

Linear Programming Exercises

Exercise 1 Consider the case of the Betta Machine Products Company described in the lecture notes.

- (a) Use a graphical method to obtain the new optimal solution when the selling price of product 2 changes to (i) 55 pounds or (ii) 54 pounds.
- (b) A third product is being considered which would take 8, 3, 2 minutes of casting, machinery, and assembly time per unit. The unit cost would be 20 pounds and the selling price 38 pounds. It is thought possible that Betta might be short of casting capability so there is the possibility of subcontracting some or all of the casting of product 3. For items cast at the sub-contractors the unit cost would be 25 pounds. Formulate, but do not try to solve, the enlarged problem.

Exercise 2 Solve the following problems graphically.

(a)

$$\begin{aligned} &\text{maximise } z = x_1 + x_2 \\ &\text{subject to:} \\ &x_1 + 2x_2 \leq 10 \\ &2x_1 + x_2 \leq 16 \\ &-x_1 + x_2 \leq 3 \\ &x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} &\text{minimise } z = x_1 + 3x_2 \\ &\text{subject to:} \\ &x_1 + 2x_2 \geq 6 \\ &x_1 - x_2 \leq 3 \\ &x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(c) Minimise $2x_1 - x_2$ subject to the constraints in (b).

Exercise 3 A factory makes 3 components, A, B and C using the same production process for each. A unit of A take 1 hr, a unit of B takes 0.75 hrs and a unit of C takes 0.5 hrs. In addition, C has to be hand finished, an activity taking 0.25 hrs per unit. Each week total production time (excluding

hand finishing) must not exceed 300 hrs and hand finishing must not exceed 45 hrs.

The components are finally assembled to make two finished products. One product consists of 1 unit of A and 1 unit of C selling for 30 pounds whilst the other consists of 2 units of B and 1 unit of C and sells for 45 pounds. At most 130 of the first product and 100 of the second product can be sold each week. Formulate the problem of planning weekly production to maximise total proceeds as a linear programming problem in 2 variables and obtain the solution graphically.

Exercise 4 The Claverton Police Force has the following minimum daily requirements for policemen on duty.

0.00 – 4.00	4.00 – 8.00	8.00 – 12.00	12.00 – 16.00	16.00 – 20.00	20.00 – 24.00
15	35	65	80	40	25

Each policeman comes on duty at 0.00, 4.00, 8.00, 12.00, 16.00 or 20.00 hrs and works for eight consecutive hours. Formulate the problem of finding the duty schedule that minimises the total number of policemen required. Assume that the same schedule is repeated day after day. Do not attempt to solve the problem.

Exercise 5 Consider a linear programming problem given in standard form (S). That is to say,

$$\begin{aligned} &\text{maximise } z = \mathbf{c} \cdot \mathbf{x} \\ &\text{subject to:} \\ &\mathbf{Ax} \leq \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0}_n \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. Write down the associated problem in canonical form (C). Prove that if (C) has an optimal solution then its restriction to \mathbb{R}^n is also the optimal solution to (S). Conversely suppose that (S) has an optimal solution $\mathbf{x} \in \mathbb{R}^n$, then there exists an optimal solution to (C) whose restriction to \mathbb{R}^n is \mathbf{x} .

Exercise 6 Put the following problem in canonical form and solve by determining all its basic solutions. Interpret graphically.

$$\begin{aligned} &\text{maximise } z = 3x_1 + 2x_2 \\ &\text{subject to:} \\ &x_1 + x_2 \leq 4 \\ &x_1 + 2x_2 \leq 6 \\ &x_1, x_2 \geq 0. \end{aligned}$$

Exercise 7 Prove that if the system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a feasible solution it has a basic feasible solution. Use a simplified version of the proof of the fundamental theorem.

Exercise 8 A mining company produces 100 tons of red ore and 80 tons of black ore each week. These can be treated in different ways to produce three different alloys, Soft, Hard or Strong. To produce 1 ton of Soft alloy requires 5 tons of red ore and 3 tons of black. For the Hard alloy the requirements are 3 tons of red and 5 tons of black, whilst for the Strong alloy they are 5 tons of red and 5 tons of black. The profit per ton from selling the alloys (after allowing for production but not mining costs, which are regarded as fixed) are £250, £300 and £400 for Soft, Hard and Strong respectively.

Formulate the problem of deciding how much of each alloy to make each week as a L.P. problem and use the Simplex method to find the optimal solution.

Exercise 9 Prove that if for some basic solution and for some value l , there is an entry in the objective row $o_l > 0$ and also all values $a_{il} \leq 0$, then the problem has no finite maximising solution. (See pages 30–31 of lecture notes for notation as well as the example ending on p33 for a numerical example of such a phenomena).

Exercise 10 Use the Simplex method to show that the following problem has no finite maximising solution.

$$\begin{aligned} &\text{maximise } z = -x_1 + 2x_2 + x_3 \\ &\text{subject to:} \\ &3x_1 + x_2 - 4x_3 \leq 4 \\ &x_1 - x_2 - x_3 \leq 10 \\ &x_1 - 2x_2 + 6x_3 \leq 9 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

Find a particular solution with $z > 1000$.

Exercise 11 Solve the following problem by the two phase method.

$$\begin{aligned} &\text{maximise } z = x_1 + x_2 - 2x_3 + 2x_4 \\ &\text{subject to:} \\ &x_1 - x_2 - x_3 - 2x_4 \geq 2 \\ &x_1 + x_2 + x_4 \leq 8 \\ &x_1 + 2x_2 - x_3 = 4 \\ &x_1, \dots, x_4 \geq 0. \end{aligned}$$

Exercise 12 A health food shop packages three types of snack foods; chewy, crunchy and nutty. These are made by mixing sunflower seeds, rasins, and peanuts. The specifications for each food being given in the following table.

Mixture	Sunflower seeds	Rasisns	Peanuts	Retail price/kg
Chewy	-	at least 60%	at most 25 %	£2.00
Crunchy	at least 60 %	-	-	£1.60
Nutty	at most 20%	-	at least 60%	£1.20

The suppliers of the ingredients can deliver each week at most 100kg of sunflower seeds at £1.00/kg, 80kg of rasins at £1.50/kg and 60kg of peanuts at £0,80/kg. Assuming there is no limit to what can be sold, formulate the problem of finding the mixing scheme that maximises weekly profit.

Exercise 13 A coffee packer blends Brazilian coffee and Colombbian coffee to prepare two products, **super** and **deluxe** brands. Each kilogram of super coffee contains 0.5 kg of Brazilian coffee and 0.5 kg of Colombbian coffee, whereas each kilogram of deluxe coffee contains 0.25 kg of Brazilian coffee and 0.75 kg of Colombbian coffee. The packer has 120kg of Brazilian coffee and 160kg of Colombbian coffee on hand. If the profit one each kilogram of super coffee is 22 cents and the profit on each kilogram of Deluxe coffee is 30 cents, how many kilograms of each type of coffee should be blended to maximise profits? Formulate and solve.

Exercise 14 Major oil companies use linear programming to model many phases of their operations. Consider the following simplified version of part of a refinery operation. Two kinds of aviation gasoline, **high-octane** and **low-octane**, are made by blending four components from the refinery output. For the low octane gasoline the components are augmented with a small amount of tetraethyllead (TEL) to give the low-octane ratings shown in the accompanying table. The high-octane gasoline is made from the same components when these have been augmented with a larger amount of TEL, giving the high-octane ratings in the table. Assume that the octane rating (OR) of the mixture is the volumetric average of the octane ratings of the components. That is, letting V denote volume, we have

$$OR_{mix} = \frac{OR_{comp1} \times V_{comp1} + OR_{comp2} \times V_{comp2} + \cdots}{V_{comp1} + V_{comp2} + \cdots}.$$

The vapour pressure (a measure of the tendency of the gasoline to evaporate) of both gasolines must be 7. Assume that the vapour pressure of a mixture is the volumetric average of the vapour pressures of the components. Vapour pressure and octane rating are the only two physical properties for which there are constraints. Data for the components and desired mixtures are given in the accompanying table.

	Vapour pressure	OR		Demand	Supply	Revenue	Cost
		High	Low				
Component							
Alkylate	5	108	98	-	700	-	7.20
Catalytic cracked	6.5	94	87	-	600	-	4.35
Straight run	4	87	80	-	900	-	3.80
Isopentane	18	108	100	-	500	-	4.30
Mixture							
High octane	7	100	-	1300	-	6.50	-
Low octane	7	-	90	800	-	7.50	-

Assuming that demands must be met exactly, maximise revenue minus cost. **Formulate this problem in the form of a linear programming problem.**

Exercise 15 (Bookwork). Show that the dual of

$$\begin{aligned}
 &\text{maximise } z = \mathbf{c} \cdot \mathbf{x} \\
 &\text{subject to:} \\
 &\mathbf{Ax} = \mathbf{b} \\
 &\mathbf{x} \geq \mathbf{0}_n
 \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are given. is the linear programming problem

$$\begin{aligned}
 &\text{minimise } z = \mathbf{b} \cdot \mathbf{w} \\
 &\text{subject to:} \\
 &\mathbf{A}^T \mathbf{w} \geq \mathbf{c} \\
 &\mathbf{w} \in \mathbb{R}^m
 \end{aligned}$$

(ie no restriction on the sign of each entry of \mathbf{w}).

Exercise 16 Consider the linear programming problem in canonical form

$$\begin{aligned}
 &\text{maximise } z = \mathbf{c} \cdot \mathbf{x} \\
 &\text{subject to:} \\
 &\mathbf{Ax} = \mathbf{b} \\
 &\mathbf{x} \geq \mathbf{0}_n
 \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are given. Recall from Section 5.3 of the lecture notes that if we have a basic feasible solution in the form

$$\mathbf{x} = \begin{pmatrix} \mathbf{0}_{n-m} \\ \mathbf{x}_B \end{pmatrix}$$

and partitioning \mathbf{A} accordingly as $(\mathbf{A}^o | \mathbf{B})$ and similarly with \mathbf{c} then the tableau associated with the next step of the simplex algorithm may be rearranged as follows:

$\mathbf{B}^{-1}\mathbf{A}^o$	\vdots	\mathbf{I}_m	\mathbf{x}_B
$(\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}^o - \mathbf{c}^o)^T$	\vdots	$\mathbf{0}_m^T$	$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$

Suppose now that \mathbf{x} is optimal.

- (a) Deduce that $\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}^o \geq \mathbf{c}^o$.
- (b) Now suppose that we define $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$ and note that $\mathbf{w} \in \mathbb{R}^m$. Show that $\mathbf{w}^T \mathbf{A} \leq \mathbf{c}^T$.
- (c) Finally show that $\mathbf{b} \cdot \mathbf{w} = \mathbf{c} \cdot \mathbf{x}$.

Use the weak duality theorem to show that \mathbf{w} is the optimal solution to the asymmetric dual of the canonical linear programming problem.

Note that this question also tells you how to solve explicitly the dual of a given linear programming problem - this is useful in the next two questions.

Exercise 17 The problem

$$\begin{aligned} &\text{maximise } z = 3x_1 + 6x_2 + 2x_3 \\ &\text{subject to:} \\ &3x_1 + 4x_2 + 2x_3 \leq 200 \\ &x_1 + 3x_2 + 2x_3 \leq 100 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

has its optimal solution with x_1 and x_2 as basic variables. Use this information to obtain the simplex tableau for the optimal solution directly without using the simplex algorithm. (ie calculate \mathbf{B}^{-1} etc).

Write down the dual problem and use the tableau to obtain both the primal and the dual optimal solutions.

Check that the solutions are (i) feasible (ii) have equal values of the objectives and (iii) **(for later)** satisfy the complementary slackness conditions.

Exercise 18 Consider the problem of the mining company in Sheet 2, Exercise 3. Write down the dual problem and use your simplex tableau corresponding to the optimal primal solution to obtain the optimal dual solution. Give an economic interpretation of the dual variables and in terms of this explain why none of the hard alloy was made. By how much would the profit per ton for this alloy have to increase before it could be produced economically?

Exercise 19 Suppose that we consider the asymmetric dual to the primal canonical linear programming problem

$$\begin{aligned} &\text{maximise } z = \mathbf{c} \cdot \mathbf{x} \\ &\text{subject to:} \\ &\mathbf{Ax} = \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0}_n \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are given. Prove that the conclusion of the Weak Duality Theorem is still valid. That is to say, for any feasible solution \mathbf{x} to the primal and feasible solution to the dual \mathbf{y} it is still true that $\mathbf{c} \cdot \mathbf{x} \leq \mathbf{b} \cdot \mathbf{y}$. Further, it is still true that if $\mathbf{c} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{y}$ then \mathbf{x} and \mathbf{y} are optimal for the primal and dual respectively.

Exercise 20 Go back and do part (iii) of Exercise 17.

Exercise 21 Show without using the Simplex Method that

$$\mathbf{x}^T = \left(\frac{5}{26}, \frac{5}{2}, \frac{27}{26} \right)$$

is an optimal solution to the following linear programming problem:

$$\begin{aligned} &\text{maximise } z = 9x_1 + 14x_2 + 7x_3 \\ &\text{subject to:} \\ &2x_1 + x_2 + 3x_3 \leq 6 \\ &5x_1 + 4x_2 + x_3 \leq 12 \\ &2x_2 \leq 5 \\ &x_1, x_2, x_3 \text{ unrestricted.} \end{aligned}$$

Exercise 22 Consider the linear programming problem

$$\begin{aligned} &\text{maximise } z = 3x_1 + 4x_2 \\ &\text{subject to:} \\ &x_1 + 2x_2 \leq 10 \\ &x_1 + x_2 \leq 8 \\ &3x_1 + 5x_2 \leq 26 \\ &x_1, x_2 \geq 0. \end{aligned}$$

By using the principle of complementary slackness, show that $y_1 = 0$ in any optimal solution \mathbf{y} to the dual.

Exercise 23 Solve the transportation problem having the following cost/requirement/availability table. Start with the N.W. corner solution.

	55	70	35	40
80	13	11	18	17
100	2	14	10	1
20	5	8	18	11

Exercise 24 It is required to move machines from factories A, B and C to warehouses X, Y and Z. There are 5 required at X, 4 at Y and 3 at Z, whilst there are 8 available at A, 5 at B and 3 at C. The transport cost in £ between the sites is as follows.

	X	Y	Z
A	50	60	30
B	60	40	20
C	40	70	30

Plan the movements to minimise total transport cost.

Exercise 25 Suppose in question 2 there is now an increase in transport cost of £10 per machine taken from B for any machines above three (i.e. for the 4th and 5th). Does this alter the best plan? To solve this you will need to split source B into two sources. This device only works because the cost is **increased** for higher quantity. Why?

Exercise 26 Suppose now in question 2 there is a **reduction** in transport cost of £10 per machine taken from A for any machines above six. (This

is separate from and not in addition to the change described question 3.) Use the same device as in question 3 to find the new optimal solution, but because this time there has been a decrease in price for higher quantity, you will need to solve two problems and also use pricing out¹. Sort out the details yourself.

Exercise 27 Examine the effect on solving question 1 of using the cheapest route method to obtain a starting solution.

Exercise 28 A factory uses a raw material whose price and availability vary seasonally. The price, availability and factory requirement for each quarter of the next year are given in the following table.

Quarter	1	2	3	4
Price/ton in £	110	100	120	130
Availability (tons)	1000	1700	800	400
Requirement (tons)	750	900	1000	850

The cost per ton of storing the material from one quarter to the next is £4 + 10% of the purchase price. The material may also be stored for two quarters at double the above cost per ton, but will not keep for longer than two quarters. No stock is held initially and none is required at the end.

Find the pattern of buying and storing that minimises the total cost. State any assumptions that you make.

Exercise 29 Solve the following transportation problem starting with the N.W.corner solution. The point here is that care is needed in handling degenerate solutions.²

	5	10	10
10	4	2	3
5	6	5	8
10	1	4	3

¹Recall the pricing out is a technique by which one imposes a very high transportation cost in order to avoid such a route appearing in the solution

²Recall that a degenerate solution occurs when one of the basic variables is equal to zero.

Exercise 30 Find a maximal flow and a minimal cut for the following network. Show explicitly the flows in the individual arcs.

$$\begin{pmatrix} 0 & 9 & 6 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 & 5 & 0 \\ 0 & 6 & 3 & 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 5 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 31 For the example problem given in the lecture notes, consider applying the following sequence of flows

$$\begin{aligned} 1 &\rightarrow 3 \rightarrow 5 \rightarrow 6 : \text{flow } 5 \\ 1 &\rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 6 : \text{flow } 2 \\ 1 &\rightarrow 2 \rightarrow 5 \rightarrow 6 : \text{flow } 1 \\ 1 &\rightarrow 3 \rightarrow 6 : \text{flow } 1 \\ 1 &\rightarrow 2 \rightarrow 4 \rightarrow 6 : \text{flow } 2. \end{aligned}$$

First adjust the residual capacities **only** in the direction of the flows of the paths above and you will see that the flow appears optimal. Now, correctly, adjust the residual capacities in **both** directions and you will see that there **is** an unsaturated path and the flow can be augmented.

In doing this, you may be able to calculate the residual capacities in one step or prefer to do it in several steps.

Exercise 32 Show that a maximal flow problem may be expressed as a linear programming problem.

Exercise 33 Prove that for any cut (A, B)

$$A(f) = f(A, B)$$

by showing that (a) $A(f) = f(A, A) + f(A, B) - f(A - s, N)$ and (b) $f(A, A) = 0$ and (c) $f(A - s, N) = 0$.