Eigenvalues and eigenvectors of tridiagonal uniform matrices

In numeric calculus is common to encounter symmetric, tridiagonal, uniform matrices like the following. For this kind, there is a nice close formula giving all eigenvalues for any size of the matrix dimension.

$$\begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ b & a & b & 0 & 0 & 0 \\ 0 & b & a & b & 0 & 0 \\ 0 & 0 & b & a & b & 0 \\ 0 & 0 & 0 & b & a & b \\ 0 & 0 & 0 & 0 & b & a \end{bmatrix}$$

If the symmetric matrix has n x n dimension, eigenvalues are:

$$l_k = a + 2b \cdot \cos\left(\frac{kp}{n+1}\right)$$

where k = 1, 2… n

We can do the following observations:

- All eigenvalues are real and distinct being the matrix symmetric
- All eigenvalues are symmetric around the point “a”
- For n odd exists the trivial eigenvalues $\lambda = a$
- All roots lie into the interval $a - 2b < \lambda_k < a + 2b$

Also the eigenvectors matrix can be written in a closed compact form.

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{1n} \\ \dots & \dots & \dots & \dots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix}$$

If the symmetric matrix has n x n dimension, the elements of the eigenvectors matrix are:

$$u_{ik} = \sin\left(i \cdot k \frac{p}{n+1}\right)$$

Where i = 1, 2… n , k = 1, 2… n

The unsymmetrical tridiagonal uniform case can be led back to the above one.
We distingue two cases:
1) The sub-diagonals have the same sign. In that case we can demonstrate that all roots are real and distinct.

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 & \dots \\ c & a & b & 0 & 0 & \dots \\ 0 & c & a & b & 0 & \dots \\ 0 & 0 & c & a & b & \dots \\ 0 & 0 & 0 & c & a & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

If the matrix has n x n dimension, and bc > 0 , the eigenvalues are:

$$l_k = a + 2\sqrt{bc} \cdot \cos\left(\frac{kp}{n+1}\right)$$

where k = 1, 2… n

All roots lie on the interval:

$$a - 2\sqrt{bc} < l_k < a + 2\sqrt{bc}$$

2) The sub-diagonals have different sign. In that case we can demonstrate that all root are complex conjugate for n even; for n odd exists only one real root λ = a .

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 & \dots \\ c & a & b & 0 & 0 & \dots \\ 0 & c & a & b & 0 & \dots \\ 0 & 0 & c & a & b & \dots \\ 0 & 0 & 0 & c & a & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

If the matrix has n x n dimension, and bc < 0 , the eigenvalues are complex:

$$l_k = a + i \cdot 2\sqrt{-bc} \cdot \cos\left(\frac{kp}{n+1}\right) = a + id_k$$

where k = 1, 2… n

All roots lie on the segment:

$$re(l_k) = a \qquad -2\sqrt{-bc} < im(l_k) < 2\sqrt{-bc}$$

Eigenvectors can be computed by the following iterative algorithm

$$x_k = l_k - a$$

$$u_{ik} = \frac{1}{b}\left(x_k \cdot u_{(i-1)k} - c \cdot u_{(i-2)k}\right)$$

Where : k = 1, 2… n , i = 1, 2… n

$$u_{1k} = 1 \ , \ u_{2k} = \frac{1}{b} x_k$$

Prof:

Taking the following n x n matrix U, we are researching all its eigenvalues

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & ... \\ 1 & 0 & 1 & 0 & 0 & ... \\ 0 & 1 & 0 & 1 & 0 & ... \\ 0 & 0 & 1 & 0 & 1 & ... \\ 0 & 0 & 0 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix} \qquad U - tI = \begin{bmatrix} t & 1 & 0 & 0 & 0 & ... \\ 1 & t & 1 & 0 & 0 & ... \\ 0 & 1 & t & 1 & 0 & ... \\ 0 & 0 & 1 & t & 1 & ... \\ 0 & 0 & 0 & 1 & t & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

The characteristic polynomial ***U-tI*** can be computed by the iterative process

$P_n = t \cdot P_{n-1} - P_{n-2}$, with $P_0 = 1$, $P_1 = t$

Note that it is not necessary that both subdiagonals are equal. It necessary only that their product makes one. In that case the iterative process gives the same characteristic polynomial. So, the following matrices have the same eigenvalues

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & ... \\ 1 & 0 & 1 & 0 & 0 & ... \\ 0 & 1 & 0 & 1 & 0 & ... \\ 0 & 0 & 1 & 0 & 1 & ... \\ 0 & 0 & 0 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 1/k & 0 & 0 & 0 & ... \\ k & 0 & 1/k & 0 & 0 & ... \\ 0 & k & 0 & 1/k & 0 & ... \\ 0 & 0 & k & 0 & 1/k & ... \\ 0 & 0 & 0 & k & 0 & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

The first 6 polynomials are:

$P_0 = 1$

$P_1 = t$

$P_2 = t^2 - 1$

$P_3 = t^3 - 2t$

$P_4 = t^4 - 3t^2 + 1$

$P_5 = t^5 - 4t^3 + 3t$

$P_6 = t^6 - 5t^4 + 6t^2 - 1$

We can do the following observations:

- Odd polynomials contain only odd powers
- Even polynomial contain only even powers
- All roots are symmetric around the zero
- All roots are real being the matrix U symmetric
- For n odd exists the trivial root t=0
- For all roots is $|t_i| < 2$ (Gersgorin's Theorem)

The above iterative process can be re-written as a difference equation

$P_n - t \cdot P_{n-1} + P_{n-2} = 0$

Taking its associate equation, we have

$x^2 - t \cdot x + 1 = 0 \quad \Rightarrow \quad \Delta = t^2 - 4$

Being

$|t| < 2 \;\Rightarrow\; t^2 < 4 \;\Rightarrow\; t^2 - 4 < 0 \;\Rightarrow\; \Delta < 0$

Then, the equation has two complex conjugate roots

$$x_{1,2} = \frac{t}{2} \pm i\,\frac{\sqrt{4-t^2}}{2} = \quad e^{\pm iq}$$

Where:

$$\cos(q) = \frac{\mathrm{t}}{2} \quad , \quad \sin(q) = \frac{\sqrt{4-t^2}}{2} \quad , \quad \tan(q) = \frac{\sqrt{4-t^2}}{t}$$

The general solution of this difference equation is

$$P_n = A_1 \cos(nq) + A_2 \sin(nq)$$

The coefficients A_1 and A_2 can be determined by the starting condition.

$$P_0 = A_1 = 1 \qquad \Rightarrow \quad A_1 = 1$$

$$P_1 = A_1 \cos(q) + A_2 \sin(q) = t$$

Substituting A_1, cos(θ), sin(θ), into the last equation we have

$$\frac{t}{2} + A_2 \cdot \frac{\sqrt{4-t^2}}{2} = t \quad \Rightarrow \quad A_2 = \frac{t}{\sqrt{4-t^2}} = \frac{1}{\tan(q)}$$

Then, the solution is

$$P_n = \cos(nq) + \frac{\sin(nq)}{\tan(q)}$$

$$q = \operatorname{atan}\left(\frac{\sqrt{4-t^2}}{t}\right)$$

P_n is the characteristic polynomial of the n x n uniform tridiagonal matrix U
These equations give the value of the n^{th}-degree polynomial for a given value of t

In order to find the eigenvalues of matrix U, we have to solve the equation $P_n = 0$

$$P_n = 0 \quad \Rightarrow \quad \cos(nq) + \frac{\sin(nq)}{\tan(q)} = 0$$

Rearranging we have

$$\sin(q)\cdot\cos(nq) + \cos(q)\cdot\sin(nq) = 0 \quad \Rightarrow \quad \sin(q + nq) = 0$$

$$\Rightarrow \quad q + nq = kp \quad \Rightarrow \quad q_k = \frac{kp}{n+1}$$

Being:

$$\tan(q) = \frac{\sqrt{4-t^2}}{t}$$

Solving for t variable, we have

$$\tan^2(q) = \frac{4-t^2}{t^2} \quad \Rightarrow \quad t^2 = \frac{4}{1+\tan^2(q)} = 4\cos^2(q) \quad \Rightarrow \quad t = 2\cos(q)$$

Substituting the angle θ_k in the above equation we get, finally, the close formula for the polynomial roots of $P_n(t)$ and the eigenvalues of matrix U:

$$t_k = 2\cos\left(\frac{kp}{n+1}\right)$$

Where:
k = 1, 2, …n

Note that, if n is odd, the above formula gives also the trivial root: t = 0

Taking now the following n x n matrix U, we are researching all its eigenvalues

$$Q = \begin{bmatrix} a & 1 & 0 & 0 & 0 & ... \\ 1 & a & 1 & 0 & 0 & ... \\ 0 & 1 & a & 1 & 0 & ... \\ 0 & 0 & 1 & a & 1 & ... \\ 0 & 0 & 0 & 1 & a & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix} \qquad Q - lI = \begin{bmatrix} a-l & 1 & 0 & 0 & 0 & ... \\ 1 & a-l & 1 & 0 & 0 & ... \\ 0 & 1 & a-l & 1 & 0 & ... \\ 0 & 0 & 1 & a-l & 1 & ... \\ 0 & 0 & 0 & 1 & a-l & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Substituting $t = a - \lambda$, we fall in the case previous discussed. So its eigenvalues are given by the formula:

$$t_k = 2\cos\left(\frac{kp}{n+1}\right)$$

And substituting, we have:

$$t_k = a + 2\cos\left(\frac{kp}{n+1}\right)$$

Where:
k = 1, 2, …n

Taking finally the n x n matrix A, tridiagonal, symmetric and uniform

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 & ... \\ b & a & b & 0 & 0 & ... \\ 0 & b & a & b & 0 & ... \\ 0 & 0 & b & a & b & ... \\ 0 & 0 & 0 & b & a & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Dividing al values for b we normalize the upper and lower sub-diagonals having the new matrix M

$$M = \begin{bmatrix} a/b & 1 & 0 & 0 & 0 & ... \\ 1 & a/b & 1 & 0 & 0 & ... \\ 0 & 1 & a/b & 1 & 0 & ... \\ 0 & 0 & 1 & a/b & 1 & ... \\ 0 & 0 & 0 & 1 & a/b & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Eigenvalues of matrix M are the same of matrix A scaled by factor 1/b. That is:

$\mu_k = \lambda_k /b$ (1)

Eigenvalues of matrix M can be computed with the formula:

$$t_k = \frac{a}{b} + 2\cos\left(\frac{kp}{n+1}\right)$$

Where:
k = 1, 2, …n

Substituting the eigenvalues relation (1) we have

$$l_k = a + 2b \cdot \cos\left(\frac{kp}{n+1}\right)$$

Where:
k = 1, 2, …n

Eigenvectors

For symmetric uniform tridiagonal matrix we can obtain a closed compact formula for all eigenvectors

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 & ... \\ b & a & b & 0 & 0 & ... \\ 0 & b & a & b & 0 & ... \\ 0 & 0 & b & a & b & ... \\ 0 & 0 & 0 & b & a & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix} \qquad A - lI = \begin{bmatrix} a-l & b & 0 & 0 & 0 & ... \\ b & a-l & 0 & 0 & 0 & ... \\ 0 & 0 & a-l & 0 & 0 & ... \\ 0 & 0 & 0 & a-l & 0 & ... \\ 0 & 0 & 0 & 0 & a-l & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Each eigenvector is the solution if the homogeneous linear system

$$(A - l_k I) \cdot u_k = 0$$

Substituting the eigenvalue λ_k given from the above formula we have

$$\begin{bmatrix} -2b\cos(a_k) & b & 0 & ... \\ b & -2b\cos(a_k) & b & ... \\ 0 & b & -2b\cos(a_k) & ... \\ ... & ... & ... & ... \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ ... \end{bmatrix} = 0$$

That can be rearranged as

$$\begin{cases} -2b\cos(a_k)\cdot u_1 + b\cdot u_2 = 0 \\ b\cdot u_1 - 2b\cos(a_k)\cdot u_2 + b\cdot u_3 = 0 \\ \ldots \\ b\cdot u_{i-1} - 2b\cos(a_k)\cdot u_i + b\cdot u_{i+1} = 0 \end{cases} \qquad \begin{cases} u_2 = 2\cos(a_k)\cdot u_1 \\ u_3 = 2\cos(a_k)\cdot u_2 - u_1 \\ \ldots \\ u_{i+1} = 2\cos(a_k)\cdot u_i - u_{i-1} \end{cases}$$

The first value u_1 is arbitrary. If we choose $u_1 = \sin(\alpha)$, all expressions change in a nice form, using the following recurrent formula

$$\sin(n\cdot a) = 2\cos(a)\cdot\sin((n-1)\cdot a) - \sin((n-2)\cdot a)$$

$$\begin{cases} u_1 = \sin(a_k) \\ u_2 = 2\cos(a_k)\cdot\sin(a_k) \\ \ldots \\ u_{i+1} = 2\cos(a_k)\cdot u_i - u_{i-1} \end{cases} \qquad \begin{cases} u_1 = \sin(a_k) \\ u_2 = \sin(2a_k) \\ \ldots \\ u_i = \sin(ia_k) \end{cases}$$

We have obtained the general eigenvector for the eigenvalue λ_k. Repeating the computation for each eigenvalue we have

$$u_{ik} = \sin(ia_k) = \sin\left(i\cdot k\frac{p}{n+1}\right)$$

Where i = 1, 2,…n and k = 1, 2, …n

Unsymmetrical matrix

The case of unsymmetrical matrix can be led back to the above one if the sub-diagonals have the <u>same sign</u>. Taking the n x n matrix A, tridiagonal, unsymmetrical and uniform, we divide each element for $\sqrt{bc}$.

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 & \ldots \\ c & a & b & 0 & 0 & \ldots \\ 0 & c & a & b & 0 & \ldots \\ 0 & 0 & c & a & b & \ldots \\ 0 & 0 & 0 & c & a & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \qquad z = \begin{bmatrix} \frac{a}{\sqrt{bc}} & \frac{b}{\sqrt{bc}} & 0 & \ldots \\ \frac{c}{\sqrt{bc}} & \frac{a}{\sqrt{bc}} & \frac{b}{\sqrt{bc}} & \ldots \\ 0 & \frac{c}{\sqrt{bc}} & \frac{a}{\sqrt{bc}} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

Eigenvalues of matrix A and matrix Z are related by the following :

$$z_i = \frac{l_i}{\sqrt{bc}}$$

Now the product of sub-diagonal elements of matrix Z is 1, so we can apply the following formula:

$$z_k = \frac{a}{\sqrt{bc}} + 2\cos\left(\frac{kp}{n+1}\right)$$

Substituting the eigenvalues relation we have

$$l_k = a + 2\sqrt{bc} \cdot \cos\left(\frac{kp}{n+1}\right)$$

Where:
k = 1, 2, …n

q.e.d.

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Example

Find all eigenvalues of the following tridiagonal uniform 8 x 8 matrix

10	**1**	0	0	0	0	0	0
4	**10**	**1**	0	0	0	0	0
0	**4**	**10**	**1**	0	0	0	0
0	0	**4**	**10**	**1**	0	0	0
0	0	0	**4**	**10**	**1**	0	0
0	0	0	0	**4**	**10**	**1**	0
0	0	0	0	0	**4**	**10**	**1**
0	0	0	0	0	0	**4**	**10**

We observe that the subdiagonal values have the same sign so all eigenvalues are real and distinct. They can be obtained by the following close formula:

$$l_k = a + 2\sqrt{bc} \cdot \cos\left(\frac{kp}{n+1}\right)$$

for k = 1, 2, …8 and where a = 10, b = 1, c = 4, n= 8

Giving the following 8 eigenvalues

λ_1	13.7587704831436
λ_2	13.0641777724759
λ_3	12
λ_4	10.6945927106677
λ_5	9.30540728933228
λ_6	8
λ_7	6.93582222752409
λ_8	6.24122951685637

All eigenvalues are contained into the interval (a – 4, a + 4) = (6, 14)

Example

Find all eigenvalues of the following <u>tridiagonal uniform</u> 7 x 7 matrix

10	**2**	0	0	0	0	0
-1	**10**	**2**	0	0	0	0
0	**-1**	**10**	**2**	0	0	0
0	0	**-1**	**10**	**2**	0	0
0	0	0	**-1**	**10**	**2**	0
0	0	0	0	**-1**	**10**	**2**
0	0	0	0	0	**-1**	**10**

We observe that the subdiagonal values have different sign and the dimension n is odd, then all eigenvalues are complex conjugate except only one real trivial root $\lambda = 10$.
They can be obtained by the following close formula:

$$l_k = a + i \cdot 2\sqrt{-bc} \cdot \cos\left(\frac{kp}{n+1}\right) = a + id_k$$

for k = 1, 2, …7   and where a = 10, b = 2, c = –1, n= 7

Giving the following 7 eigenvalues.

	real	imm
λ_1	10	2.6131259297528
λ_2	10	2
λ_3	10	1.0823922002924
λ_4	10	0
λ_5	10	-1.0823922002924
λ_6	10	-2
λ_7	10	-2.6131259297528

Example

Find all eigenvalues of the following <u>tridiagonal uniform</u> 8 x 8 matrix

1	**1**	0	0	0	0	0	0
-1	**1**	**1**	0	0	0	0	0
0	**-1**	**1**	**1**	0	0	0	0
0	0	**-1**	**1**	**1**	0	0	0
0	0	0	**-1**	**1**	**1**	0	0
0	0	0	0	**-1**	**1**	**1**	0
0	0	0	0	0	**-1**	**1**	**1**
0	0	0	0	0	**0**	**-1**	**1**

We observe that the subdiagonal values have different sign and the dimension n is even, then no real eigenvalues exist and all eigenvalues are complex conjugate.
They can be obtained by the following close formula:

$$l_k = a + i \cdot 2\sqrt{-bc} \cdot \cos\left(\frac{kp}{n+1}\right) = a + id_k$$

for k = 1, 2, … 8 and where a = 1, b = 1, c = –1, n= 8

Giving the following 8 eigenvalues.

	real	imm
λ_1	1	1.8793852415718
λ_2	1	1.5320888862380
λ_3	1	1
λ_4	1	0.3472963553339
λ_5	1	-0.3472963553339
λ_6	1	-1
λ_7	1	-1.5320888862380
λ_8	1	-1.8793852415718

Example

Find all eigenvalues of the following <u>tridiagonal uniform</u> 8 x 8 matrix

-2	**1**	0	0	0	0	0	0
1	**-2**	**1**	0	0	0	0	0
0	**1**	**-2**	**1**	0	0	0	0
0	0	**1**	**-2**	**1**	0	0	0
0	0	0	**1**	**-2**	**1**	0	0
0	0	0	0	**1**	**-2**	**1**	0
0	0	0	0	0	**1**	**-2**	**1**
0	0	0	0	0	0	**1**	**-2**

We observe that the matrix is symmetric so all eigenvalues are real and distinct.
They can be obtained by the following close formula:

$$l_k = a + 2b \cdot \cos\left(\frac{kp}{n+1}\right)$$

for k = 1, 2, …8   and where a = –2, b = 1, c = 1, n= 8

Giving the following 8 eigenvalues

λ_1	-0.1206147584282
λ_2	-0.4679111137620
λ_3	-1
λ_4	-1.6527036446661
λ_5	-2.34729635533386
λ_6	-3
λ_7	-3.53208888623796
λ_8	-3.87938524157182

All eigenvalues are contained into the interval (a – 2, a + 2) = (–4, 0)
We observe that they are all negative

The eigenvectors matrix can be obtained in a very fast way using the formula

$$u_{ij} = \sin\left(i \cdot j \frac{p}{n+1}\right)$$

$$U = \begin{bmatrix} \sin(a) & \sin(2a) & \dots & \sin(8a) \\ \sin(2a) & \sin(4a) & \dots & \sin(16a) \\ \dots & \dots & \dots & \dots \\ \sin(8a) & \sin(16a) & \dots & \sin(64a) \end{bmatrix}$$

That gives the following approximate matrix

0.342020143	0.64278761	0.866025404	0.984807753	0.984807753	0.866025404	0.64278761	0.342020143
0.64278761	0.984807753	0.866025404	0.342020143	-0.34202014	-0.8660254	-0.9848078	-0.64278761
0.866025404	0.866025404	0	-0.8660254	-0.8660254	0	0.8660254	0.866025404
0.984807753	0.342020143	-0.8660254	-0.64278761	0.64278761	0.866025404	-0.3420201	-0.98480775
0.984807753	-0.34202014	-0.8660254	0.64278761	0.64278761	-0.8660254	-0.3420201	0.984807753
0.866025404	-0.8660254	0	0.866025404	-0.8660254	0	0.8660254	-0.8660254
0.64278761	-0.98480775	0.866025404	-0.34202014	-0.34202014	0.866025404	-0.9848078	0.64278761
0.342020143	-0.64278761	0.866025404	-0.98480775	0.984807753	-0.8660254	0.64278761	-0.34202014

Note that the column-vectors are orthogonals.