# Nested radicals

It is well known from basic algebraic courses that the nested radical $\sqrt{a \pm b\sqrt{c}}$
can be split in two simple radical by the following formulas

$$\sqrt{a \pm b\sqrt{c}} = \sqrt{\tfrac{1}{2}(a+\sqrt{q})} \pm \sqrt{\tfrac{1}{2}(a-\sqrt{q})}$$

if the quantity $q = a^2 - b^2 c$ is a perfect square.
For example

$$\sqrt{3+\sqrt{5}} = \sqrt{\tfrac{1}{2}(3+\sqrt{4})} + \sqrt{\tfrac{1}{2}(3-\sqrt{4})} = \sqrt{\frac{5}{2}} + \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{10} + \frac{1}{2}\sqrt{2}$$

$$\sqrt{11+4\sqrt{7}} = \sqrt{\tfrac{1}{2}(11+\sqrt{9})} + \sqrt{\tfrac{1}{2}(11-\sqrt{9})} = \sqrt{7} + \sqrt{4} = \sqrt{7} + 2$$

$$\sqrt{8+4\sqrt{3}} = \sqrt{\tfrac{1}{2}(8+\sqrt{16})} - \sqrt{\tfrac{1}{2}(8-\sqrt{16})} = \sqrt{6} - \sqrt{2}$$

Quite unknown is the method for splitting the cubic nested radicals of the type $\sqrt[3]{a+b\sqrt{c}}$ .
Similarly to the previous formula, we define the auxiliary quantity $q = a^2 - b^2 c$ .
If q is a perfect cube, then we form the associate rational cubic equation

$$n^3 + p\,n = 2a \quad \text{, where} \quad p = \sqrt[3]{-27q} \qquad (1)$$

The integer solutions of this equation (if any) solve the given problem, being

$$\sqrt[3]{a+b\sqrt{c}} = \frac{n}{2} + \sqrt{\frac{8a-n^2}{12n}} \qquad (2)$$

Let's see same examples

$\sqrt[3]{26+15\sqrt{3}}$

$q = -1 \;\Rightarrow\; p = -3 \;\Rightarrow\; n^3 - 3n = 52$

the integer solution of the above equation, if exist, must belong to the set of the exact divisors of 52 that are: { 1, 2, 4, 13, 26, 52}. It is easy to verify that n = 4 is the solution.
Tip: one simple trick to restrict the number of tests is starting from the integer $n_0$ nearest to the initial guess $x_0 = \sqrt[3]{2a}$ . If $n_0$ does not verify the equation, we pick up the second nearest one and so on until the entire set has been explored. In that case the initial guess is $x_0 \cong 3.73...$ and the nearest integer of the divisors-set is 4, that is just the exact root of the equation.
Applying the formula (2) we have the final split radical

$$\sqrt[3]{26+15\sqrt{3}} = \frac{4}{2} + \sqrt{\frac{8\cdot 26 - 4^2}{12\cdot 4}} = 2 + \sqrt{3}$$

**Decompose the following nested cubic radicals**

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$\sqrt[3]{7+5\sqrt{2}}$

$q=1 \;\Rightarrow\; p=3 \;\Rightarrow\; n^3+3n=14$

The divisor-set is: { 1, 2, 7, 14}, and the starting geuss is $x_0 \cong 3.73$.... The exact root is n = 2

$$\sqrt[3]{7+5\sqrt{2}} = \frac{2}{2}+\sqrt{\frac{8\cdot 7-2^2}{12\cdot 2}} = 1+\sqrt{3}$$

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$\sqrt[3]{-99+70\sqrt{2}}$

$q=-1 \;\Rightarrow\; p=-3 \;\Rightarrow\; n^3-3n=-198$

In that case the divisor-set is quite wide { −1, −2, −3, −6, −9, −11, −18...}, but using the initial guess $x_0 \cong -5.82$..., we can easily discover the exact root n = −6

$$\sqrt[3]{-99+70\sqrt{2}} = \frac{-6}{2}+\sqrt{\frac{8\cdot(-99)-6^2}{12\cdot(-6)}} = -3+\sqrt{8}$$

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$\sqrt[3]{44-18\sqrt{6}}$

$q=8 \;\Rightarrow\; p=6 \;\Rightarrow\; n^3+6n=88$

In that case the divisor-set is {1, 4, 8, 11, 22...}, the initial guess $x_0 \cong 4.44$..., that gives us the exact root n = 4

$$\sqrt[3]{44-18\sqrt{6}} = \frac{4}{2}+\sqrt{\frac{8\cdot 44-4^2}{12\cdot 4}} = 2+\sqrt{6}$$

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$\sqrt[3]{459-300\sqrt{3}}$

$q=59319=39^3 \;\Rightarrow\; p=117 \;\Rightarrow\; n^3+117n=918$

In that case the divisor-set is wide {1, 2, 3, 6, 9,17, 18...}, the initial guess $x_0 \cong 9.71$..., that gives us the first guess n = 9, but is not a root; therefore we take the second nearest number, n = 6 that, finally, verify the equation .

$$\sqrt[3]{459-300\sqrt{3}} = \frac{6}{2}+\sqrt{\frac{8\cdot 459-6^2}{12\cdot 6}} = 3+\sqrt{48} = 3+4\sqrt{3}$$

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