(k)} : n \in \mathbb{N}\}$ be bounded and $\underline{u} \in \mathcal{R}_k \cap \mathcal{Z}^*$. Let P_1 be the set of the primes $p \in \mathbb{P}$ dividing infinitely many $a_n^{(k)}$. Then $t_{\underline{u}}(\mathbb{T}) = T \oplus G$, where*

(a) G is a torsion-free subgroup of $t_{\underline{u}}(\mathbb{T})$,

(b) $T = t(t_{\underline{u}}(\mathbb{T}))$ is isomorphic to $\mathbb{Z}(m) \oplus \bigoplus_{p \in P_\infty} \mathbb{Z}(p^\infty)$ with $P_\infty \subseteq P_1$ and $p \not| m$ for all $p \in P_\infty$.

Moreover, if the sequences $\{a_n^{(i)} : n \in \mathbb{N}, i = 1, \ldots, k\}$ are bounded, then $r(G) < k$. If also $a_n^{(k)} \neq 0$ $(n \in \mathbb{N})$ and $r(G) = r > 0$, then there exists a free subgroup $F_0 \cong \mathbb{Z}^r$ of G such that $G/F_0 \cong \bigoplus_{p \in P_1} \mathbb{Z}(p^\infty)^{m_p}$, where $m_p \leq r$ for every $p \in P_1$.

Note that in the above theorem the set P_1 is finite since $\{a_n^{(k)} : n \in \mathbb{N}\}$ is bounded, hence $t_{\underline{u}}(\mathbb{T})$ may have only finitely many p-primary components of the form $\mathbb{Z}(p^\infty)$.

3.2 Linear recurrences of second order

In this section, let $a_n \in \mathbb{N}$ and $b_n \in \mathbb{N} \cup \{0\}$ $(n > 2)$. We will study $t_{\underline{u}}(\mathbb{T})$ for $\underline{u}$ satisfying

$$u_n = a_n u_{n-1} + b_n u_{n-2} \ \ (n > 2) \ \text{ with } u_1,\, u_2 \in \mathbb{N}.$$

According to Proposition 1.2 (b), to get TB-sequences on $\mathbb{Z}$ it is sufficient to know when the group $t_{\underline{u}}(\mathbb{T})$ is not torsion. Indeed, if $t_{\underline{u}}(\mathbb{T})$ is not torsion, then obviously it is infinite, hence $\underline{u}$ is a TB-sequence. The next theorem gives some conditions which guarantee that $t_{\underline{u}}(\mathbb{T})$ is not torsion.

Theorem 3.3 [2, Corollary 4.8] *Let $\underline{u} \in \mathbb{N}^{\mathbb{N}}$ satisfy $u_n = a_n u_{n-1} + b_n u_{n-2}$ with $u_1, u_2 \in \mathbb{N}$.*

(a) If a_n, $b_n \in \mathbb{N}$ and $a_n \geq b_n$ for every $n \in \mathbb{N}$, then $t_{\underline{u}}(\mathbb{T})$ is not torsion.

(b) If $a_n < b_n$ for every $n \in \mathbb{N}$ and the sequence b_n is bounded, then $t_{\underline{u}}(\mathbb{T})$ is torsion.

The following example shows that if $a_n < b_n$ for every $n \in \mathbb{N}$, then the subgroup $t_{\underline{u}}(\mathbb{T})$ can be even trivial.

Example 3.4 Let $u_1 = u_2 = 1$ and $u_n = u_{n-1} + pu_{n-2}$ for some $p \in \mathbb{P}$. Then $n_q(\underline{u}) = 0$ for every $q \in \mathbb{P}$ and by Theorem 3.3 (b), $t(t_{\underline{u}}(\mathbb{T})) = t_{\underline{u}}(\mathbb{T}) = \{0\}$.

Question 3.5 Is $\underline{u}$ from Example 3.4 a T-sequence?

It is proved in [33, Theorem 14] that if $a_n \in \mathbb{N}$ for every $n \geq 1$, then the sequence $\underline{u}$ determined by $u_n = a_n u_{n-1} + u_{n-2}$ with $u_1 = 1$ and $u_2 = a_1$ is a T-sequence. By Theorem 3.3 (a) $\underline{u}$ is actually a TB-sequence.

If $b_n = 1$ and (a_n) is bounded, then it is possible to describe $t_{\underline{u}}(\mathbb{T})$.

Theorem 3.6 [2, Theorem 4.10] *Let $(a_n) \in \mathbb{N}^{\mathbb{N}}$ be bounded and $\underline{u} \in \mathbb{N}^{\mathbb{N}}$ satisfy $u_n = a_n u_{n-1} + u_{n-2}$. Let d be the greatest commom divisor of u_1, u_2 and $l, k \in \mathbb{N} \cup \{0\}$ such that $d = |lu_1 - ku_2|$. Let $\underline{r}$ be the sequence satisfying $r_n = a_n r_{n-1} + r_{n-2}$ with $r_1 = k, r_2 = l$. Then the sequence $\frac{r_n}{u_n}$ converges to an irrational number θ and $t_{\underline{u}}(\mathbb{T}) = \langle \varphi(\theta) \rangle \oplus \langle \varphi(\frac{1}{d}) \rangle \cong \mathbb{Z} \times \mathbb{Z}(d)$.*

The next theorem shows that the size of $t_{\underline{u}}(\mathbb{T})$ is determined by the asymptotic behaviour of $\frac{u_n}{u_{n-1}}$ whenever $a_n \geq b_n$ (cf. Theorem 2.1).

Theorem 3.7 [2, Theorem 4.11] *Let $u_n = a_n u_{n-1} + b_n u_{n-2}$ with $a_n, u_1, u_2 \in \mathbb{N}$, $b_n \in \mathbb{N} \cup \{0\}$ and $a_n \geq b_n$ for every $n \in \mathbb{N}$. Then $|t_{\underline{u}}(\mathbb{T})| \leq \aleph_0$ if and only if $\frac{u_n}{u_{n-1}}$ is bounded.*

3.3 Linear recurrences of second order with constant coefficients

In this section we study sequences $\underline{u} \in \mathcal{Z}$ satisfying a second order recurrence relation $u_n = au_{n-1} + bu_{n-2}$ with constant coefficients a, b and we are looking for conditions on a, b which guarantee that the subgroup $t_{\underline{u}}(\mathbb{T})$ is infinite.

The next proposition follows from Theorem 3.3.

Proposition 3.8 *Let $\underline{u} \in \mathcal{Z}$ satisfy $u_n = au_{n-1} + bu_{n-2}$ $(\forall n > 2)$ with $a, b, u_1, u_2 \in \mathbb{N}$. Then $a \geq b$ if and only if $t_{\underline{u}}(\mathbb{T})$ is not torsion.*

In the next theorem we completely describe the sequences satisfying a second order recurrence relation with constant coefficients for which the torsion subgroup $t(t_{\underline{u}}(\mathbb{T}))$ of $t_{\underline{u}}(\mathbb{T})$ is infinite.

Theorem 3.9 *Let $\underline{u} \in \mathcal{Z}$ satisfy $u_n = au_{n-1} + bu_{n-2}$ $(\forall n > 2)$ with $a, b, u_1, u_2 \in \mathbb{N}$. Then $t(t_{\underline{u}}(\mathbb{T}))$ is infinite if and only if $(a, b) > 1$ or $\underline{u}$ is a geometric progression, i.e. $u_n = u_1 \lambda^{n-1}$ $(n \in \mathbb{N})$ with $\lambda \in \mathbb{Z}$.*

Proof. If $(a, b) > 1$ and $p \in \mathbb{P}$ with $p|(a, b)$, then one can prove by induction that $p^n | u_{2n+1}$ and $p^n | u_{2n+2}$, $(n \in \mathbb{N})$. Therefore $n_p(\underline{u}) = \infty$. If $u_n = u_1 \lambda^{n-1}$ with $\lambda \in \mathbb{Z}$, then $t_{\underline{u}}(\mathbb{T})$ is infinite by [15]; in fact if $p|\lambda$, then $n_p(\underline{u}) = \infty$ and therefore $t(t_{\underline{u}}(\mathbb{T}))$ contains a subgroup isomorphic to $\mathbb{Z}(p^\infty)$.

Conversely, if $t(t_{\underline{u}}(\mathbb{T}))$ is infinite, then $n_p(\underline{u}) = \infty$ for some $p \in \mathbb{P}$ and $p|b$ by [2, Proposition 3.3 (c)]. If $p|a$, then $(a, b) > 1$. Assume that $p \not| a$. The roots of the characteristic polynomial $p(x) = x^2 - ax - b$ are $\lambda_1 = \frac{a+\sqrt{a^2+4b}}{2}$ and $\lambda_2 = \frac{a-\sqrt{a^2+4b}}{2}$. Since $\lambda_1 \neq \lambda_2$, there are constants α and β such that $u_n = \alpha\lambda_1^n + \beta\lambda_2^n$ for every $n \in \mathbb{N}$. Let v_p be a valuation on $\mathbb{R}$ extending the p-adic valuation on $\mathbb{Q}$. Since $\lambda_1 + \lambda_2 = a$ and $\lambda_1\lambda_2 = -b$, the assumption $p \not| a$ and $p|b$ implies that $0 = v_p(\lambda_1 + \lambda_2) \geq \min\{v_p(\lambda_1), v_p(\lambda_2)\}$ and $v_p(\lambda_1) + v_p(\lambda_2) \geq 1$. Then either $v_p(\lambda_1) \geq 1$ and $v_p(\lambda_2) \leq 0$ or $v_p(\lambda_1) \leq 0$ and $v_p(\lambda_2) \geq 1$. Assume that $v_p(\lambda_1) \geq 1$. Then $v_p(\lambda_1^n) \to \infty$. Since $n_p(\underline{u}) = \infty$, $u_n \to 0$ with respect to the p-adic topology on $\mathbb{Z}$. Therefore $v_p(\beta\lambda_2^n) = v_p(u_n - \alpha\lambda_1^n) \to \infty$. Since $v_p(\lambda_2) \leq 0$, this happens only if $\beta = 0$, i.e. $u_n = \alpha{\lambda_1}^n$. As $u_n \in \mathbb{N}$, one obtains that $\lambda_1 \in \mathbb{Q}$ and consequently $\lambda_1 \in \mathbb{Z}$ (as the polynomial $p(x) = x^2 - ax - b$ is monic). QED

Corollary 3.10 *Let $\underline{u} \in \mathcal{Z}$ satisfy $u_n = au_{n-1} + bu_{n-2}$ $(\forall n > 2)$ with a, b, u_1, $u_2 \in \mathbb{N}$. Then $t_{\underline{u}}(\mathbb{T})$ is finite if and only if $a < b$, $(a, b) = 1$ and $\underline{u}$ is not a geometric progression.*

4 Topologizing TB-sequences of integers

By definition, for every TB-sequence $\underline{u} \in \mathcal{Z}$, $\sigma_{\underline{u}}$ is the *finest* precompact group topology on $\mathbb{Z}$ that makes $\underline{u}$ converge to 0. The properties of the topology $\sigma_{\underline{u}}$ and the completion $K_{\underline{u}}$ of $(\mathbb{Z}, \sigma_{\underline{u}})$ will be discussed in this section (with particular emphasis on metrizability). We recall that, by Proposition 1.2 (c), metrizability of $\sigma_{\underline{u}}$ is equivalent to countability of the subgroup $t_{\underline{u}}(\mathbb{T})$ of $\mathbb{T}$.

In the next rather surprizing theorem we show that *every* metrizable precompact group topology on $\mathbb{Z}$ has the form $\sigma_{\underline{u}}$ for an appropriate sequence of integers $\underline{u}$. This should be compared with the following fact proved in [33]: for every T-sequence $\underline{u}$ in $\mathbb{Z}$ the finest *Hausdorff* group topology on $\mathbb{Z}$ that makes $\underline{u}$ converge to 0 is never metrizable.

Theorem 4.1 *Let τ be a metrizable precompact group topology on $\mathbb{Z}$. Then there exists a sequence $\underline{u} \in \mathcal{Z}$ such that $\tau = \sigma_{\underline{u}}$.*

Proof. Let τ be a metrizable precompact group topology on $\mathbb{Z}$. Then by [11, Theorem 1.2] there exists a countable subgroup H of $\mathbb{T}$ such that $\tau = T_H$. By [6, Theorem 2] there exists a sequence $\underline{u} \in \mathcal{Z}$ such that $H = t_{\underline{u}}(\mathbb{T})$, i.e. $\tau = T_H = \sigma_{\underline{u}}$. QED

In the sequel we denote by $c(K_{\underline{u}})$ the connected component of $K_{\underline{u}}$ and by $\dim K_{\underline{u}}$ the *dimension* of $K_{\underline{u}}$ (the reader is referred to [24] for dimension theory in compact groups).

Proposition 4.2 *Let $\underline{u} \in \mathcal{Z}$ be a TB-sequence and $K_{\underline{u}}$ be the completion of $(\mathbb{Z}, \sigma_{\underline{u}})$. Then $(\mathbb{Z}, \sigma_{\underline{u}})$ is metrizable if and only if $c(K_{\underline{u}})$ is metrizable. In particular, $K_{\underline{u}}$ is metrizable whenever $\dim K_{\underline{u}} < \infty$.*

Proof. If $(\mathbb{Z}, \sigma_{\underline{u}})$ is metrizable, then obviously $K_{\underline{u}}$ is metrizable and so does its subgroup $c(K_{\underline{u}})$.

Conversely, assume that $c(K_{\underline{u}})$ is metrizable. Since $K := K_{\underline{u}}/c(K_{\underline{u}})$ is compact and totally disconnected, its Pontryagin dual $\widehat{K}$ is torsion (cf. [25, Theorem 24.26]). Morover, by [25, Corollary 24.20], $\widehat{K}$ is isomorphic to the subgroup $t(\widehat{K_{\underline{u}}})$ of $\widehat{K_{\underline{u}}}$. As $K_{\underline{u}}$ is a compact group with the same character group as $(\mathbb{Z}, \sigma_{\underline{u}})$, its Pontraygin dual $\widehat{K_{\underline{u}}}$ coincides with $t_{\underline{u}}(\mathbb{T})$ endowed with the discrete topology. Therefore, $\widehat{K}$ is isomorphic to the subgroup $t(t_{\underline{u}}(\mathbb{T})) \leq \mathbb{Q}/\mathbb{Z}$, hence it is countable. Then K is metrizable by [25, Theorem 8.3]. From [25, 5.38(e) on p.47] it follows that $K_{\underline{u}}$ is metrizable too.

Let $K_{\underline{u}}$ be finite-dimensional. Then by [25, Theorem 24.28] $r := r(\widehat{c(K_{\underline{u}})}) = \dim c(K_{\underline{u}}) = \dim K_{\underline{u}} < \infty$. Hence, $\widehat{c(K_{\underline{u}})}$ has finite rank and it is torsion-free by [25, Theorem 24.25]. Therefore, $\widehat{c(K_{\underline{u}})} \hookrightarrow \mathbb{Q}^r$, so it is countable and, consequently, $c(K_{\underline{u}})$ is metrizable. QED

Remark 4.3 The conclusion of Proposition 4.2 regarding finite-dimensionality of $K_{\underline{u}}$ cannot be reversed. Indeed there exist many precompact metrizable group topologies τ on $\mathbb{Z}$ such that the completion K of $(\mathbb{Z}, \tau)$ is infinite-dimensional. Indeed let τ be the topology induced on a dense cyclic subgroup of the monothetic group $\mathbb{T}^{\mathbb{N}}$ [25]. Then τ is precompact and metrizable. By Theorem 4.1 there exists a sequence $\underline{u}$ such that $\tau = \sigma_{\underline{u}}$. Hence $K_{\underline{u}}$ is metrizable and $\dim K_{\underline{u}} = \infty$.

4.1 TB-sequences satisfying a linear recurrence

According to Theorem 1.1 (see also [34]) $\mathbb{Z}$ has some faithfully indexed sequence $\underline{u}$ such that $\mathbb{Z}$ admits $2^{\mathfrak{c}}$ many pairwise non-homeomorphic precompact group topologies of weight $\mathfrak{c}$ making $\underline{u}$ converge to 0. We show that for some sequences $\underline{u}$ (as in Theorem 4.4) the family $\mathcal{F}$ of precompact group topologies on $\mathbb{Z}$ making $\underline{u}$ converge to 0 is countable. More precisely, each $\tau \in \mathcal{F}$ is metrizable and all they are pairwise homeomorphic (as every countable metric space without isolated points is homeomorphic to $\mathbb{Q}$ by Sierpinski's theorem [35]).

Theorem 4.4 *Let $\{a_n^{(i)} : n \in \mathbb{N}, i = 1, ..., k\}$ be bounded and $\underline{u} \in \mathcal{Z}$ be a TB-sequence satisfying $u_n = a_n^{(1)} u_{n-1} + a_n^{(2)} u_{n-2} + \ldots + a_n^{(k)} u_{n-k}$. Then $\mathbb{Z}$ has precisely countably many precompact group topologies that make $\underline{u}$ converge to 0. Moreover*

- *(a) $K_{\underline{u}}$ is a compact metrizable Abelian group with $\dim K_{\underline{u}} < k$;*
- *(b) $K_{\underline{u}} \cong c(K_{\underline{u}}) \times N$, where the group N is a finite product of p-adic integers (for distinct primes p) and a finite cyclic group.*
- *(c) If $a_n^{(k)} \neq 0$ $(n \in \mathbb{N})$ and $r = \dim K_{\underline{u}} > 0$, then there exists a continuous surjective homomorphism $f : c(K_{\underline{u}}) \to \mathbb{T}^r$ such that $\ker f \cong \prod_{p \in P_1} \mathbb{Z}_p^{m_p}$ where $P_1 := \{p \in \mathbb{P} : p$ divides infinitely many $a_n^{(k)}\}$, and $m_p \leq r$ for every $p \in P_1$.*
- *(d) Let $a_n^{(k)} = 1$ for all n and d be the greatest commom divisor of $u_1, \ldots, u_k$. Then $K_{\underline{u}} \cong \mathbb{T}^r \times \mathbb{Z}(d)$. Consequently $(\mathbb{Z}, \sigma_u)$ has no proper open subgroups when $d = 1$.*

Proof. By Theorem 3.2, $t_{\underline{u}}(\mathbb{T}) \cong T \oplus G$ where G is a torsion-free group with $r(G) = r < k$. Moreover, since $K_{\underline{u}}$ has the same character group as $(\mathbb{Z}, \sigma_{\underline{u}})$, one has $\widehat{K_{\underline{u}}} = t_{\underline{u}}(\mathbb{T})$ equipped with the discrete topology.

For every precompact group topology T_H on $\mathbb{Z}$ that makes $\underline{u}$ converge to 0 the group H is necessarily contained in $t_{\underline{u}}(\mathbb{T})$ (cf. [1, Proposition 2.2]), so that to prove the first statement we have to see that $t_{\underline{u}}(\mathbb{T})$ has countably many infinite subgroups. This can be seen directly or in [4].

(a) Since $\dim K_{\underline{u}} = r(G)$, $K_{\underline{u}}$ is metrizable by Proposition 4.2.

(b) Since by Theorem 3.2 $t_{\underline{u}}(\mathbb{T}) \cong G \oplus \bigoplus_{p \in P_\infty} \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(m)$ where P_∞ is a finite subset of $\mathbb{P}$ and $m \in \mathbb{N}$, Pontryagin duality gives $K_{\underline{u}} \cong \widehat{G} \times \prod_{p \in P_\infty} \mathbb{Z}_p \times \mathbb{Z}(m)$. Since G is torsion-free, $\widehat{G}$ is connected by [25, Theorem 24.25] and therefore $\widehat{G} \cong c(K_{\underline{u}})$.

(c) According to Theorem 3.2 (b), there exists a short exact sequence

$$0 \to F_0 \to G \to \oplus_{p \in P_1} \mathbb{Z}(p^\infty)^{m_p} \to 0 \tag{1}$$

where $F_0 \cong \mathbb{Z}^r$ is a free group of rank $r = \dim K_{\underline{u}}$ and $0 \leq m_p \leq r$ for every $p \in P_1$. By applying the Pontryagin duality to (1), one obtains the exact sequence

$$0 \to \prod_{p \in P_1} \mathbb{Z}_p^{m_p} \to \widehat{G} \to \mathbb{T}^r \to 0.$$

To conclude observe as in (b) that $c(K_{\underline{u}}) \cong \widehat{G}$.

(d) It follows from Theorem 3.1 that $\widehat{K_{\underline{u}}} \cong \mathbb{Z}^r \times \mathbb{Z}(d)$ with $r < k$. According to Pontryagin duality we have $K_{\underline{u}} \cong \mathbb{T}^r \times \mathbb{Z}(d)$. Since $\mathbb{T}^r$ has no proper open subgroups, so does its dense subgroup $(\mathbb{Z}, \sigma_{\underline{u}})$. QED

The next theorem follows from Corollary 3.3 (a) and Proposition 1.2.

Theorem 4.5 *Let $\underline{u} \in \mathbb{N}^{\mathbb{N}}$ satisfy $u_n = a_n u_{n-1} + b_n u_{n-2}$ with $a_n, b_n \in \mathbb{N}$ for every $n \in \mathbb{N}$. If $a_n < b_n$ for every $n \in \mathbb{N}$ and (b_n) is bounded, then $\sigma_{\underline{u}}$ is linear.*

Proof. The group $t_{\underline{u}}(\mathbb{T})$ is torsion by Corollary 3.3 (b). Hence $\sigma_{\underline{u}}$ is linear by Proposition 1.2 (d). QED

Theorem 4.6 *Let $\underline{u}$ be a nontrivial sequence satisfying $u_n = a_n u_{n-1} + b_n u_{n-2}$ with a_n, u_1, $u_2 \in \mathbb{N}$, $b_n \in \mathbb{N} \cup \{0\}$ and $a_n \geq b_n$ for every $n \in \mathbb{N}$. Then*

(a) $\underline{u}$ is a TB-sequence.

(b) $w(\mathbb{Z}, \sigma_{\underline{u}}) = \mathfrak{c}$ if and only if the sequence $\frac{u_n}{u_{n-1}}$ is not bounded.

(c) $\sigma_{\underline{u}}$ is metrizable if and only if the sequence $\frac{u_n}{u_{n-1}}$ is bounded.

(d) $\sigma_{\underline{u}}$ is linear if and only if the sequence $\frac{u_n}{u_{n-1}}$ is bounded and $b_n = 0$ infinitely many times.

Proof. (a) By Proposition 1.2 (b) we have to show that $t_{\underline{u}}(\mathbb{T})$ is infinite. If $b_n > 0$ for every $n \in \mathbb{N}$, by Corollary 3.3 (a), the subgroup $t_{\underline{u}}(\mathbb{T})$ of $\mathbb{T}$ is non-torsion and therefore infinite. The same is true if $b_n = 0$ only finitely many times. Suppose that $b_n = 0$ infinitely many times. Let $b_m = 0$ for some $m \in \mathbb{N}$, then $u_{m-1} | u_k$ for every $k \geq m - 1$, hence $\mathbb{Z}(u_{m-1}) \subseteq t_{\underline{u}}(\mathbb{T})$, therefore $u_{m-1} \leq |t_{\underline{u}}(\mathbb{T})|$. Since this holds true for infinitely many indices $m \in \mathbb{N}$, we get that $t_{\underline{u}}(\mathbb{T})$ is infinite.

(b) and the implication $\Longrightarrow$ of (c) follow from Theorem 3.7 and Proposition 1.2 (a) and (c).

(c) $\Longleftarrow$: $\sigma_{\underline{u}}$ is Hausdorff by (a). By Theorem 3.7, we have $|t_{\underline{u}}(\mathbb{T})| \leq \aleph_0$. Hence σ_u is metrizable by Proposition 1.2.

(d) $\Longrightarrow$: Suppose that $\sigma_{\underline{u}}$ is linear. By Proposition 1.2 (d) $t_{\underline{u}}(\mathbb{T})$ is torsion; as a subgroup of $t(\mathbb{T})$, $t_{\underline{u}}(\mathbb{T})$ is at most countable. Therefore by Theorem 3.7 the sequence $\frac{u_n}{u_{n-1}}$ is bounded. By way of contradiction, suppose $b_n = 0$ only finitely many times, then Corollary 3.3 (a) implies that $t_{\underline{u}}(\mathbb{T})$ is not torsion. Therefore $\sigma_{\underline{u}}$ is not linear.

(d) $\Longleftarrow$: Suppose that $b_n = 0$ infinitely many times and the sequence $\frac{u_n}{u_{n-1}}$ is bounded. Let $\theta \in \mathbb{R}$ with $\varphi(\theta) \in t_{\underline{u}}(\mathbb{T})$. By [2, Remark 2.6] there exists $\underline{r} \in \mathcal{Z}$ such that $\frac{r_n}{u_n} \to \theta$. By [2, Theorem 3.6] there exists $n_0 \in \mathbb{N}$ such that $r_n = a_n r_{n-1} + b_n r_{n-2}$ for $n \geq n_0$. Let $k > n_0$ with $b_k = 0$, then $\frac{r_k}{u_k} = \frac{r_{k-1}}{u_{k-1}}$ and therefore we obtain by induction that $\frac{r_n}{u_n} = \frac{r_{k-1}}{u_{k-1}}$ for every $n \geq k$, hence $\theta \in \mathbb{Q}$. Therefore $t_{\underline{u}}(\mathbb{T})$ is torsion. By Proposition 1.2(d) $\sigma_{\underline{u}}$ is linear. QED

Applying Theorem 4.6 with $b_n = 0$ for every $n \in \mathbb{N}$, we obtain the following corollary.

Corollary 4.7 *Let $\underline{u}$ be a nontrivial sequence in $\mathbb{N}$ such that u_{n-1} divides u_n. Then the following conditions are equivalent:*

(a) The sequence $\frac{u_n}{u_{n-1}} \in \mathbb{N}$ is bounded.

(b) $\sigma_{\underline{u}}$ is linear.

(c) $\sigma_{\underline{u}}$ is metrizable.

(d) $w(\mathbb{Z}, \sigma_{\underline{u}}) = \aleph_0$.

Corollary 4.8 *Let $a > 1$ be an integer number. Then every precompact group topology on $\mathbb{Z}$ that makes (a^n) converge to 0 is metrizable.*

This was proved by Raczkowski [34, Theorem 13] in the special case when $a = p$ is a prime number. Note that in this special case σ_u coincides with the p-adic topology and this is the unique precompact topology that makes p^n converge to 0 (as the p-adic topology on $\mathbb{Z}$ is minimal).

In the next theorem we describe the topology $\sigma_{\underline{u}}$ for sequences $\underline{u}$ satisfying a second order recurrence relation with coefficients $b_n = 1$ ($n \in \mathbb{N}$).

Theorem 4.9 *Let (a_n) be bounded and $\underline{u} \in \mathbb{N}^{\mathbb{N}}$ satisfy $u_n = a_n u_{n-1} + u_{n-2}$, $n > 2$. Let d be the greatest common divisor of u_1, u_2. Then $\underline{u}$ is a TB-sequence and the completion $K_{\underline{u}}$ of the group $(\mathbb{Z}, \sigma_{\underline{u}})$ is isomorphic to $\mathbb{T} \times \mathbb{Z}(d)$. In particular, when $d = 1$, $\sigma_{\underline{u}}$ is the group topology on $\mathbb{Z}$ that makes the homomorphism $(\mathbb{Z}, \sigma_{\underline{u}}) \hookrightarrow \mathbb{T}$ defined by $1 \mapsto \varphi(\theta)$ a topological group embedding where $\langle \varphi(\theta) \rangle = t_{\underline{u}}(\mathbb{T})$.*

Proof. The first assertion follows from Theorems 3.6 and 4.4 (d). Assume that $d = 1$. Then $t_{\underline{u}}(\mathbb{T}) = \langle \varphi(\theta) \rangle$, where θ is an irrational number. Since $\sigma_{\underline{u}}$ is the weakest group topology on $\mathbb{Z}$ such that all characters of $t_{\underline{u}}(\mathbb{T})$ are continuous, the diagonal homomorphism $(\mathbb{Z}, \sigma_{\underline{u}}) \hookrightarrow \mathbb{T}^{t_{\underline{u}}(\mathbb{T})}$ defined by $1 \mapsto (k\varphi(\theta))_{k \in \mathbb{Z}}$ is a topological group embedding. Clearly, the homomorphism $(\mathbb{Z}, \sigma_{\underline{u}}) \hookrightarrow \mathbb{T}$ defined by $1 \mapsto \varphi(\theta)$ is a topological group embedding. QED

Example 4.10 If $a_n = 1$ for every $n \in \mathbb{N}$ and $u_1 = u_2 = 1$, then $u_n = u_{n-1} + u_{n-2}$, $n > 2$, is the Fibonacci's sequence. By Theorem 4.9, $\underline{u}$ is a TB-sequence with cyclic subgroup $t_{\underline{u}}(\mathbb{T}) = \langle \varphi(\frac{1+\sqrt{5}}{2}) \rangle$ and the completion $K_{\underline{u}}$ of $(\mathbb{Z}, \sigma_{\underline{u}})$ is isomorphic to $\mathbb{T}$.

Proposition 3.8 and Theorem 3.9 allow us to describe completely the TB-sequences satisfying a second order recursive relation with constant coefficients.

Theorem 4.11 *Let $\underline{u} \in \mathcal{Z}$ satisfy $u_n = au_{n-1} + bu_{n-2}$ ($n > 2$) with $a, b, u_1, u_2 \in \mathbb{N}$.*

(1) Then $\sigma_{\underline{u}}$ has a countable 0-neighbourhood base.

(2) The following conditions are equivalent:

(i) $\sigma_{\underline{u}}$ is metrizable.

(ii) $\underline{u}$ is a TB-sequence.

(iii) $a \geq b$ or $(a, b) > 1$ or $\underline{u}$ is a geometric progression.

(3) $\sigma_{\underline{u}}$ is linear if and only if $a < b$. If $\underline{u}$ is a TB-sequence and $a < b$, then the completion $K_{\underline{u}}$ of $(\mathbb{Z}, \sigma_{\underline{u}})$ is a compact totally disconnected group isomorphic to $\mathbb{Z}(m) \times \prod_{p \in P_\infty} \mathbb{Z}_p$ where $P_\infty \subseteq \{p \in \mathbb{P} : p|b\}$ and $p \not| m$ for every $p \in P_\infty$.

Proof. (1) follows from Theorem 2.1(b) and Proposition 1.2.

(2) follows from (1), Corollary 3.10 and Proposition 1.2 (b).

(3) The first assertion follows from Theorem 4.5. If $a < b$, then $t_{\underline{u}}(\mathbb{T})$ is torsion by Proposition 3.8. Therefore it follows from Theorem 3.2 that $t_{\underline{u}}(\mathbb{T})) \cong \mathbb{Z}(m) \oplus \bigoplus_{p \in P_\infty} \mathbb{Z}(p^\infty)$ where $P_\infty \subseteq \{p \in \mathbb{P} : p|b\}$ and $p \not| m$ for every $p \in P_\infty$. If $\underline{u}$ is a TB-sequence, i.e. $\sigma_{\underline{u}}$ is Hausdorff, then $\widehat{K_{\underline{u}}} = \widehat{(\mathbb{Z}, \sigma_{\underline{u}})} = t_{\underline{u}}(\mathbb{T})$ and applying the Pontryagin duality one obtains

$$K_{\underline{u}} \cong \widehat{t_{\underline{u}}(\mathbb{T})} \cong \mathbb{Z}(m) \times \prod_{p \in P_\infty} \mathbb{Z}_p.$$

QED

4.2 Compatible TB-sequences

Let $\underline{u}$ be a sequence of integers such that the ratio $\frac{u_n}{u_{n-1}}$ converges to ∞. Then, by Theorem 2.1, there exist $2^{\mathfrak{c}}$-many pairwise non homeomorphic precompact group topologies of weight $\mathfrak{c}$ that makes $\underline{u}$ converge to 0. For the same reason the sequence $\underline{v}$ defined by $v_n := u_n + 1$ has the same property. Nevertheless, there exists no *Hausdorff* group topology that makes *both* these sequences converge to 0. This motivates the following:

Definition 4.12 Let G be an Abelian group.

(a) A pair of TB-sequences $\underline{u}, \underline{v}$ in G is *compatible* if there exists a precompact group topology τ on G such that $u_n \to 0$ and $v_n \to 0$ in (G, τ).

(b) A family of TB-sequences $\mathcal{U} = \{\underline{u}^{(\nu)} : \nu \in I\}$ in G is *compatible* if there exists a precompact group topology τ on G such that $u_n^{(\nu)} \to 0$ in (G, τ) for every $\nu \in I$. We denote by $\sigma_{\mathcal{U}}$ the finest precompact group topology on G that makes all sequences $u_n^{(\nu)} \in \mathcal{U}$ converge to 0.

Lemma 4.13 *Let $\mathcal{U} = \{\underline{u}^{(\nu)} : \nu \in I\}$ be a family of compatible TB-sequences of integers. Then $\sigma_{\mathcal{U}} = T_H$ with $H = \bigcap_{\underline{u} \in \mathcal{U}} t_{\underline{u}}(\mathbb{T})$.*

Clearly, a pair of TB-sequences $\underline{u}, \underline{v}$ is compatible if and only if the infimum of $\sigma_{\underline{u}}$ and $\sigma_{\underline{v}}$ is Hausdorff. Analogously, a family of TB-sequences $\{\underline{u}^{(\nu)} : \nu \in I\}$ is compatible if $\inf\{\sigma_{\underline{u}^{(\nu)}} : \nu \in I\}$ is Hausdorff.

In particular, a pair of TB-sequences $\underline{u}, \underline{v}$ in $\mathbb{Z}$ is compatible if and only if the subgroup $t_{\underline{u}}(\mathbb{T}) \cap t_{\underline{v}}(\mathbb{T})$ of $\mathbb{T}$ is infinite.

In this section we study compatibility of TB-sequences satisfying a second order recurrence relation $u_n = a_n u_{n-1} + b_n u_{n-2}$ where a_n, b_n are *fixed* sequences in $\mathbb{N}$ with $a_n \geq b_n$ and the initial values (u_1, u_2) run over $\mathbb{N}^2$. We denote by $\mathcal{R}_2$ the group of all sequences $\underline{r} \in \mathcal{Z} + \mathcal{Q}_0$ satisfying $r_n = a_n r_{n-1} + b_n r_{n-2}$ (compare with §3.1).

It is proved in [2, Proposition 4.18] that if (a_n) is bounded, then for every $\underline{u} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$ the subfield $F_{\underline{u}}$ of $\mathbb{R}$ generated by $\tau_{\underline{u}}(\mathbb{R})$ *does not depend* on the initial values u_1, u_2. Moreover, $F_{\underline{u}} = \mathbb{Q}(\theta)$ for every $\underline{u} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$ and every $\theta \in \tau_{\underline{u}}(\mathbb{R}) \setminus \mathbb{Q}$. This property allows us to denote by $F_{\underline{a},\underline{b}}$ the field $F_{\underline{u}}$ for any $\underline{u} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$.

The following theorem shows that, for $\underline{u}, \underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$, the behaviour of the intersection $t_{\underline{u}}(\mathbb{T}) \cap t_{\underline{v}}(\mathbb{T})$ depends on the degree $(F_{\underline{a},\underline{b}} : \mathbb{Q})$.

Theorem 4.14 [2, Theorem 4.19] *Let (a_n) be a bounded sequence and $a_n \geq b_n \geq 1$ for every $n \in \mathbb{N}$.*

- *(a) $t_{\underline{u}}(\mathbb{T}) \cap t_{\underline{v}}(\mathbb{T})$ is a rank-one subgroup of $\mathbb{T}$ for any pair of dependent sequences $\underline{u}, \underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$.*
- *(b) The following conditions are equivalent:*
  - *(b_1) $(F_{\underline{a},\underline{b}} : \mathbb{Q}) > 2$.*
  - *(b_2) $t_{\underline{u}}(\mathbb{T}) \cap t_{\underline{v}}(\mathbb{T})$ is torsion for every pair of independent sequences $\underline{u}, \underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$.*
  - *(b_3) There exists a pair $\underline{u}, \underline{v}$ of independent sequences of $\mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$ such that $t_{\underline{u}}(\mathbb{T}) \cap t_{\underline{v}}(\mathbb{T})$ is torsion.*

For reader's convenience we give in (a) of the next corollary the equivalence of (b_1) and (b_2) in counter-positive form.

Corollary 4.15 [2, Corollaries 4.20 and 4.21] *Let (a_n) be a bounded sequence and $a_n \geq b_n \geq 1$ for every $n \in \mathbb{N}$.*

- *(a) Then $(F_{\underline{a},\underline{b}} : \mathbb{Q}) = 2$ if and only if $t_{\underline{u}}(\mathbb{T}) \cap t_{\underline{v}}(\mathbb{T})$ is a rank-one subgroup of $\mathbb{T}$ for any pair of sequences $\underline{u}, \underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$.*
- *(b) If $b_n = 1$ for every $n \in \mathbb{N}$, then $(F_{\underline{a},\underline{b}} : \mathbb{Q}) > 2$ if and only if $t_{\underline{u}}(\mathbb{T}) \cap t_{\underline{v}}(\mathbb{T})$ is a finite subgroup of $\mathbb{T}$ for every pair of independent sequences $\underline{u}, \underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$.*

With these observations and Theorem 4.14 we obtain:

Theorem 4.16 *Let (a_n) be bounded and $a_n \geq b_n \geq 1$ for every $n \in \mathbb{N}$.*

- *(a) Any pair of dependent sequences $\underline{u}$, $\underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$, is a compatible pair of TB-sequences.*
- *(b) If $b_n = 1$ for every $n \in \mathbb{N}$, then the following conditions are equivalent:*
  - *(b_1) $(F_{\underline{a},\underline{b}} : \mathbb{Q}) > 2$.*
  - *(b_2) There exists a pair of non-compatible independent TB-sequences $\underline{u}$, $\underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$.*
  - *(b_3) Any pair of independent elements $\underline{u}$, $\underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$, is a non-compatible pair of TB-sequences.*

Proof. (a) follows from Theorem 4.14 (a).

(b) Observe that by Theorem 4.9 every sequence $\underline{u} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$ is a TB-sequence and apply Theorem 4.14 and Corollary 4.15 (b). QED

Now we give the above theorem in counter-positive form that allows us to add one more equivalent condition. Namely, $(F_{\underline{a},\underline{b}} : \mathbb{Q}) = 2$ if and only if there exists a single precompact group topology that topologizes all sequences satisfying $u_n = a_n u_{n-1} + u_{n-2}$ (independently on the initial terms u_1, u_2).

Theorem 4.17 *Let (a_n) be bounded and $b_n = 1$ for every $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) $(F_{\underline{a},\underline{b}} : \mathbb{Q}) = 2$;

(b) there exists a pair of compatible TB-sequences $\underline{u}$, $\underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$;

(c) any pair $\underline{u}$, $\underline{v} \in \mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$ is a compatible pair of TB-sequences;

(d) $\mathcal{R}_2 \cap \mathbb{N}^{\mathbb{N}}$ is a family of compatible TB-sequences.

Proof. The implications $(d) \Rightarrow (c) \Rightarrow (b)$ are trivial. The implication $(b) \Rightarrow (a)$ follows from Corollary 4.15 (b).

To prove the implication $(a) \Rightarrow (d)$ use the following fact proved in [2, Theorem 4.24] under the hypothesis of our theorem:

$(F_{\underline{a},\underline{b}} : \mathbb{Q}) = 2$ if and only if $\bigcap_{\underline{w} \in \mathcal{R}_2} t_{\underline{w}}(\mathbb{T})$ is an infinite cyclic subgroup of $\mathbb{T}$. QED

5 The general case of precompact Abelian groups

In this section we study TB-sequences of discrete Abelian groups (in particular of the Prüfer group $\mathbb{Z}(p^\infty)$) and precompact group topologies without nontrivial convergent sequences.

Definition 5.1 [16, Definition 2.1] Let G be an Abelian group and $\widehat{G} = Hom(G, \mathbb{T})$. Let $\underline{u} = (u_n)$ be a sequence in $\widehat{G}$. Then we set $s_{\underline{u}}(G) := \{x \in G : u_n(x) \to 0 \ \text{ in } \ \mathbb{T}\}$.

Clearly, $s_{\underline{u}}(G)$ is a subgroup of G and, for $G = \mathbb{T}$, one has $s_{\underline{u}}(\mathbb{T}) = t_{\underline{u}}(\mathbb{T})$ for every sequence $\underline{u} \in \mathcal{Z}$.

Definition 5.1 can be extended to countably infinite subsets U of $\widehat{G}$ by setting $s_U(G) := s_{\underline{u}}(G)$, where $\underline{u} = (u_n)$ is a one-to-one enumeration of U (it is easy to see that the subgroup $s_{\underline{u}}(G)$ does not depend on the specific enumeration of U).

In the sequel we identify a locally compact Abelian group G with its second Pontryagin dual. This will allows us to speak of $s_{\underline{u}}(\widehat{G})$ for $\underline{u} \in G^{\mathbb{N}}$.

The following lemma generalizes [1, Proposition 2.2]:

Lemma 5.2 *Let G be an Abelian group and let $H \leq \widehat{G}$. Then for a sequence $\underline{u} = (u_n) \in G^{\mathbb{N}}$ one has $u_n \to 0$ in (G, T_H) if and only if $H \leq s_{\underline{u}}(\widehat{G})$.*

Let $\sigma_{\underline{u}}$ be the supremum of all totally bounded group topologies on G for which $\underline{u}$ converge to 0. Then $\sigma_{\underline{u}}$ is a totally bounded group topology and $\sigma_{\underline{u}} = T_{s_{\underline{u}}(\widehat{G})}$.

The next proposition shows the relationship between the topology $\sigma_{\underline{u}}$ and the corresponding subgroup $s_{\underline{u}}(\widehat{G})$ of $\widehat{G}$.

Theorem 5.3 *Let G be a discrete Abelian group and let $\underline{u} \in G^{\mathbb{N}}$ be a sequence. Then:*

(a) $w(G, \sigma_{\underline{u}}) = |s_{\underline{u}}(\widehat{G})|$;

(b) $\sigma_{\underline{u}}$ is Hausdorff if and only if $s_{\underline{u}}(\widehat{G})$ is dense in $\widehat{G}$, in such a case $w(G, \sigma_{\underline{u}}) \geq |G|$;

(c) $\sigma_{\underline{u}}$ is metrizable if and only if $s_{\underline{u}}(\widehat{G})$ is countable and dense in $\widehat{G}$.

Proof. (a), (b) (without the inequality) and (c) are proved in [16, Proposition 3.2]. Let $\lambda = w(G, \sigma_{\underline{u}})$ and assume that $\sigma_{\underline{u}}$ is Hausdorff. To prove the inequality in (b) denote by K the compact Pontryagin dual of the discrete group G, and let $H = s_{\underline{u}}(K)$. Then H is dense in K and $\sigma_{\underline{u}} = T_H$. By (a) $|H| = \lambda$. Let $N = \bigcap_n \ker u_n$. Since every subgroup $\ker u_n$ is a G_δ-set in K, also N is a G_δ-set in K and obviously $N \subseteq H$. Since $|H| = \lambda$ we can find at most λ open sets $\{O_i : i < \lambda\}$ of K containing 0 such that $H \cap \bigcap_{i<\lambda} O_i = \{0\}$. This yields that $\{0\}$ can be obtained as the intersection of at most λ open sets in K. Since K is a compact group, this yields $w(K) \leq \lambda$. As $|G| = w(K)$ we are done. QED

Clearly, for $G = \mathbb{T}$, (a)-(c) of Proposition 1.2 are a corollary of the above results.

The question of when a given precompact group topology on an Abelian group G has the form $\sigma_{\underline{u}}$ for some TB-sequence $\underline{u} \in G^{\mathbb{N}}$ may be considered as a counterpart of Theorem 4.1. If this occurs, we shall refer to $\underline{u}$ as *characterizing sequence* of τ. Precompact group topologies admitting a characterizing sequence must satisfy the necessary condition imposed by the inequality in item (b) of Theorem 5.3 that we prefer to state explicitly for the case of countable weight in the next corollary.

Corollary 5.4 *If a precompact metrizable Abelian group (G, τ) admits a characterizing sequence $\underline{u} \in G^{\mathbb{N}}$, then G is countable.*

In the opposite direction we have the following theorem, due to recent unpublished results of Bíró [5]:

Theorem 5.5 *Let (G, τ) be a countable precompact Abelian group such that the completion of (G, τ) is isomorphic to $\mathbb{T}^n$ for some $n \in \mathbb{N}$. Then there exists $\underline{u} \in G^{\mathbb{N}}$ such that $\tau = \sigma_{\underline{u}}$.*

Proof. Let K be the compact Pontryagin dual of the discrete group G. Then K is metrizable by [25, Theorem 24.15]. According to [11, Theorems 1.2 and 1.3], the subgroup H of τ-continuous characters of G is dense in K and $\tau = T_H$. Since the completion N of (G, τ) has the same group of characters as (G, τ), we conclude that $H \cong \widehat{N} \cong \mathbb{Z}^n$. From Theorem 1 of [5] one can easily deduce (see for example [14, Corollary 5.4]) that $H = s_{\underline{u}}(K)$ for some $\underline{u} \in G^{\mathbb{N}}$. QED

It is not clear whether the somewhat artificial additional restraint on the completion of (G, τ) cannot be replaced by the weaker and more natural restraint of metrizability of (G, τ). It can be shown that the positive answer to this question is equivalent to the positive answer to

Question 5.6 Does every dense countable subgroup of a metrizable compact abelian group K necessarily have the form $s_{\underline{u}}(K)$ for some $\underline{u} \in \widehat{K}^{\mathbb{N}}$?

5.1 TB-sequences in $\mathbb{Z}(p^\infty)$

Let p be a prime number. We consider the Prüfer group $\mathbb{Z}(p^\infty)$ (let us recall that its dual group is the group of p-adic integers $\mathbb{Z}_p$).

Denote by c_n the element $1/p^n + \mathbb{Z}$ of $\mathbb{Z}(p^\infty)$, so that the sequence $\underline{c} = (c_n)$ is nothing else but the canonical set of generators of $\mathbb{Z}(p^\infty)$. (One can look at c_n also as the character $\mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}(p^n) \leq \mathbb{T}$ obtained from the canonical map $\mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p$.)

Theorem 5.7 [16, Theorem 3.3] *Let n_k be a strictly increasing sequence of naturals and let $\underline{c}^* = (c_{n_k})$ be a subsequence of $\underline{c}$. Then $\underline{c}^*$ is a TB-sequence of $\mathbb{Z}(p^\infty)$ and the following are equivalent:*

(a) $s_{\underline{c}^*}(\mathbb{Z}_p) = \mathbb{Z}$,

(b) $s_{\underline{c}^*}(\mathbb{Z}_p)$ *is countable;*

(c) $|s_{\underline{c}^*}(\mathbb{Z}_p)| < \mathfrak{c}$;

(d) the differences $n_{k+1} - n_k$ are bounded;

(e) $\{n_k : k \in \mathbb{N}\}$ is a large subset of $\mathbb{N}$ (i.e., there exists a finite set $F \subseteq \mathbb{N}$ such that $\mathbb{N} \subseteq F \cup (F + \{n_k : k \in \mathbb{N}\})$;

(f) $\sigma_{\underline{c}^*}$ is metrizable;

(g) $\sigma_{\underline{c}^*}$ has weight $< \mathfrak{c}$;

(h) every precompact group topology τ on $\mathbb{Z}(p^\infty)$ that makes c_{n_k} converge to 0 is metrizable.

Remark 5.8 (a) Let p be a prime number, $p > 2$, and consider the sequence (a_n) of $\mathbb{Z}(p^\infty)$ defined by $a_n := \frac{b_n}{p^n}$ where $b_n = \frac{p^n-1}{2}$ for every $n \in \mathbb{N}$. Then $a_n \to \frac{1}{2}$ with respect to the topology on $\mathbb{Z}(p^\infty)$ induced by $\mathbb{T}$. Therefore, also the subsequence $\underline{a} = (a_{n!}) \to \frac{1}{2}$ with respect to the same topology. On the other hand, $(a_{n!})$ is a TB-sequence of $\mathbb{Z}(p^\infty)$. Indeed, let $\xi \in \mathbb{Z}_p$ be the character of $\mathbb{Z}(p^\infty)$ defined by $\xi := 1 + p + 2\sum_{k=2}^{\infty} p^{k!}$. Then one can prove that $\xi(a_{n!}) \to 0$ in $\mathbb{T}$ so that $\xi \in s_{\underline{a}}(\mathbb{Z}_p)$. Since $\langle \xi \rangle$ is dense in $\mathbb{Z}_p$, Proposition 5.3 applies to conclude that $(a_{n!})$ is a TB-sequence.

(b) A first step towards a complete description of the TB-sequences in $\mathbb{Z}(p^\infty)$ can be to see when a sequence (u_n) in $\mathbb{Z}(p^\infty)$ converging to a point a in $\mathbb{T}$ in the usual topology of $\mathbb{T}$, admits a subsequence converging to 0 in some precompact group topology on $\mathbb{Z}(p^\infty)$.

Remark 5.8 motivates the following

Question 5.9 Is it true that every one-to-one sequence (u_n) in an infinite Abelian group G admits a TB-subsequence ?

The answer is positive when $G = \mathbb{Z}$.

5.2 Precompact topologies without convergent sequences

Theorem 5.3 implies that a sequence $\underline{u}$ of a discrete Abelian group G is a TB-sequence if and only if $s_{\underline{u}}(\widehat{G})$ is a dense subgroup of the compact group $\widehat{G}$. Analogously one can describe, via an appropriate notion of density, the precompact group topologies on G without nontrivial convergent sequences. To this end we recall the following notion (proposed in [13]):

Definition 5.10 [16, Definition 2.4] For a subgroup H of a topological group G the $\mathfrak{g}$-*closure* of H is

$$\mathfrak{g}_G(H) := \bigcap\{s_{\underline{u}}(G) : \underline{u} \in \widehat{G}^{\mathbb{N}},\ H \leq s_{\underline{u}}(G)\},$$

H is said to be $\mathfrak{g}$-*dense* if $\mathfrak{g}_G(H) = G$.

The $\mathfrak{g}$-*closed* subgroups (defined by $H = \mathfrak{g}_G(H)$) were extensively studied in [16]; here we shall be interested only in $\mathfrak{g}$-density (see Theorem 5.11). It will be useful to note that a $\mathfrak{g}$-dense subgroup H of a precompact Abelian group G is also dense (it suffices to observe that for every character $\chi : G \to \mathbb{T}$ the kernel of χ coincides with the subgroup $s_{\underline{u}}(G)$ of G, where $u_n = \chi$ for every n; see [16, Lemma 2.12] for more detail).

Let us see now that one can study precompact group topologies without nontrivial convergent sequences on an Abelian group G by studying $\mathfrak{g}$-density of subgroups of the compact Pontryagin dual $\widehat{G}$ of the (discrete) group G.

Theorem 5.11 [14, Theorem 4.22] *Let G be a discrete Abelian group. Then a subgroup H of $\widehat{G}$ is $\mathfrak{g}$-dense if and only if the group (G, T_H) has no nontrivial convergent sequences.*

Proof. According to Lemma 5.2, for a sequence $\underline{u} \in G^{\mathbb{N}}$ one has $s_{\underline{u}}(\widehat{G}) = \widehat{G}$ if and only if $u_n \to 0$ with respect to the maximal precompact group topology $T_{\widehat{G}}$ of G. Since by Glicksberg's Theorem the topology $T_{\widehat{G}}$ has no non-trivial convergent sequences, it follows that $s_{\underline{u}}(\widehat{G}) = \widehat{G}$ if and only if $\underline{u}$ is eventually null. Therefore, $s_{\underline{u}}(\widehat{G})$ is a proper subgroup of $\widehat{G}$ for every nontrivial sequence $\underline{u}$ of G. This observation and Lemma 5.2 conclude the proof. QED

It easily follows from the definition that for every sequence $\underline{u} \in G^{\mathbb{N}}$, the subgroup $s_{\underline{u}}(\widehat{G})$ of $\widehat{G}$ is an $F_{\sigma\delta}$-set, hence it is measurable with respect to the Haar meausure of $\widehat{G}$. The measure properties of the $\mathfrak{g}$-closed subgroups $s_{\underline{u}}(\widehat{G})$ of $\widehat{G}$ are studied in [12] where the following theorem is proved:

Theorem 5.12 [12, Lemma 3.10] *Let G be an infinite discrete Abelian group and let $K := \widehat{G}$. Then for every countably infinite subset U of G, $s_U(K)$ is a Haar measure-zero subgroup of K.*

From the point of view of topologies, if some precompact group topology τ on an Abelian group G admits a nontrivial convergent sequence (u_n), then $H = \widehat{(G,\tau)}$ is a dense measure-zero subgroup of $\widehat{G}$ as a subgroup of the Haar measure-zero subgroup $s_U(K)$, where the set U is the support of the sequence $\underline{u}$.

This paradigm was used by Raczkowski and coauthors [34, 10] to produce (many) precompact group topologies without nontrivial convergent sequences (cf. [10, Theorems 4.1 and 5.5]).

In terms of $\mathfrak{g}$-density, a non $\mathfrak{g}$-dense subgroup H of a compact group K must have measure zero by Theorem 5.12. The question of whether the converse holds true was raised by Raczkowski in [34, Question 3.1] in terms of convergent sequences. Using Theorem 5.11 we can reformulate her question in terms of $\mathfrak{g}$-density as follows

Question 5.13 Can a $\mathfrak{g}$-dense subgroup of an infinite compact Abelian group have measure-zero ?

An affermative answer to this question was given in [1], where, under the assumption of Martin's Axiom, $2^{\mathfrak{c}}$-many measure-zero, $\mathfrak{g}$-dense subgroups of $\mathbb{T}$ are produced (cf. [1, Theorem 4.1]). The construction of [1] was extended to arbitrary connected locally compact Abelian groups that are metrizable or compact of weight $\leq \mathfrak{c}$ (cf. [3, Corollary 5.6]).

Later Hart and Kunen [22] found an effective ZFC counter-example in the case of the circle group $\mathbb{T}$. For a countably infinite subset B of $\mathbb{T}$ let $s_B(\mathbb{T}) = \{x \in \mathbb{T} : \lim_{b \in B} bx = 0 \text{ in } \mathbb{T}\}$. To every non-fixed filter $\mathcal{F}$ on ω let $s_{\mathcal{F}}(\mathbb{T}) = \bigcup_{B \in \mathcal{F}} s_B(\mathbb{T})$.

Theorem 5.14 [22] *There exists a filter $\mathcal{F}$ on ω such that the subgroup $s_{\mathcal{F}}(\mathbb{T})$ of $\mathbb{T}$ is $\mathfrak{g}$-dense with Haar measure zero.*

For the proof one takes $\mathcal{F} \subseteq [\omega]^\omega$, generated by all sets $\{k! + 1 : k \in D\}$, where $D \subseteq \omega$ has asymptotic density 1. Then $\mathcal{F} \subseteq [\omega]^\omega$ is a Borel set, hence $s_{\mathcal{F}}(\mathbb{T})$ is measurable.

Recently this example was extended in [23] to arbitrary compact Abelian groups:

Theorem 5.15 [23] *Every infinite compact Abelian group G admits a $\mathfrak{g}$-dense subgroup of Haar measure zero.*

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