How to transform unsymmetrical tridiagonal matrix to symmetrical one

In the field of numerical calculus there are many efficient algorithms to solve the eigen-problem for symmetrical matrices. Thus, when we have a problem involving an unsymmetrical matrix we should try to transform the unsymmetrical matrix into a symmetrical one having the same eigenvalues.
For **tridiagonal matrices** there is a very handy and short way to make this transformation. Remember that a symmetrical matrix has always real eigenvalues. Thus this method can be useful to test if a general tridiagonal matrix has all real eigenvalues or not.

Giving the unsymmetrical tridiagonal matrix A

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ c_2 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & c_3 & a_3 & b_3 & \dots & 0 \\ 0 & 0 & c_4 & a_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & c_n & a_n \end{bmatrix}$$

We can apply the similarity transform $S = D^{-1} \cdot A \cdot D$
Where D is a diagonal matrix

$$S = \begin{bmatrix} d_1^{-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & d_2^{-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & d_3^{-1} & 0 & \dots & 0 \\ 0 & 0 & 0 & d_4^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & d_n^{-1} \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ c_2 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & c_3 & a_3 & b_3 & \dots & 0 \\ 0 & 0 & c_4 & a_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & c_n & a_n \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & d_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & d_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & d_n \end{bmatrix}$$

The diagonal element [d_1, d_2, d_3…d_n] can be obtained by the following iterative formula (***square root formula***)

$$d_1 = 1 \Rightarrow d_2 = \sqrt{\frac{c_2}{b_1}} \cdot d_1 \;\Rightarrow d_3 = \sqrt{\frac{c_3}{b_2}} \cdot d_2 \;\; \dots \Rightarrow d_n = \sqrt{\frac{c_n}{b_{n-1}}} \cdot d_{n-1}$$

As we can see, this process can be applied if the square root argument is positive. Thus must be:

$$\frac{c_k}{b_{k-1}} > 0 \qquad k = 1,2 \dots n$$

Conditions for symmetrical tridiagonal transformation, and, thus for all real eigenvalues

The above conditions are sufficient to transform the unsymmetrical tridiagonal matrix **A** into a symmetrical tridiagonal matrix **S**. As known, the similarity transform does not alter the eigenvalues. Thus the matrices **S** and **A** are the same eigenvalues (similar matrices).

Because of the symmetry, the matrix **S** has all real eigenvalues and then, also the matrix **A** has all real eigenvalues

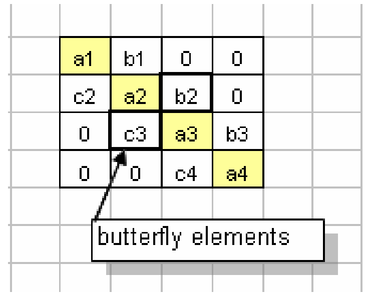
Test for real eigenvalues of tridiagonal matrix

If each elements couple of a tridiagonal matrix:

c_k , b_{k-1} (this couple, like a “butterfly”, is put in evidence in the figure)

have the same sign, then the eigenvalues are all real.

We can say, as well, that the matrix can be converted in symmetrical form by similarity transform.



Example

Given the following (8 x 8) tridiagonal matrix, find a symmetric similar matrix

A =

5	1	0	0	0	0	0	0
4	6	4	0	0	0	0	0
0	1	-7	8	0	0	0	0
0	0	2	-1	-4	0	0	0
0	0	0	-16	-5	5	0	0
0	0	0	0	5	8	18	0
0	0	0	0	0	2	2	27
0	0	0	0	0	0	3	-1

Test

4 x 1 > 0
1 x 4 > 0
2 x 8 > 0
(–16) x (–4) > 0
5 x 5 > 0
2 x 18 > 0
3 x 27 > 0
result: positive

The “butterfly test” is positive so the matrix can be converted into symmetric one by similar transform

With the iterative formula we find the elements of diagonal matrix

d1 = 1, ⇒ d2 = 1* sqrt(4/1) = 2, ⇒ d3 = 2* sqrt(1/4) = 1 …

Obtainig

L =

1	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	1/2	0	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	1/3	0
0	0	0	0	0	0	0	1/9

$\mathbf{L}^{-1}$ **=**

1	0	0	0	0	0	0	0
0	1/2	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	2	0	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	3	0
0	0	0	0	0	0	0	9

Performing the similarity transform

$$S = D^{-1} \cdot A \cdot D$$

We get the symmetrical tridiagonal matrix

5	2	0	0	0	0	0	0
2	6	2	0	0	0	0	0
0	2	-7	4	0	0	0	0
0	0	4	-1	-8	0	0	0
0	0	0	-8	-5	5	0	0
0	0	0	0	5	8	6	0
0	0	0	0	0	6	2	9
0	0	0	0	0	0	9	-1

Breaking down method

When the tridiagonal matrix has one or more zero elements into sub diagonals the above process cannot be applied. Fortunately in this case the eigen-problem can be simplified breaking the given matrix into one or more sub-matrix. Let'see this example

Element $b_4 = 0$, so the given (8 x 8) matrix can be broken into two matrices of (4 x 4) dimension

5	2	0	0	0	0	0	0
2	6	2	0	0	0	0	0
0	4	-7	4	0	0	0	0
0	0	1	-1	0	0	0	0
0	0	0	3	-5	5	0	0
0	0	0	0	1	8	3	0
0	0	0	0	0	1	2	5
0	0	0	0	0	0	9	-1

5	2	0	0				
2	6	2	0				
0	4	-7	4				
0	0	1	-1	0			
			3	-5	5	0	0
				1	8	3	0
				0	1	2	5
				0	0	9	-1

For each sub matrix we can now repeat the process of similarity transform for finding the symmetrical form. Then, we find the eigenvalues of the first and the second matrix separately The union of two set of eigenvalues gives all the eigenvalues of the original matrix

Eigenvalues of the upper matrix

λ1	7.921770128
λ2	3.710149293
λ3	-8.138611526
λ4	-0.493307896

Eigenvalues set of the lower matrix

λ5	9.348469228
λ6	-5.348469228
λ7	6.480740698
λ8	-6.480740698

Eigenvalues of the given matrix

λ1	-8.138611526
λ2	-6.480740698
λ3	-5.348469228
λ4	-0.493307896
λ5	3.710149293
λ6	6.480740698
λ7	7.921770128
λ8	9.348469228