Tridiagonal uniform matrices (T o e p l i t z)

$$P=\begin{pmatrix} 3 & -\sqrt{2} & & & \\ -\sqrt{2} & 3 & -\sqrt{2} & & \\ & -\sqrt{2} & \mathbf{O} & \mathbf{O} & \\ & & \mathbf{O} & 3 & -\sqrt{2} \\ & & & -\sqrt{2} & 1 \end{pmatrix}_{n\times n}$$

and

$$G=\begin{pmatrix} 3 & -\sqrt{2} & & & \\ -\sqrt{2} & 3 & -\sqrt{2} & & \\ & -\sqrt{2} & \mathbf{O} & \mathbf{O} & \\ & & \mathbf{O} & 3 & -\sqrt{2} \\ & & & -\sqrt{2} & 3 \end{pmatrix}_{n\times n}$$

are two tridiagonal matrices. They are different only in the last entry. Prove that:

1. The eigenvalues of P are all single eigenvalue .
2. The largest eigenvalue of G is bigger than the largest eigenvalue of P.

Before beginning to solve the given problem, we will prove the following lemmas:
(In the following examples we will use finite dimension matrices only for clarity)

Lemma 1:
The matrices A end B

$$A=\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & a & b & 0 & 0 \\ 0 & c & a & b & 0 \\ 0 & 0 & c & a & b \\ 0 & 0 & 0 & c & k \end{bmatrix} \qquad B=\begin{bmatrix} k & b & 0 & 0 & 0 \\ c & a & b & 0 & 0 \\ 0 & c & a & b & 0 \\ 0 & 0 & c & a & b \\ 0 & 0 & 0 & c & a \end{bmatrix}$$

have the same eigenvalues.
Prof:
Matrices A and B can be transformed each other by similarly transform with the orthogonal unitary matrix

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U^{-1} = U^T = U$$

$$A = U^{-1} \cdot B \cdot U = U \cdot B \cdot U$$

So the eigenvalues of A are the same of B
QED

Lemma 2
The max eigenvalue of the (n x n) tridiagonal uniform matrix is always greater than t he same ((n-1) x (n-1)) matrix

Given the matrix (n x n)

$$P_5 = \begin{bmatrix} a & b & 0 & \dots & 0 \\ c & a & b & \dots & 0 \\ 0 & c & a & \dots & 0 \\ \dots & \dots & \dots & \dots & b_{n-1} \\ 0 & 0 & 0 & c_{n-1} & a_n \end{bmatrix}$$

The max eigenvalues is

$$l_{\max}(n) = a + 2bc \cdot \cos\left(\frac{np}{n+1}\right)$$

While the max eigenvalue of the ((n-1) x (n-1)) matrix is

$$l_{\max}(n-1) = a + 2bc \cdot \cos\left(\frac{(n-1)p}{n}\right)$$

We want to prove that, for n > 2, is

$$l_{\max}(n-1) < l_{\max}(n)$$

Substituting, we have

$$a + 2bc \cdot \cos\left(\frac{(n-1)p}{n}\right) < a + 2bc \cdot \cos\left(\frac{np}{n+1}\right)$$

$$\cos\left(\frac{(n-1)p}{n}\right) < \cos\left(\frac{np}{n+1}\right)$$

being: $\cos\left(\frac{(n-1)p}{n}\right) = \cos\left(p - \frac{p}{n}\right) = -\cos\left(\frac{p}{n}\right)$

We have:

$$-\cos\left(\frac{p}{n}\right) < \cos\left(\frac{np}{n+1}\right)$$

For n>2 , the first member is always negative while the second one is always positive, so the disequation is always true
QED

Lemma 3
Two consecutive characteristic polynomials of tridiagonal uniform matrices have opposite sign
That is:

$$G_n(l) = a_n l^n + a_{n-1} l^{n-1} + a_{n-2} l^{n-2} + \dots a_1 l + a_0$$

$$a_n = \begin{cases} +1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

The proof comes immediately from the recurrence formula

$$G_0 = 1$$
$$G_1 = a - l$$

$$G_n(l) = (a - l) \cdot G_{n-1} - bc \cdot G_{n-2}$$

Example for $a = 3 \quad b = c = -\sqrt{2}$

$$G_0 = 1$$

$$G_1 = 3 - l$$

$$G_2 = 7 - 6l + l^2$$

$$G_3 = 15 - 23l + 9l^2 - l^3$$

Etc.

Now we shell prove the given problem

For $a=3$, $b=-\sqrt{2}$, given the following matrix

$$P=\begin{bmatrix} 1 & b & 0 & 0 & 0 & 0 \\ b & a & b & 0 & 0 & 0 \\ 0 & b & a & b & 0. & 0 \\ 0 & 0 & b & a & b & 0 \\ 0 & 0 & 0 & b & a & b \\ 0 & 0 & 0 & 0 & b & a \end{bmatrix}$$

and the following uniform tridiagonal matrix

$$G=\begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ b & a & b & 0 & 0 & 0 \\ 0 & b & a & b & 0. & 0 \\ 0 & 0 & b & a & b & 0 \\ 0 & 0 & 0 & b & a & b \\ 0 & 0 & 0 & 0 & b & a \end{bmatrix}$$

We shell prove that

1. The eigenvalues of P are all single eigenvalue .
2. The largest eigenvalue of G is bigger than the largest eigenvalue of P.

The first point comes immediately being the matrices P and G symmetric; so theirs eigenvalues are all real, single and distinct.

For proving the second point we study the sign of the characteristic polynomial of P

$$|P-lI|=\begin{bmatrix} 1-l & -\sqrt{2} & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 3-l & -\sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 3-l & -\sqrt{2} & 0. & 0 \\ 0 & 0 & -\sqrt{2} & 3-l & -\sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 3-l & -\sqrt{2} \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 3-l \end{bmatrix}$$

Appling the Laplace's develop to the first row, we have

$$P_n=(1-l)\cdot G_{n-1}-2G_{n-2} \qquad (1)$$

where G_{n-1} and G_{n-2} are the uniform matrices of (n-1) and (n-2) dimension respectively

Appling the same develop to the uniform tridiagonal G matrix we have

$$G_n = (3 - l) \cdot G_{n-1} - 2G_{n-2} \qquad (2)$$

Subtracting (2) from (1) we have:

$$G_n - P_n = (3 - l) \cdot G_{n-1} - 2G_{n-2} - (1 - l) \cdot G_{n-1} + 2G_{n-2}$$

$$P_n = G_n - 2G_{n-1}$$

Now we write the polynomial G in the factorized form

Calling the “n” eigenvalues of G_n

$$s_1 < s_2 < s_3 < ... < s_n \quad \Rightarrow \quad G_n = a_n(l - s_1)(l - s_2)(l - s_3)...(l - s_n)$$

And calling the “n-1” eigenvalues of G_{n-1}

$$t_1 < t_2 < t_3 < ... < t_n \quad \Rightarrow \quad G_{n-1} = a_{n-1}(l - t_1)(l - t_2)(l - t_3)...(l - t_{n-1})$$

So we have

$$P_n = a_n(l - s_1)(l - s_2)(l - s_3)...(l - s_n) - 2a_{n-1}(l - t_1)(l - t_2)(l - t_3)...(l - t_{n-1})$$

$$P_n = a_n\left((l - s_1)(l - s_2)(l - s_3)...(l - s_n) - 2\frac{a_{n-1}}{a_n}(l - t_1)(l - t_2)(l - t_3)...(l - t_{n-1}) \right)$$

and for the Lemma 3 $\frac{a_{n-1}}{a_n} = -1$

$$P_n = a_n\left((l - s_1)(l - s_2)(l - s_3)...(l - s_n) + 2(l - t_1)(l - t_2)(l - t_3)...(l - t_{n-1})\right) \qquad (3)$$

P_n is the characteristic polynomial of the matrix P. The s_n is the biggest eigenvalues of the matrix G_n that for the lemma 2 is also bigger that the max eigenvalues t_{n-1} of the matrix G_{n-1}. That is:

$$t_{n-1} < s_n$$

We shell prove that P_n cannot have eigenvalues bigger or equal to s_n

Or, in other words:

$$P_n(l) \neq 0 \quad \Leftrightarrow \quad l \geq s_n \qquad (4)$$

In fact, for $l \geq s_n$

$$(l - s_j) \geq 0 \qquad j = 1...n$$

And also

$$(l - t_j) > 0 \qquad j = 1...(n-1)$$

So, the right member of (3) is always positive or negative (depending by a_n)

$$P_n \begin{cases} > 0 & \text{n even} \\ < 0 & \text{n odd} \end{cases}$$

That proves the statement (4)

Consequently, having P_n "n" real distinct eigenvalues, its max eigenvalues must be always less than s_n

QED

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