Cognitive Growth in Elementary and Advanced Mathematical Thinking

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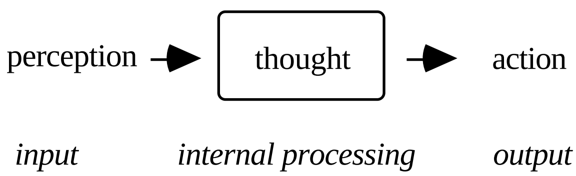
This paper addresses the development of mathematical thinking from elementary beginnings in young children to university undergraduate mathematics and on to mathematical research. It hypothesises that mathematical growth starts from **perceptions of***, and* **actions on***, objects in the environment. Successful "perceptions of" objects lead through a Van Hiele development in* **visuo-spatial** *representations with increasing verbal support to visually inspired verbal proof in geometry. Successful "actions on" objects use* **symbolic** *representations flexibly as "***procepts***" —* ***pro****cesses to do and con****cepts*** *to think about — in arithmetic and algebra. The resulting cognitive structure in elementary mathematical thinking becomes advanced mathematical thinking when the* **concept images** *in the cognitive structure are reformulated as* **concept definitions** *and used to construct* **formal concepts** *that are part of a systematic body of shared mathematical knowledge. The analysis will be used to highlight the changing status of mathematical concepts and mathematical proof, the difficulties occurring in the transition to advanced mathematical thinking and the difference between teaching and learning the full process of advanced mathematical thinking as opposed to the systematic product of mathematical thought.*

Perception, thought and action

I find it useful to separate out three components of human activity as input (perception), internal activity (thought) and output (action):



This simple observation allows us to see mathematical activities as perceiving objects, thinking about them, and performing actions upon them. I shall begin by considering input and output before moving on to the nature of the internal processing.

Input and output – objects and action

Elementary mathematics begins with *perceptions of* and *actions on* objects in the external world. The perceived objects are at first seen as visuo-spatial gestalts, but then, as they are analysed and their properties are teased out, they are described verbally,

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leading in turn to classification (first into collections, then into hierarchies) the beginnings of verbal deduction relating to the properties and the development of systematic verbal proof (Van Hiele, 1959).

On the other hand, *actions* on objects, such as counting, lead to a different kind of development. Here the *process* of counting is developed using number words and symbols which become conceptualised as number *concepts*. This leads to fundamentally different kind of development, described by Piaget, as follows:

> ... mathematical entities move from one level to another; an operation on such "entities" becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by "stronger" structures.
>
> (Piaget, 1972, p. 70)

Such an idea has led to a number of theories which highlight the duality of process and concept. Davis (1975) noted children may not distinguish between the name of a symbol and the underlying process. Skemp (1979) proposed a general "varifocal theory" in which a schema seen as a whole is a concept and a concept seen in detail is a schema. Greeno (1983) focused on the notion of "conceptual entities" which may be used as inputs to other procedures. More recently, Dubinsky (1991) speaks of *encapsulation* of process as object, Sfard (1991) of *reification* of process as object, and Gray & Tall (1994) see the symbol as pivot between ***pro***cess and con***cept***—the notion of ***procept***.

The two sequences of development beginning with object and action are quite distinct. I therefore hypothesise that, rather than view growth in elementary mathematics as a single development in the manner of a neo-Piagetian stage theory, an alternative theory is to see two different developments which occur at the same time. One is visuo-spatial becoming verbal and leading to proof, the other uses symbols both as processes to do things (such as counting, addition, multiplication) and also concepts to think about (such as number, sum, product).

It is interesting to note that these developments can occur quite independently. The Ancient Greeks developed a theory of geometry (including geometric constructions of arithmetic) without any symbolism for algebra and arithmetic, and it is possible to develop arithmetic and algebra without any reference to geometry. However, many useful links have been made between visual and manipulative symbolic methods and it is clearly opportune to take advantage of them to develop a versatile approach which uses each to its best advantage.

In the advanced stages of such a development, certain subtle difficulties occur which mean that advanced mathematical thinking must expunge itself of possible hidden assumptions that occur when visual ideas are verbalised. In the nineteenth century a number of flaws became apparent in Euclidean geometry and theoretical developments in algebra (such as non-commutative quaternions) were over-stretching simple beliefs in the manipulation of symbols. Research mathematics took a new direction using set-theoretic definition and logical deduction. Theorems inspired by geometric perception and symbolic manipulation were reformulated to give a new axiomatic approach to mathematics that led on to greater generality.

This theory is also flawed. The axiomatic method asks us to write down finite lists of set-theoretic definitions and axioms and to deduce theorems in a finite number of steps. But if we do this with an infinite set, such as the natural numbers, Gödel showed that there are theorems that must be true but which cannot be proven in a finite number of steps. Essentially, there will always be "too many theorems" to prove. Thus the existence of a systematic body of formal mathematical knowledge is not the final quest in mathematics, although it does offer a vital foundation upon which even more sophisticated ideas can be built.

Advanced mathematical thinking today involves using cognitive structures produced by a wide range of mathematical activities to construct new ideas that build on and extend an ever-growing system of established theorems.

The cognitive growth from elementary to advanced mathematical thinking in the individual may therefore be hypothesised to start from "perception of" and "action on" objects in the external world, building through two parallel developments—one visuo-spatial to verbal-deductive, the other successive process-to-concept encapsulations using manipulable symbols—leading to a use of all of this to inspire creative thinking based on formally defined objects and systematic proof (figure 1).

Internal processing and external representations

The cognitive growth that occurs in mathematics is implicitly designed to make maximum use of the facilities available to *homo sapiens*. The two parallel developments described relate to the complementary roles of perception (input) and action (output). In between is the internal mental processing which is far more difficult to describe and analyse. Crick suggests that:

> The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) "object(s)" can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once). (Crick, 1994, p. 61)

Brain activity therefore has two highly contrasting features:

- a huge store of experiences and simultaneous activity,
- a small focus of attention,

(where the latter need not be a *place* in the brain to store items as in a computer but a mental activity which is temporarily linked to conscious thought processes).

To minimise the cognitive strain it is essential to do two things:

- compress knowledge appropriately for the small focus of attention,
- construct linkages to other mental data to make it easy to use.

The first is an essential characteristic of mathematics:

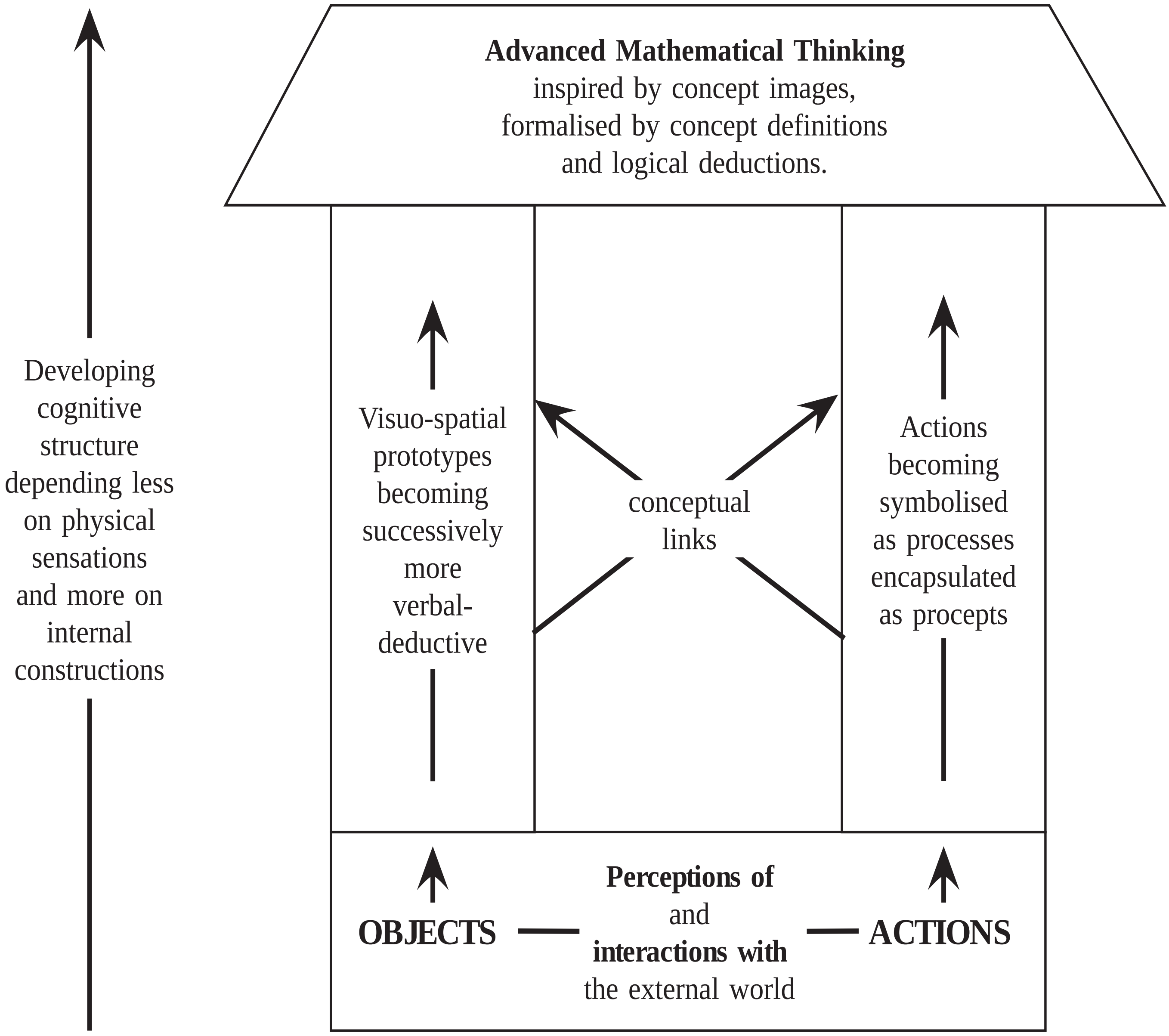

Figure 1: Outline cognitive development from child to research mathematician

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990, p. 847)

It is achieved in a variety of ways—routinising processes so that they occupy little conscious attention, using pictures to allow the viewer to focus at whatever level and on whatever detail is desired, and using words and symbols (particularly procepts) to compress the notation into small, mentally manipulable entities.

The second involves the development of *conceptual knowledge* with many links to maximise retrieval. This also involves concept-process links enabling the successful individual to carry out mathematical procedures to find answers to problems. However, if the mathematics places too great a cognitive strain, either through failure to compress or failure to make appropriate links, the fall-back position resorts to the more primitive method of routinising sequences of activities —rote-learning of *procedural knowledge*.

The status of mental objects

In cognitive growth, the mental objects we think about are constructed in several different ways, each having a different status. The visual objects we see are direct perceptions of the outside world, or rather, our own personal constructions of what we think we see in the outside world. Later in geometry, objects such as a "point" or a "line" take on a more abstract meaning. A point is no longer a pencil mark with finite size (so that a child may imagine a finite number of points on a line segment (e.g. Tall, 1980)), but an abstract concept that has "position but no size". A straight line is no longer a physical mark made using pencil and ruler, but an imagined, perfectly straight line, with no thickness which can be continued as far as required in either direction. In Euclid, a line is defined as "breadthless length" and a straight line "lies evenly with the points on itself". These words do not *define* a straight line in any absolute sense, but they help to convey the meaning of the perfect Platonic object which we may "see" lying behind any inadequate physical picture. As Hardy observed:

> Let us suppose that I am giving a lecture on some system of geometry, such as the ordinary Euclidean geometry, and that I draw figures on the blackboard to stimulate the imagination of my audience, rough drawings of straight lines or circles or ellipses. It is plain, first, that the truth of the theorems which I prove is in no way affected by the quality of my drawings. Their function is merely to bring home my meaning to my hearers, and, if I can do that, there would be no gain in having them redrawn by the most skilful draughtsman. They are pedagogical illustrations, not part of the real subject-matter of the lecture.
>
> (Hardy, 1940/1967, p. 125.)

The mental "objects" constructed by process-object encapsulation have a very different status. As Dörfler suggested:

> … my subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariably comes to my mind are certain patterns of dots or other units, a pentagon, the symbol 5 or V, relations like 5+5=10, 5*5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever could find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless I deem myself able to talk about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object "5".
>
> (Dörfler, 1993, pp. 146–147.)

In terms of the notion of *concept image* of Vinner (Vinner & Hershkowitz, 1980), there is no conflict here. Within our mental structure we have both recognition structures that recognise, say, the perceptions of a physical object, such as a drawing of a triangle, and we also have connected sequences of mental actions that are triggered to carry out processes in time. The concept image of a *procept* uses the symbol to links to suitable processes and relationships in the cognitive structure. Thus, although we may not have anything in our mind which is like a physical object, we have symbols that we can manipulate *as if* they were mental objects.

I do not believe in my own case that I have things in my mind that correspond to visualisations either. Despite working for many years on visualisations in mathematics in which I can produce good external pictures on the computer screen to represent mathematical concepts, the pictures I conjure up in my mind are very different from the external representations. It is different with words. As I type this, I can hear the words in my mind and if I start saying them as I type, what I hear out loud is what I hear in my head. But *homo sapiens* has no "picture-projecting facility" for communication in the same way as it has a verbal "sound-making facility".

I have a theory therefore that when we visualise, we use not "picture-making" facilities, but "picture-*recognising* facilities" which we have in plenty. We have many structures that resonate with incoming visual stimuli to recognise them and we simply use these recognition's to attempt to build up our visuo-spatial imagery. The result is a vague "sense" of a picture. Certainly in my case it is vague. I do not know what *you* see when you think of a visualisation, perhaps you see an eidetic image in full colour. Then again, what we all see may be just the emperor's new clothes!

Many mathematicians say that they think in "vague" visuo-spatial ways as a springboard for more abstract thinking. Hadamard (1945) reported that most of the mathematicians he consulted did so. In his own case he even saw formulae in this way:

> I see not the formula itself, but the place it would take if written: a kind of ribbon, which is thicker or darker at the place corresponding to the possibly important terms; or (at other moments), I see something like a formula, but by no means a legible one, as I should see it (being strongly long-sighted) if I had no eye-glasses on, with letters seeming rather more apparent (though still *not legible*) at the place which is supposed to be the important one.
>
> (Hadamard, 1945, p. 78.)

Representations

In considering the kind of mental "objects" we have in different mathematical contexts, it is interesting to return to the ideas of Bruner (1966) who formulated his theory of three different types of representation of human knowledge:

- enactive,
- iconic,
- symbolic.

One of these is essentially a physical *process* (enactive) whilst the other two produce physical objects that are drawn or written[1] (iconic, symbolic). Iconic representations drawn by hand, such as a free-hand graph, also have enactive elements in them, suggesting a broader "visuo-spatial" concept. (For instance, one senses enactively that a "continuous" graph going from negative to positive must pass through zero.)

Symbols as procepts in arithmetic, algebra etc., also have dual process-object meanings. This in turn suggests that the symbolic mode of presentation needs

[1]Verbal symbolism can, of course, also be spoken, but the written word has great value as a permanent record that can be scanned and reflected upon.

subdividing as Bruner himself hinted when he mentioned, “language in its natural form” and the two “artificial languages of number and logic”, (Bruner, 1966, pp. 18, 19).

Natural language occurs throughout mathematics to set the mathematical activity in context. In the visuo-spatial to verbal developm ent, natural language becomes a vehicle for describing iconic images and formulating proof. It can also be used to describe properties of numbers, for instance, that addition is commutative because it is observed to be always independent of the order (a fact easier seen by visualising the change in order than carrying out the counting procedure—a valuable use of the interrelations between visual and symbolic.) Meanwhile the “artificial language of number” has mental objects which are procepts and the “artificial language of logic” in advanced mathematics has concepts which are formally *defined*.

It is essential to distinguish between elementary mathematics, (including geometry) where objects are *described* and advanced mathematics where objects are *defined*. In both cases language is used to formulate the properties of objects, but in elementary mathematics the description is constructed from experience of the object, in advanced mathematics, the properties of the object are constructed from the definition—a reversal which causes great difficulties of accommodation for novices in advanced mathematical thinking.

This gives a range of different types of representation in mathematics, including:

- enactive (physical process),
- iconic (visual),

and three forms of symbolic representation:

- verbal (description),
- formal (definition),
- proceptual (process-object duality).

The notion of “procept” helps in the analysis of cognitive difficulties related to symbolism. When Eddie Gray and I first coined the term I felt, in a moment of self-doubt, that all we had done was to give a name to something that was well-known to the mathematics education community. Subsequently I realised it was more. By giving it a name, we had essentially *encapsulated the process of encapsulation.* This enables us to discuss different kinds of encapsulation in different contexts and to see how learners face cognitive difficulties when procepts behave differently in different contexts.

For instance, in the development from the process of counting to the number concept, the sequence of number words initially only function as utterances in the schema of pointing and counting, but then the last word becomes the name for the number of objects in the collection. In arithmetic of whole numbers, symbols such as 4+3 initially evoke a counting procedure (count-all) which is then compressed via “count-on” (which uses 4 as a number concept and +3 as a count-on procedure) to a “known” fact where 4+3 is the number 7. In this encapsulation there is a new *concept*, namely the *sum*, 4+3, but it relates to a known *object* (the number “7”). However, for the process of equal sharing for 3/4 (divide into four equal parts and take three) to be encapsulated requires the construction of a *new* mental object — a fraction. Hence the considerable increased difficulty with fractions as a succession of encapsulations and mental constructions.

Arithmetic procepts such as 4+3, 3×4, have a built-in algorithm to compute the result, which children come to expect. Such procepts are genuinely *operational*, in the sense that one can operate on them to get an answer. But in algebra, procepts such as $4+3x$ certainly have a process of evaluation (add four to three times x) but cannot be evaluated until x is known. Such symbols are termed *template* procepts, in that they are templates for operations which can be evaluated only when the variables are given appropriate values. However, the symbols can still be manipulated as objects, in simplifying, factorising, solving equations, and so on. The shift in focus from the symbolism of arithmetic where the aim is to obtain numerical answers to the manipulation of template procepts in algebra is one which causes severe difficulties for many learners.

Likewise, in the beginnings of calculus there are symbols which act dually as process and concept. For instance, the *limit* procept:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

dually represents both *process* (as h gets small) and *concept* (the limit itself). This causes further difficulties because it is not computed by a finite set of calculations (Cornu, 1991). Instead, in a specific case such as $f(x)=x^2$, first the simplification is performed assuming $h\neq 0$, then the final result is computed by setting $h=0$. For other expressions this confusing limit process soon becomes too complicated and derivatives are computed by a collection of rules.

The limit concept causes great difficulties for students (Cornu, 1991, Williams, 1991). The majority seem to continue to treat a limit as a *process* getting close rather than a *concept* of limit. The usual default behaviour to cope with lack of meaning is to use the rules of differentiation procedurally. It at least has the familiar quality that it is an algorithm giving a result, albeit a symbolic one, making the limit procept operational.

Figure 2 uses this analysis of different forms of representation to show how they feature in different mathematical topics. It outlines the visuo-spatial to verbal development in geometry, the proceptual development in arithmetic and algebra, and the relationships between them in measurement, trigonometry and cartesian coordinates.

At the top of the figure are the subjects which begin the transition to advanced mathematical thinking. All of these require significant cognitive reconstructions. Euclidean proof requires the realisation of the need of systematic organisation, and agreed ways of verbal deduction for visually inspired proof (the use of congruent triangles). The move into calculus has the difficulties caused by the limit procept. The move into more advanced algebra (such as vectors in three and higher dimensions) involves such things as the vector product which violates the commutative law of multiplication, or the idea of four or more dimensions, which overstretches and even severs the visual link between equations and imaginable geometry.

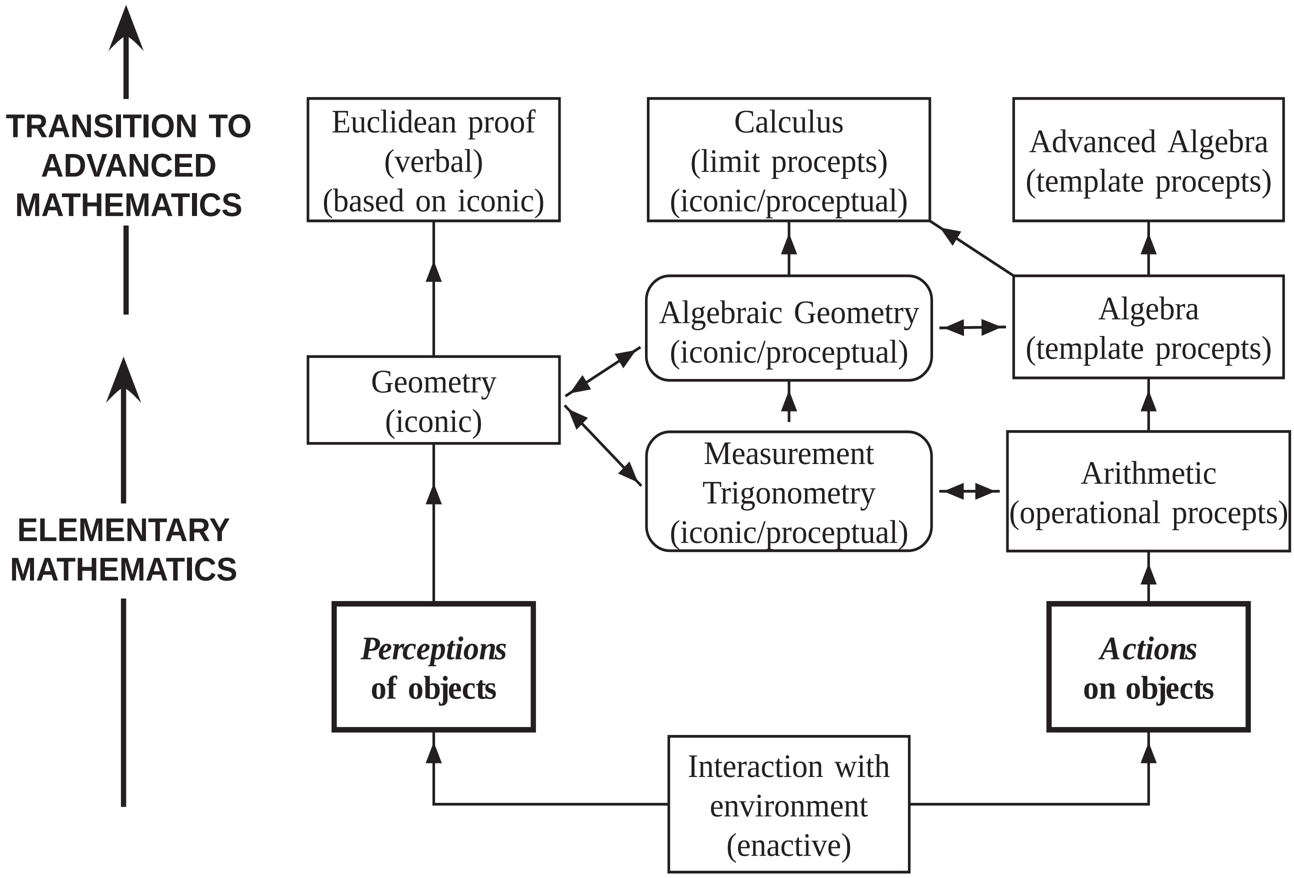

Figure 2: Actions and objects in the building of various mathematical knowledge structures

The transition in all three subjects therefore requires considerable cognitive reconstruction involving a struggle to understand. However, there is an even greater leap to be made in advanced mathematical thinking to formal definitions (which changes the status of the objects being studied) and formal deduction (which changes the nature of proof). To see just how much change is required, let us briefly look at how the nature of proof is dependent on the representations available and on the mathematical context.

The Status of Proof

Given different types of representation and different ways of thinking about them, it follows that there are likely to be different kinds of proof. In the *enactive* mode, proof is by prediction and physical experiment: to show two triangles with equal sides have equal angles, put them one on top of another and see. In the *iconic* mode, a picture is often seen as a *prototype*, that can be thought of as representing not only a single specific case, but others in the same class. The picture in figure 3, which demonstrates that four times three is three times four will work for any other whole numbers and so may be visualised as a *generic* proof that whole number multiplication does not depend on the order:

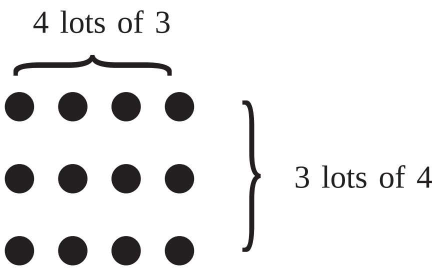

Figure 3: Multiplication is independent of order

Visual proofs, however, begin to fail when pictorial prototypes cease to represent the full meaning of the class of objects to which the proof refers. For instance, the difference between real numbers and rational numbers is difficult to represent visually (although I simulate it in some of my own software for schools, (Tall, 1991)). Here are two pictures. The one on the left is of a *continuous* function on the rationals (the formula reads "if $x^2>2$, then the value is 1 else it is –1, on the domain where x is rational). The one on the right is the real function taking the value $x^2(x^2-1)+1$ if x is rational, and 1 if x is irrational. It is continuous only at $x=-1$, 0 and 1. (It is even differentiable at $x=0$.)

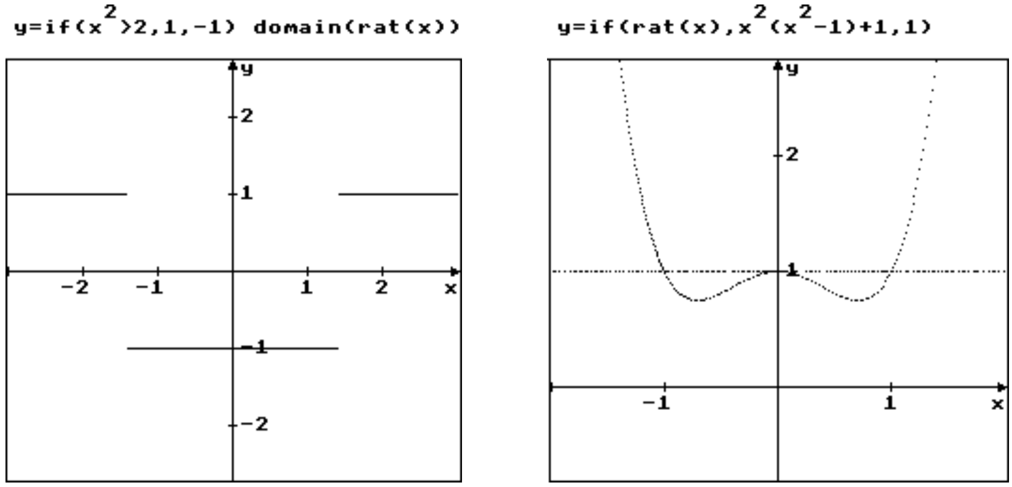

Figure 4: The first function is continuous, the second continuous at –1, 0, 1.

Developing meaning for pictures so that they give correct intuitions is a sophisticated business, which is even more difficult to turn into formal proof. However, for some professional mathematicians visualisation gives valuable insight in the sense of Dreyfus (1991), whilst others, aware of the possible pitfalls, distrust them completely.

Verbal proof depends on the context in which it occurs. For instance, in Euclidean geometry it is essentially a translation of visual generic proofs for triangles and circles based on the pivotal notion of congruent triangles. It is *not* logical proof in the sense accepted in modern axiomatic mathematics. However, it does introduce the learner to a most important aspect of axiomatic proof, that of *systematic organisation*, proving theorems in order, so that each depends only on previously established theorems.

Proof involving *procepts* is usually performed through using the built-in processes. For instance, proof in arithmetic is either through a generic computation, "typical" of a class of examples, or using algebraic computation, e.g. the proof that the sum of two successive odd numbers (2*n*+1 and 2*n*+3) is a multiple of 4 (4*n*+4).

Proof at the *formal* level consists essentially of rearranging the content of a given set of quantified statements to give another quantified statement. These statements relate to definitions of formal mathematical concepts, so if certain properties of concepts are given, others are deduced. The logical part of the deduction is just the tip of the iceberg. The part under the water is a hazard for many trying to navigate for the first time. Expert mathematical thinkers use so much more of their experience to choose concepts worth studying, to formulate them in the most productive way and to select likely lines of attack for proof.

> A group of mathematicians interacting with each other can keep a collection of mathematical ideas alive for a period of years, even though the recorded version of their mathematical work differs from their actual thinking, having much greater emphasis on language, symbols, logic and formalism. But as new batches of mathematicians learn about the subject they tend to interpret what they read and hear more literally, so that the more easily recorded and communicated formalism and machinery tend to gradually take over from other modes of thinking. (Thurston, 1994, p. 167).

The move to advanced mathematical thinking, using a full range of personal mental imagery to develop new theories formulated in terms of systematic proof is more than just the appreciation of a formal development from definitions and axioms. It builds in the kind of structure exhibited in figure 5, with the advanced mathematical thinker using visuo-spatial ideas, symbol-sense and all kinds of intuitions to develop new theories that can be woven into the Bourbaki-like systematic development that forms the solid theoretical basis of the subject.

Where is the transition to advanced mathematical thinking?

In the description so far, the place where elementary mathematical thinking becomes advanced has yet to be precisely defined. In figure 1, the “transition to advanced mathematics” includes systematic Euclidean geometry, calculus and advanced algebra. Certainly these subjects all involve inherent difficulties requiring considerable cognitive reconstruction and, at various times in history (ancient Greece, the seventeenth and nineteenth centuries, respectively), they were topics of mathematical research by the most creative minds of their generation. Calculus and advanced algebra also contain a significant quantity of the mathematics taught at university for students as service subjects, so it would be politic to include these subjects as “advanced mathematics”.

In the deliberations of the Advanced Mathematical Thinking Group of PME at its first meeting in 1987, we found it impossible to come to an agreement and decided pragmatically to take our brief to study mathematical thinking in topics beyond regular mathematics from the age of sixteen. Pragmatism suggests that it would be pertinent to include Euclidean geometry, calculus and advanced algebra above the line. However, whereas each of these subjects has its own idiosyncratic difficulties, the universal cognitive change occurs with the introduction of the axiomatic method, where mathematical objects have a new cognitive status as defined concepts constructed from

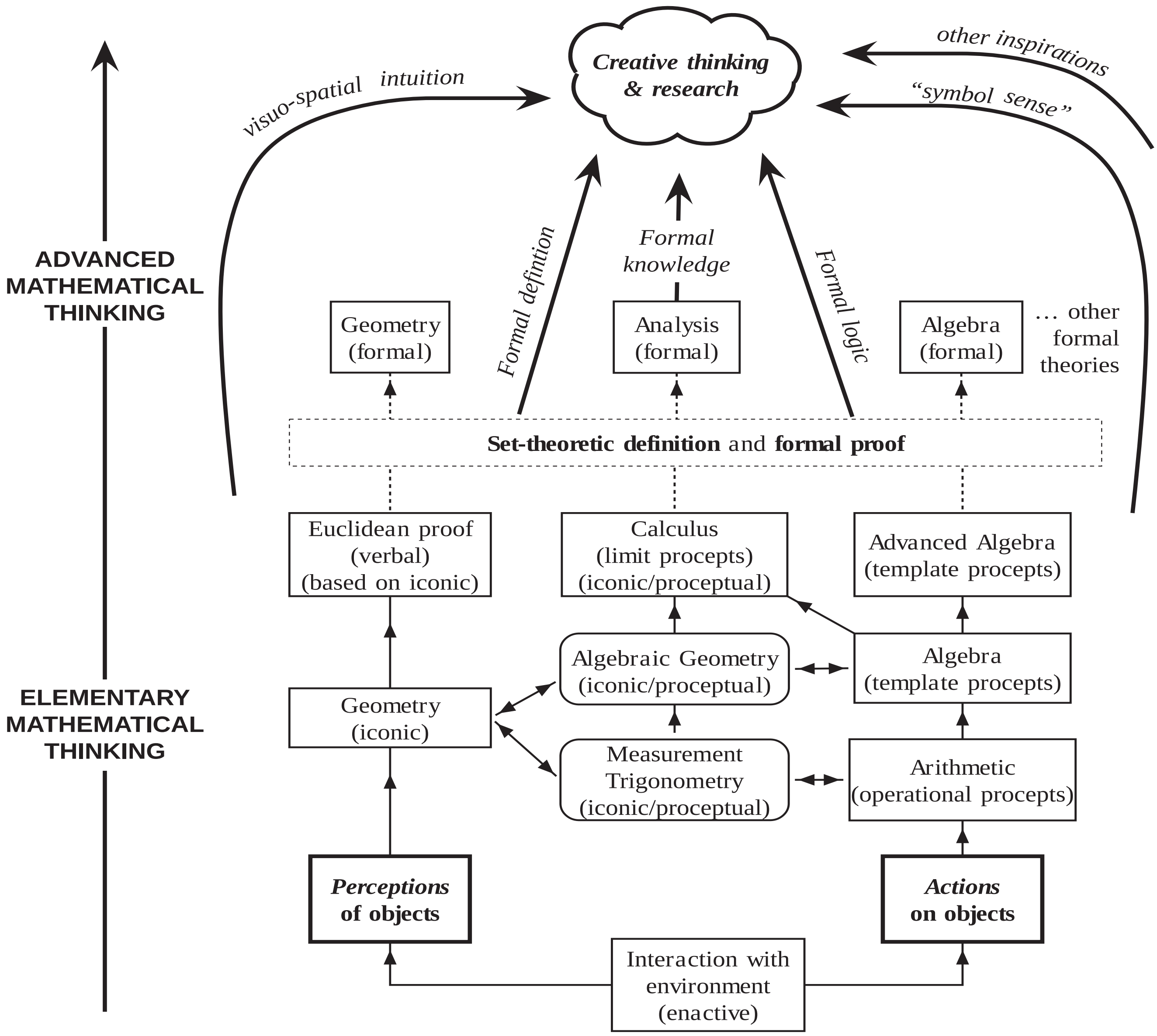

Figure 5 : The development of advanced mathematical thinking

verbal definitions. This is therefore a more natural place to draw the line between elementary and advanced mathematical thinking. It is essentially a change in cognitive stage from the equilibrium of visual conviction and proceptual manipulation to defined objects and formal deduction.

It remains valuable to consider the first level beyond elementary school mathematics to be a preliminary stage of advanced mathematical thinking, in which elementary ideas are stretched to their limits (literally!) before the theoretical crisis they generate requires the reconstruction of a formal view. Many will not require the full range of formal mathematics, being fully occupied with the proceptual complexities of the manipulation of symbols in calculus and algebra. The full range of creative advanced mathematical thinking is mainly the province of professional mathematicians and their students.

The relationship between elementary and advanced mathematical thinking

The changes in status of mathematical objects and mathematical proof at various stages of development offer an alternative viewpoint to consider the relationship between elementary and advanced mathematical thinking. Indeed, it reveals very different forms of mathematics in school and university. The "New Math" of the nineteen sixties was an attempt to introduce a set-theoretic deductive approach in elementary mathematics and it failed. Now mathematics educators involved with mathematics in school are operating in an age of democratic equality of opportunity which is predicated on a broad curriculum suitable for the needs of the wide population. There are signs that the curriculum in elementary mathematics is producing students less ready to study mathematics at university.

A recent report (Pozzi & Sutherland, 1995) has highlighted perceived shortcomings in students arriving in the UK to study engineering. An exchange of letters in the English national press has revealed serious concerns about "falling standards" related to changes in the English curriculum. For example, Sykes & Whittaker (1994) report that in 1994, only 50% of the entrants to their business studies course could multiply 1/2 by 2/3 and, whereas 66% could correctly calculate the square of 0.3 in 1987, by 1994 this had fallen to 16%. The general consensus amongst university mathematicians in England is that students arrive at university to study mathematics with less understanding of proof, less proficiency in handling arithmetic (particularly fractions and decimals) and less facility with algebraic manipulation.

The decline of Euclidean geometry in English schools has led to a loss of experience with systematic proof. The increase in practical links with real world problems and loss of manipulative practice seems to lead to less meaning *within* mathematics. Procepts, such as fractions, involve many conceptual encapsulations, including the encapsulation of counting as the concept of number, addition of whole numbers as sum, repeated addition as product and the process of equal sharing as the concept of fraction. There is little wonder that fractions proves difficult for a wide range of the population. Likewise, the meaningless manipulation of symbols in algebra is a consequence of inability to give them meaning as process and concept (Sfard & Linchevski, 1994).

It would be pertinent for a proportion of the mathematics education community to focus on the learning of those students in elementary mathematics who might develop the potential for advanced mathematical thinking, to analyse whether their learning environment is suitable for their long-term development. Short-term it would be possible to consider the ways in which the "more successful" do mathematics, to see if they need a different environment from others. Perhaps some of the educational devices for introducing mathematics in an elementary way, such as physical balances to introduce equations, introduce cognitive baggage which is not in the long-term helpful for cognitive compression.

Advanced mathematical thinking and undergraduate mathematics

At college level, mathematics is usually still taught in the "definition-theorem-proof-illustration" sequence with little opportunity for developing a full range of advanced mathematical thinking.

> The huge quantities of work covered by each course, in such a short space of time, make it extremely difficult to take it in and understand. … From personal experience I know that most courses do not have any lasting impression and are usually forgotten directly after the examination. This is surely not an ideal situation, where a maths student can learn and pass and do well, but not have an understanding of his or her subject.
>
> *Final Year Undergraduate Mathematics Student*

Rote-learning at university is even worse than procedural learning in school. At least procedures can be *used*, even if the range of application is narrow, but a rote-learnt proof that has no link to anything else has little value other than for passing examinations. Regrettably, students who are good at routine problems in advanced mathematics often fail when faced with something a little different (e.g. Selden, Mason & Selden, 1994).

Mathematicians seem to face a dilemma:

> … we should not expect students to (re-)invent what has taken centuries of corporate mathematical activity to achieve. Yet if we do not encourage them to participate in the generation of mathematical ideas as well as their routine reproduction, we cannot begin to show them the full range of advanced mathematical thinking. (Ervynck, 1991, p. 53)

Fortunately, it *is* possible to encourage students to *think* in a mathematical way at university level, as is shown by problem-solving approaches such as Mason *et al* (1982), Schoenfeld (1987), Rogers (1988), and the "proof debates" of the Grenoble school (Alibert, 1988). Following the problem-solving approach of Mason *et al*, Mohd Yusof (1995) has shown that such a problem-solving approach changed student attitudes in a way that university professors desired, whereas the adherence to traditional lecture methods and the vast quantity of rote-learnt content caused students to change attitude in the opposite direction. Typical responses from professors and students were as follows:

> I see mathematics as something that needs *doing* rather than learning where I should participate actively in making conjectures, constructing arguments to convince others, reflecting on my problem-solving and so on. But I think the maths course at the university does not encourage this. *Student A*

> We work under pressure and often feel anxious that we can't do maths. Not because we can't do it, because we can't do it in time. *Student B*

> The experience of making conjectures, generalising and the like I think students can get themselves on their own, from doing their project work. We do not have the time to teach them everything. *Professor C*

> To me mathematics is a mental activity, but I should say that at this level I present it more as a formal system. Because we are confined by the syllabus and also depending on the students' background. … I would like to change. How do I do that? I don't know.
>
> *Professor D*

Paradoxically, traditional ways of teaching are, for most students, causing precisely the opposite effect that university mathematicians desire. The sheer difficulty and volume of material to be covered in a university mathematics degree makes it difficult for students to cope with formal mathematical content in a limited time. But does this mean that we must accept the status quo of a huge formal syllabus with widespread rote learning, or is it not possible to modify courses to allow students to develop ways of thinking more mathematically? The acquisition of a wide repertoire of advanced mathematical thinking is a challenge which now faces university mathematicians. Is it a challenge which will be accepted?

References

Bruner, J. S. (1966). *Towards a Theory of Instruction*, New York: Norton.
Alibert, D. (1988). Towards New Customs in the Classroom, *For the Learning of Mathematics*, **8** (2), 31–35.
Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 153–166). Dordrecht: Kluwer.
Crick, F. (1994). *The Astonishing Hypothesis*. London: Simon & Schuster.
Davis, R. (1975). Cognitive processes involved in solving simple arithmetic equations, *Journal of Children's Mathematical Behavior*, **1** (3), 7–35.
Dörfler, W. (1993). Fluency in a discourse or manipulation of mental objects, *Proceedings of PME 17*, Tsukuba, Japan, II, 145–152.
Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education, *Proceedings of PME 15*, Assisi, I, 32–48.
Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 95–123). Dordrecht: Kluwer.
Ervynck, G. (1991). Creativity. In D. O. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 42–53). Dordrecht: Kluwer.
Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal of Research in Mathematics Education*, **26** (2), 115–141.
Greeno, G. J. (1983). Conceptual entities. In Genter & Stevens (Eds.), *Mental Models*, 227–252.
Hadamard, J. (1945). *The Psychology of Invention in the Mathematical Field*, reprinted Dover.
Hardy, G. H. (1940). *A Mathematician's Apology* (2nd edition, 1967). Cambridge: CUP.
Mason, J., Burton, L. & Stacey, K. (1982): *Thinking Mathematically*, London: Addison Wesley.
Mohd Yusof, Y. (1995). *Thinking mathematically: A framework for developing positive attitudes amongst undergraduates*. Unpublished Ph. D. Thesis, University of Warwick.
Piaget, J. (1972). *The principles of Genetic Epistemology*, (W. Mays trans.). London: Routledge & Kegan Paul.
Pozzi, S. & Sutherland, R. (1995). *The Changing Mathematical Background of Undergraduate Engineers*, Engineering Council, 110 Maltravers St, London WC2R 3ER.
Rogers, P. (1988). Student-sensitive teaching at the tertiary level: a case study, *Proceedings of PME 15*, Veszprem, Hungary, 536–543.
Selden, J., Mason, A. & Selden, A. (1994). Even good calculus students can't solve non-routine problems. In J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning*. MAA, 33, 119–26.
Schoenfeld, A. H., (1985), *Mathematical Problem Solving*, Orlando: Academic Press.
Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, **22** (1), 1–36.
Sfard, A. & Linchevski, L. (1994). The gains and pitfalls of reification – the case of algebra, *Educational Studies in Mathematics*, **26**, 191–228.
Skemp, R. R. (1979). *Intelligence, Learning and Action*, London: Wiley.
Sykes, A. M. & Whittaker, E. T. (1994). Letter to the *Guardian*, 31 December, 1994.
Tall, D. O. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity, *Educational Studies in Mathematics*, **11**, 271–284.
Tall, D. O. (1991). *Real Functions and Graphs*, [software for the Archimedes Computer]. Cambridge University Press.
Thurston, W. P. (1990). Mathematical Education, *Notices of the American Mathematical Society*, **37** (7), 844–850.
Thurston, W. P. (1994). On proof and progress in mathematics, *Bulletin of the American Mathematical Society*, **30** (2), 161–177.
Van Hiele, P. & D. (1959). The child's thought and geometry. Reprinted (1984) in D. Fuys, D. Geddes & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele*, (pp. 1–214). Brooklyn NY: Brooklyn College.
Vinner, S. & Hershkowitz, R. (1980). Concept images and some common cognitive paths in the development of some simple geometric concepts, *Proceedings of PME4*, Berkeley, 177–184.
Williams, S. R. (1991). Models of limit held by college calculus students, *Journal for Research in Mathematics Education*, **22** (3), 237–251.